Math 361, Section F1, Fall 2000
Quiz 1, August 30 **Name:** ____________________

1. 10 points Suppose that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.3$, and $\mathbb{P}(A \cap B) = 0.2$.

   (a) 4 points Compute $\mathbb{P}(A|B)$.

   (b) 4 points Compute $\mathbb{P}(B|A)$.

   (c) 2 points Compute $\mathbb{P}(A|A \cap B)$.

Answers

1. (a) $\mathbb{P}(A|B) = \frac{0.2}{0.3} = \frac{2}{3}$.
   (b) Compute $\mathbb{P}(B|A) = \frac{0.2}{0.5} = \frac{2}{5}$.
   (c) Compute $\mathbb{P}(A|A \cap B) = \frac{\mathbb{P}(A \cap A \cap B)}{\mathbb{P}(A \cap B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cap B)} = 1$.

Math 361, Section F1, Fall 2000
Quiz 2, September 8 **Name:** ____________________

1. [10 points] Consider a certain test for cancer. Suppose that 90% of those with cancer react positively to the test, while 5% of those without cancer also react positively to the test. Suppose also that 1% of people actually have cancer. Suppose you take the test and react positively. What is the probability that you actually have cancer?

Answers

1. Define the sets

$$P \stackrel{\text{def}}{=} \{\text{test positive}\} \qquad \text{and} \qquad C = \{\text{have cancer}\}.$$

Then

$$\mathbb{P}(P|C) = 0.9, \quad \mathbb{P}(P|C^c) = 0.05, \quad \mathbb{P}(C) = 0.1, \quad \text{and} \quad \mathbb{P}(C^C) = 0.99.$$

Then

$$\mathbb{P}(C|P) = \frac{\mathbb{P}(P|C)\mathbb{P}(C)}{\mathbb{P}(P|C)\mathbb{P}(C) + \mathbb{P}(P|C^c)\mathbb{P}(C^c)} = \frac{(0.9)(0.01)}{(0.9)(0.01) + (0.05)(0.99)}.$$

Math 361, Section F1, Fall 2000
Quiz 3, September 15 **Name:** ______________________

1. 10 points A mathematics department consists of

   - 15 full professors
   - 10 associate professors
   - 5 assistant professors.

   A committee of 6 is selected at random from the faculty of the department. What is the probability that the committee will consist of

   - 2 full professors
   - 2 associate professors
   - 2 assistant professors?

Answers

1.

$$\frac{\binom{15}{2}\binom{10}{2}\binom{5}{2}}{\binom{30}{6}}.$$

Math 361, Section F1, Fall 2000
Quiz 4, October 2 **Name:** ____________________

1. 10 points Suppose that we have a cumulative distribution function given by the formula

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{8} & \text{if } 0 \le t < \frac{1}{3} \\ \frac{1}{9} + \left(t - \frac{1}{3}\right)^2 & \text{if } \frac{1}{3} \le t < \frac{2}{3} \\ \frac{1}{2} & \text{it } \frac{2}{3} \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

Compute

$$\mathbb{P}\left\{\text{either } X = \frac{1}{3} \text{ or } X \ge \frac{3}{4}\right\}.$$

1.

$$\begin{aligned}\mathbb{P}\left\{\text{either } X = \frac{1}{3} \text{ or } X \geq \frac{3}{4}\right\} &= \mathbb{P}\left\{X = \frac{1}{3}\right\} + \mathbb{P}\left\{X \geq \frac{3}{4}\right\} \\ &= \mathbb{P}\left\{X = \frac{1}{3}\right\} + \left(1 - \mathbb{P}\left\{X < \frac{3}{4}\right\}\right) = F_X(1/3) - F_X(1/3-) + 1 - F_X(3/4-) \\ &= \frac{1}{9} - \frac{1}{24} + 1 - \frac{1}{2}.\end{aligned}$$

Math 361, Section F1, Fall 2000
Quiz 5, October 9 **Name:** ____________________

1. [10 points] Suppose that the random variable X has a geometric density with parameter 1/8; i.e.,

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{8}\left(1-\frac{1}{8}\right)^j & \text{if } j \in \{0,1\dots\} \\ 0 & \text{else} \end{cases}$$

Define now

$$Y \stackrel{\text{def}}{=} X + 7.$$

Find the density of Y.

Answers

1.

$$f_Y(j) = \begin{cases} \frac{1}{8}\left(\frac{7}{8}\right)^{j-7} & \text{if } j \in \{7, 8, 9 \dots\} \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 2000
Quiz 6, October 13 **Name:** ____________________

1. [10 points] Suppose that we have a random variable whose density is given by

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{4} & \text{if } j = 0 \\ \frac{1}{8} & \text{if } j = 1 \\ \frac{1}{8} & \text{if } j = 3 \\ \frac{1}{2} & \text{if } j = 4 \end{cases}$$

   (a) [3 points] Compute $\mathbb{E}[X]$.

   (b) [4 points] Compute $\mathbb{E}[X^2]$.

   (c) [3 points] Compute $\mathbb{E}[X-3]$.

Answers

1. (a)

$$\mathbb{E}[X] = \frac{(0)(2) + (1)(1) + (3)(1) + (4)(4)}{8} = \frac{20}{8}.$$

(b)

$$\mathbb{E}[X^2] = \frac{(0)^2(2) + (1)(1) + (3)^2(1) + (4)^2(4)}{8} = \frac{74}{8}.$$

(c) $\mathbb{E}[X - 3] = \mathbb{E}[X] - 3 = \frac{20}{8} - 3 = -\frac{4}{8}$.

Math 361, Section F1, Fall 2000
Quiz 7, October 25 **Name:** ____________________

1. [10 points] Suppose that the continuous random variable X has a density

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} t & \text{if } 0 \le t \le 1 \\ 2 - t & \text{if } 1 < t \le 2. \end{cases}$$

Compute

$$\mathbb{P}\left\{\frac{1}{2} \le X \le \frac{3}{2}\right\}.$$

Answers

1.

$$\mathbb{P}\left\{\frac{1}{2} \le X \le \frac{3}{2}\right\} = \int_{1/2}^{3/2} f_X(s)ds = \int_{1/2}^{1} sds + \int_{1}^{3/2}(2-s)ds$$

$$= 2\int_{1/2}^{1} sds = s^2\Big|_{s=1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}.$$

Math 361, Section F1, Fall 2000
Quiz 8, November 1 **Name:** ____________________

1. [10 points] Suppose that the continuous random variable X has a density

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} 2t & \text{if } 0 \le t \le 1 \\ 0 & \text{else.} \end{cases}$$

   (a) [5 points] Compute the cumulative density F_X of X.

   (b) [5 points] Define $Y \stackrel{\text{def}}{=} X^3$. Compute the density of Y (if one exists).

ANSWERS

1. (a)

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2 & \text{if } 0 \le t < 1 \\ 1 & \text{if } 0 \le t \ge 1 \end{cases}$$

(b)

$$f_Y(t) = \begin{cases} \frac{2}{3}t^{-1/3} & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 2000
Quiz 9, November 6 **Name:** ____________________

1. 10 points Suppose that the pair (X, Y) of continuous random variables has joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} \lambda^2 e^{-\lambda t} & \text{if } 0 \le s \le t \\ 0 & \text{else.} \end{cases}$$

Compute $\mathbb{P}\{X \le 7\}$.

Answers

1.

$$\mathbb{P}\{X \le 7\} = \int_{s=-\infty}^{7} \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)\,dt\,ds = \int_{s=0}^{7} \int_{t=s}^{\infty} \lambda^2 e^{-\lambda t}\,dt\,ds$$

$$= \int_{s=0}^{7} \lambda e^{-\lambda s} ds = 1 - e^{-7\lambda}.$$

Math 361, Section F1, Fall 2000
Quiz 10, November 8 **Name:** ______________________

1. [10 points] Suppose that the pair (X, Y) of continuous random variables has joint density

$$f_{X,Y}(s,t) \overset{\text{def}}{=} \begin{cases} \lambda^2 e^{-\lambda t} & \text{if } 0 \le s \le t \\ 0 & \text{else.} \end{cases}$$

Compute $f_Y(7)$.

Answers

1.

$$f_Y(7) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,7)ds = \int_{s=0}^{7} \lambda^2 e^{-7\lambda} ds = 7\lambda^2 e^{-7\lambda}.$$

Math 361, Section F1, Fall 2000
Quiz 11, November 13 Name: ______________

1. 10 points Suppose that X and Y are independent continuous random variables. Assume that X is uniform on $[0,1]$; i.e.,

$$f_X(s) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \le s \le 1 \\ 0 & \text{else} \end{cases}$$

and assume that Y is geometric with parameter $\lambda > 0$; i.e.,

$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \lambda e^{-\lambda t} & \text{if } t \ge 0 \\ 0 & \text{else.} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$.

(a) 5 points Compute $f_Z(5)$.

(b) 5 points Compute $f_Z(0.5)$.

1. (a)

$$
\begin{aligned}
f_Z(5) = \int_{t=-\infty}^{\infty} f_X(t) f_Y(5-t)\,dt &= \int_{t=-\infty}^{\infty} \chi_{[0,1]}(t)\chi_{[0,\infty)}(5-t)\lambda e^{-\lambda(5-t)}\,dt \\
&= \int_{t=-\infty}^{\infty} \chi_{[0,1]\cap(-\infty,5]}(t)\lambda e^{-\lambda(5-t)}\,dt \\
&= \int_{t=0}^{1} \lambda e^{-\lambda(5-t)}\,dt = \int_{t=4}^{5} \lambda e^{-\lambda t}\,dt = e^{-4\lambda} - e^{-5\lambda}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_Z(0.5) = \int_{t=-\infty}^{\infty} f_X(t) f_Y(0.5-t)\,dt &= \int_{t=-\infty}^{\infty} \chi_{[0,1]}(t)\chi_{[0,\infty)}(0.5-t)\lambda e^{-\lambda(0.5-t)}\,dt \\
&= \int_{t=-\infty}^{\infty} \chi_{[0,1]\cap(-\infty,0.5]}(t)\lambda e^{-\lambda(0.5-t)}\,dt \\
&= \int_{t=0}^{0.5} \lambda e^{-\lambda(0.5-t)}\,dt = \int_{t=0}^{0.5} \lambda e^{-\lambda t}\,dt = 1 - e^{-0.5\lambda}.
\end{aligned}
$$

Math 361, Section F1, Fall 2000
Quiz 12, November 27 **Name:** ____________________

1. 10 points Suppose that X is a random variable with moment generating function

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[e^{\theta X}\right] = \exp\left[4\theta^2 + 9\theta\right].$$

What is the density of X?

Answers

1. X is Gaussian with mean 9 and variance 16;

$$f_X(t) = \frac{1}{\sqrt{32\pi}} \exp\left[-\frac{(t-9)^2}{32}\right] \qquad t \in \mathbb{R}$$

Math 361, Section F1, Fall 2000
Exam 1, September 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [35 points] A mathematics department consists of

   - 15 full professors
   - 10 associate professors
   - 5 assistant professors.

   A committee of 6 is selected at random from the faculty of the department.

   (a) [5 points] What is the probability that the committee will consist of 1 full professor, 2 associate professors, and 3 assistant professors?

   (b) [10 points] What is the probability that the committee will consist of at least 3 full professors?

   (c) [10 points] What is the probability that the committee will consist of exactly one assistant professor and at most one full professors?

   (d) [10 points] What is the probability that the committee will consist of exactly one assistant professor given that it has at most one full professors?

2. [25 points] There are 6 balls labelled A through F are in a box. We pick them out of the box one by one.

   (a) [5 points] What is the probability that the third ball is ball B?

   (b) [10 points] What is the probability that the third ball is ball B and the fourth ball is ball A?

   (c) [10 points] What is the probability that the fourth ball is A given that the third ball is B?

3. [40 points] Suppose that we select a hand of five cards out of a standard deck of 52 cards. We assume that aces are high cards.

   (a) [10 points] What is the probability that all of the cards are greater than 5?

   (b) [15 points] What is the probability that at least two of the cards are greater than 10?

   (c) [15 points] What is the probability that all of the cards are greater than 5 given that at least two of the cards are greater than 10?

Answers

1. Set $q \stackrel{\text{def}}{=} 1/\binom{30}{6}$.
   (a) $15\binom{10}{2}\binom{5}{3}q$.
   (b)
$$1-\mathbb{P}\{0,1,\text{ or } 2 \text{ full professors}\} = 1-\left\{\binom{15}{6}+15\binom{15}{5}+\binom{15}{2}\binom{15}{4}\right\}q.$$
   (c)
$$\left\{5\binom{10}{5}+5\cdot 15\binom{10}{4}\right\}q.$$
   (d)
$$\frac{5\binom{10}{5}+5\cdot 15\binom{10}{4}}{\binom{15}{6}+15\binom{15}{5}}.$$

2. (a) $\frac{(5)_2}{(6)_3}=\frac{1}{6}$.
   (b) $\frac{(4)_2}{(6)_4}=\frac{1}{30}$.
   (c) $\frac{(1/30)}{(1/6)}=\frac{1}{5}$.

3. Sequence: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A Set $q \stackrel{\text{def}}{=} 1/\binom{52}{5}$.
   (a) $\binom{36}{5}q$
   (b)
$$\begin{aligned}1 &- \mathbb{P}\{\text{no card is greater than } 10\} - \mathbb{P}\{1 \text{ card is greater than } 10\}\\ &= 1-\mathbb{P}\{\text{all cards are 10 or less}\}\\ &\qquad -\mathbb{P}\{1 \text{ card is greater than 10, 4 cards are 10 or less}\}\\ &= 1-\binom{36}{5}q-16\binom{36}{4}q.\end{aligned}$$
   (c)
$$\begin{aligned}&\mathbb{P}\{\text{all cards greater than 5 and at least two cards greater than } 10\}\\ &\quad= \mathbb{P}\{\text{all cards greater than } 5\}\\ &\qquad -\mathbb{P}\{\text{all cards greater than 5 and no cards greater than } 10\}\\ &\qquad -\mathbb{P}\{\text{all cards greater than 5, one card greater than 10, and 4 cards 10 or less}\}\\ &\quad= \mathbb{P}\{\text{all cards greater than } 5\}\\ &\qquad -\mathbb{P}\{\text{all cards in } 6,7,8,9,10\\ &\qquad -\mathbb{P}\{4 \text{ cards in } 6,7,8,9,10, \text{ and 1 card in J,Q,K,A}\}\\ &\quad= \binom{36}{5}q-\binom{20}{5}q-16\binom{20}{4}q.\end{aligned}$$

Final answer is

$$\frac{\binom{36}{5} - \binom{20}{5} - 16\binom{20}{4}}{1 - \binom{36}{5} - 16\binom{36}{4}}.$$

Math 361, Section F1, Fall 2000
Exam 2, October 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [35 points] Let X be a geometric random variable with parameter p; i.e.,

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} p(1-p)^j & j \in \{0, 1 \dots\} \\ 0 & \text{else.} \end{cases}$$

Define a new random variable

$$Y \stackrel{\text{def}}{=} 13 - |X - 20|.$$

Find the density f_Y of Y.

2. [35 points] Suppose that X and Y have joint density

		Y: 0	1	2	3
	-1	3/27	1/27	1/27	3/27
X	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words, $f_{X,Y}(3, 0) = \frac{1}{9}$.

(a) [5 points] Compute f_X.

(b) [5 points] Compute f_Y.

(c) [10 points] Compute $\mathbb{E}[XY]$.

(d) [5 points] Compute $\mathbb{E}[X]$.

(e) [10 points] Compute $\mathbb{E}[X^2]$.

3. [30 points] Suppose that a random variable X has moment generating function

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\exp[\theta X]] = \exp\left[\frac{9\theta^2}{2} + 4\theta\right]. \qquad \theta \in \mathbb{R}$$

(a) [10 points] Compute $\mathbb{E}[X]$.

(b) [10 points] Compute $\mathbb{E}[X^2]$.

(c) [10 points] Compute the variance of X.

Answers

1. Easiest to look at graph of $t \mapsto 13 - |t-20|$. Analytically, we have as follows. If $Y = j$, then $13 - |X-20| = j$, so $|X-20| = 13-j$. Thus, either $X - 20 = 13-j$ or $X-20 = j-13$; i.e., $X = 33-j$ or $X = j+7$. Note that $33-j = j+7$ if and only if $j=13$, in which case $33-j = j+7 = 20$.

$$f_Y(j) = \begin{cases} f_X(20) & \text{if } j = 13 \\ f_X(33-j) + f_X(j+7) & \text{if } j \in \{0,1,2\dots 12\} \\ f_X(33-j) & \text{if } j \in \{-1,-2\dots\} \\ 0 & \text{else} \end{cases}$$
$$= \begin{cases} p(1-p)^{20} & \text{if } j = 13 \\ p(1-p)^{33-j} + p(1-p)^{j+7} & \text{if } j \in \{0,1,2\dots 12\} \\ p(1-p)^{33-j} & \text{if } j \in \{-1,-2\dots\} \\ 0 & \text{else} \end{cases}$$

2. (a)
$$f_X(j) = \begin{cases} \frac{6}{27} & \text{if } j = -1 \\ \frac{12}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$

(b)
$$f_Y(j) = \begin{cases} \frac{12}{27} & \text{if } j = 0 \\ \frac{1}{27} & \text{if } j = 1 \\ \frac{4}{27} & \text{if } j = 2 \\ \frac{10}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$

(c)
$$\mathbb{E}[XY] = \frac{(-1)\cdot 1 + (-2)\cdot 1 + (-3)\cdot 3 + 9\cdot 4}{27} = \frac{24}{27}.$$

(d)
$$\mathbb{E}[X] = \frac{(-1)\cdot 6 + 3\cdot 7}{27} = \frac{25}{27}.$$

(e)
$$\mathbb{E}[X^2] = \frac{(-1)^2\cdot 6 + 3^2\cdot 7}{27} = \frac{69}{27}.$$

3. We have that $\dot{\varphi}_X(\theta) = (9\theta+4)\varphi_X(\theta)$ and $\ddot{\varphi}_X(\theta) = \{(9\theta+4)^2 + 9\}\varphi_X(\theta)$ for all $\theta \in \mathbb{R}$.
   (a) $\mathbb{E}[X] = \dot{\varphi}_X(0) = 4$.

(b) $\mathbb{E}[X^2] = \ddot{\varphi}_X(0) = 4^2 + 9 = 25$.

(c) $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 25 - 4^2 = 9$.

Math 361, Section F1, Fall 2000
Exam 3, November 29

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. $\boxed{\text{47 points}}$ Let X be a uniform random variable on $[0,1]$; i.e.,

$$f_X(t) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

Define

$$Y_1 \overset{\text{def}}{=} X^3 \qquad Y_2 \overset{\text{def}}{=} X^{-3} \qquad \text{and} \qquad Y_3 \overset{\text{def}}{=} X^0.$$

(a) $\boxed{\text{5 points}}$ Compute F_X, the cumulative distribution function for X.

(b) $\boxed{\text{9 points}}$ Find the cumulative distribution function for Y_1.

(c) $\boxed{\text{5 points}}$ Find the density of Y_1 if it exists. If it doesn't exist, state so.

(d) $\boxed{\text{9 points}}$ Find the cumulative distribution function for Y_2.

(e) $\boxed{\text{5 points}}$ Find the density of Y_2 if it exists. If it doesn't exist, state so.

(f) $\boxed{\text{9 points}}$ Find the cumulative distribution function for Y_3.

(g) $\boxed{\text{5 points}}$ Find the density of Y_3 if it exists. If it doesn't exist, state so.

2. $\boxed{\text{30 points}}$ Suppose that X and Y are two continuous random variables with joint density

$$f_{X,Y}(s,t) \overset{\text{def}}{=} \begin{cases} e^{-t} & \text{if } 0 \le s \le t \\ 0 & \text{else.} \end{cases}$$

(a) $\boxed{\text{20 points}}$ Compute the cumulative distribution function of X.

(b) $\boxed{\text{10 points}}$ Compute the density of Y.

3. $\boxed{\text{23 points}}$ Suppose that X and Y are two independent continuous random variables with densities

$$f_X(s) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \le s \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) \overset{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{if } -1 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$.

(a) $\boxed{\text{10 points}}$ Compute $f_Z(0)$.

(b) $\boxed{\text{8 points}}$ Compute $\mathbb{E}[Y]$.

(c) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[Y^2]$.

1. (a)

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(b)

$$\begin{aligned} F_{Y_1}(t) = \mathbb{P}\{X^3 \le t\} = \mathbb{P}\{X \le t^{1/3}\} &= F_X(t^{1/3}) \\ &= \begin{cases} 0 & \text{if } t^{1/3} < 0 \\ t^{1/3} & \text{if } 0 \le t^{1/3} < 1 \\ 1 & \text{if } t^{1/3} \ge 1 \end{cases} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases} \end{aligned}$$

(c)

$$f_{Y_1}(t) = \begin{cases} \frac{1}{3t^{2/3}} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

(d) Look at graph of $t \mapsto t^{-3}$.

$$F_{Y_2}(t) = \mathbb{P}\{X^{-3} \le t\} = \begin{cases} \mathbb{P}\{X > t^{-1/3}\} & \text{if } t \ge 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 0 & \text{if } t < 1 \\ 1 - t^{-1/3} & \text{if } t \ge 1 \end{cases}$$

(e)

$$\begin{cases} \frac{1}{3t^{4/3}} & \text{if } t > 1 \\ 0 & \text{else} \end{cases}$$

(f) $Y_3 \equiv 1$.

$$F_{Y_3}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(g) No density.

2. Note that $f_{X,Y}$ is nonzero only in an infinite triangle bounded by the vertical axis and the line through the origin of slope 1.

(a)

$$F_X(t) = \int_{u=-\infty}^{t} \int_{v=-\infty}^{\infty} f_{X,Y}(u,v) du\, dv = \int_{u=-\infty}^{t} \int_{v=-\infty}^{\infty} e^{-v} \chi_{\{0 \le u \le v\}} du\, dv.$$

If $t < 0$, $F_X(t) = 0$. If $t \ge 0$,

$$F_X(t) = \int_{u=0}^{t} \int_{v=u}^{\infty} e^{-v} dv\, du = \int_{u=0}^{t} e^{-v} dv = 1 - e^{-t}.$$

Thus

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-t} & \text{if } t \geq 0 \end{cases}$$

(b)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \int_{s=-\infty}^{\infty} e^{-t}\chi_{\{0\leq s\leq t\}}ds.$$

If $t < 0$, $f_Y(t) = 0$. If $t \geq 0$,

$$f_Y(t) = \int_{s=0}^{t} e^{-t}ds = te^{-t}.$$

Thus

$$f_Y(t) = \begin{cases} 0 & \text{if } t > 0 \\ te^{-t} & \text{if } t \geq 0 \end{cases}$$

3. (a)

$$\begin{aligned} f_Z(t) &= \int_{s=-\infty}^{\infty} f_X(s)f_Y(t-s)ds = \frac{1}{2}\int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[-1,1]}(t-s)ds \\ &= \frac{1}{2}\int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[t-1,t+1]}(s)ds = \frac{1}{2}\int_{s=-\infty}^{\infty} \chi_{[0,1]\cap[t-1,t+1]}(s)ds \\ &= \begin{cases} \frac{1}{2}(t+1) & \text{if } 0 \leq t+1 < 1 \\ \frac{1}{2} & \text{if } t-1 < 0 \text{ and } t+1 < 1 \\ \frac{1}{2}\{1-(t-1)\} & \text{if } 0 < t-1 < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{2}(t+1) & \text{if } -1 < t < 0 \\ \frac{1}{2} & \text{if } 0 < t < 1 \\ \frac{1}{2}(2-t) & \text{if } 1 < t < 2 \\ 0 & \text{else} \end{cases} \end{aligned}$$

(b)

$$\mathbb{E}[Y] = \int_{t=-1}^{1} \frac{t}{2}dt = 0.$$

(c)

$$\mathbb{E}[Y^2] = \int_{t=-1}^{1} \frac{t^2}{2}dt = \frac{t^3}{6}\Big|_{t=-1}^{1} = \frac{2}{6} = \frac{1}{3}.$$

Math 361, Section F1, Fall 2000
Final, December 11

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [55 points] Let X and Y be two independent exponential random variables with parameters α and β, respectively, where $\beta > \alpha$; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \alpha e^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \beta e^{-\beta t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

   (a) [10 points] Find the joint density $f_{X,Y}$.

   (b) [15 points] Compute $\mathbb{P}\{Y \leq 3X\}$.

   (c) [15 points] Compute $\mathbb{P}\{\min\{X, Y\} \geq 3\}$.

   (d) [15 points] Set $Z \stackrel{\text{def}}{=} X + Y$. Compute $f_Z(t)$.

2. [15 points] Let X be a random variable with cumulative distribution function

$$F_X(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } t < 0 \\ (t+1)/4 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

   (a) [5 points] Compute $\mathbb{P}\{X \leq 1\}$.

   (b) [5 points] Compute $\mathbb{P}\{X = 1\}$.

   (c) [5 points] Compute $\mathbb{P}\{X \geq 0\}$.

3. [15 points] Suppose that we have a noisy data channel. Suppose that

$$\mathbb{P}\{0 \text{ sent}\} = \mathbb{P}\{1 \text{ sent}\} = 0.5,$$
$$\mathbb{P}\{1 \text{ received} | 1 \text{ sent}\} = 0.9, \qquad \text{and} \qquad \mathbb{P}\{0 \text{ received} | 0 \text{ sent}\} = 0.8.$$

   (a) [5 points] Compute $\mathbb{P}\{1 \text{ received} | 0 \text{ sent}\}$.

   (b) [5 points] Compute $\mathbb{P}\{1 \text{ received}\}$.

   (c) [5 points] Compute $\mathbb{P}\{1 \text{ sent} | 1 \text{ received}\}$.

4. [20 points] Suppose that

$$\mathbb{P}(A \cap C | B) = 0.04, \qquad \mathbb{P}(A|B) = 0.2, \qquad \mathbb{P}(C|B) = 0.2, \qquad \mathbb{P}(B) = 0.5$$

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}(A \cap B \cap C)$.

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}(C \cap B)$.

(c) $\boxed{\text{5 points}}$ Compute $\mathbb{P}(A|B \cap C)$.

5. $\boxed{\text{45 points}}$ Let X and Y be continuous random variables such that

$$f_{X,Y}(s,t) = \begin{cases} (t+1)e^{-(t+1)s} & \text{if } s \geq 0 \text{ and } 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

(a) $\boxed{\text{10 points}}$ Compute the density of Y.

(b) $\boxed{\text{10 points}}$ Compute $f_{X|Y}(s|t)$.

(c) $\boxed{\text{15 points}}$ Compute $\mathbb{E}[X|Y = t]$.

(d) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$.

Answers

1. (a)
$$f_{X,Y}(s,t) = \begin{cases} \alpha\beta e^{-\alpha s-\beta t} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$
   (b) $\frac{3\beta}{\alpha+3\beta}$
   (c) $e^{-3(\alpha+\beta)}$
   (d)
$$f_Z(t) = \begin{cases} \frac{\alpha\beta}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t}) & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$
2. (a) 1
   (b) $\frac{1}{2}$
   (c) 1
3. (a) 0.2
   (b) 0.55
   (c) $\frac{9}{11}$
4. (a) 0.02
   (b) 0.1
   (c) 0.2
5. (a)
$$f_Y(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$
   (b)
$$f_{X|Y}(s|t) = \begin{cases} (t+1)e^{-(t+1)s} & \text{if } s \geq 0 \text{ and } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$
   (c) $\mathbb{E}[X|Y=t] = \frac{1}{t+1}$
   (d) $\mathbb{E}[X] = \ln 2$