

1. 10 points Suppose that we have 15 women and 10 men in a room. We want to make a committee of 7. What is the probability that there is at least 2 men in the committee?

ANSWERS

1.

$$1 - \frac{\binom{15}{7}}{\binom{25}{7}} - \frac{\binom{15}{6} \binom{10}{1}}{\binom{25}{7}}$$

1. 10 points A survey found that 30% of Chicago residents are male. The survey also found that 20% of men smoke and 15% of women smoke. Suppose a smoker is randomly picked. What is the probability that the smoker is a man?

## ANSWERS

1. Define  $M \stackrel{\text{def}}{=} \{\text{male}\}$  and  $S \stackrel{\text{def}}{=} \{\text{smoker}\}$ . Then

$$\mathbb{P}(M|S) = \frac{\mathbb{P}(S|M)\mathbb{P}(M)}{\mathbb{P}(S|M)\mathbb{P}(M) + \mathbb{P}(S|M^c)\mathbb{P}(M^c)} = \frac{0.2 \times 0.3}{0.2 \times 0.3 + 0.15 \times 0.7}.$$

1. 

10 points
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 Suppose that we have 15 boxes, labelled 1 through 15. Distribute 22 balls into the boxes. What is the probability that there are exactly 4 balls in the first 7 boxes?

ANSWERS

1.

$$\frac{\binom{22}{4} 7^4 8^{18}}{15^{22}}$$

1. 10 points Suppose that we toss a fair six-sided die 4 times. Define

$$A \stackrel{\text{def}}{=} \{\text{face 2 appeared at least once}\}$$

$$B \stackrel{\text{def}}{=} \{\text{face 2 appeared exactly twice}\}$$

- (a) 5 points What is  $\mathbb{P}(B|A)$ ?
- (b) 5 points Are the sets  $A$  and  $B$  independent?

## ANSWERS

1. Since  $B \subset A$ ,  $\mathbb{P}(B \cap A) = \mathbb{P}(B)$ , so

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}.$$

We can compute that

$$\mathbb{P}(B) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$

$$\mathbb{P}(A) = 1 - \mathbb{P}\{\text{face 2 doesn't appear at all}\} = 1 - \left(\frac{5}{6}\right)^4$$

(a)

$$\mathbb{P}(B|A) = \frac{\binom{4}{2} 5^2}{6^4 - 5^4}$$

(b) Since  $\mathbb{P}(A) < 1$ ,  $\mathbb{P}(B|A) > \mathbb{P}(B)$ , so  $\mathbb{P}(B|A) \neq \mathbb{P}(A)$ . Thus they are dependent.

1. 10 points Let  $X$  be uniform on  $\{1, 2, \dots, 9, 10\}$ ; i.e.,  $X$  has density

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2, \dots, 9, 10\} \\ 0 & \text{else} \end{cases}$$

Define  $Y \stackrel{\text{def}}{=} \max\{X, 4\}$ . Compute the density of  $Y$ .

ANSWERS

1.

$$f_Y(j) = \begin{cases} \frac{1}{10} & \text{if } j \in \{5, 6 \dots 10\} \\ \frac{4}{10} & \text{if } j = 4 \\ 0 & \text{else} \end{cases}$$

1. 10 points Let  $X$  be uniform on  $\{1, 2, \dots, 9, 10\}$ ; i.e.,  $X$  has density

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2, \dots, 9, 10\} \\ 0 & \text{else} \end{cases}$$

Define  $Y \stackrel{\text{def}}{=} \min\{X, 4\}$ . Compute the density of  $Y$ .

ANSWERS

1.

$$f_Y(j) = \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2, 3\} \\ \frac{7}{10} & \text{if } j = 4 \\ 0 & \text{else} \end{cases}$$

1. 10 points Let  $X$  and  $Y$  be independent random variables with distributions

$$f_X(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{11, 12, \dots\} \\ 1 - (1-p)^{11} & \text{if } j = 10 \\ 0 & \text{else} \end{cases}$$
$$f_Y(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

Define  $Z \stackrel{\text{def}}{=} X + Y$ . Compute  $f_Z(14)$ .

ANSWERS

1.

$$f_Z(j) = 4p^2(1-p)^{14} + p(1 - (1-p)^{11})(1-p)^4.$$

1. 10 points Suppose that  $X$  is a random variable with density

$$f_X(j) = \begin{cases} \frac{1}{10} & \text{if } j = -3 \\ \frac{1}{10} & \text{if } j = -1 \\ \frac{2}{10} & \text{if } j = 0 \\ \frac{3}{10} & \text{if } j = 2 \\ \frac{3}{10} & \text{if } j = 4 \\ 0 & \text{else.} \end{cases}$$

- (a) 4 points Compute  $\mathbb{E}[X]$   
(b) 4 points Compute  $\mathbb{E}[X^2]$   
(c) 2 points Compute  $\mathbb{E}[2X]$

## ANSWERS

1. (a)  $\frac{14}{10}$
- (b)  $\frac{70}{10}$
- (c)  $\frac{28}{10}$

1. 10 points Suppose that  $X$  is a random variable with moment generating function

$$\mathbb{E}[e^{\theta X}] = \begin{cases} \frac{2}{2-\theta} & \text{if } \theta < 2 \\ \infty & \text{else.} \end{cases}$$

Define  $Y \stackrel{\text{def}}{=} X + 7$ . What is the moment generating function of  $Y$ ? In other words, compute  $\mathbb{E}[e^{\theta Y}]$  for all  $\theta \in \mathbb{R}$ .

## ANSWERS

1.

$$\mathbb{E}[e^{\theta Y}] = \begin{cases} \frac{2e^{\theta}}{2-\theta} & \text{if } \theta < 2 \\ \infty & \text{else.} \end{cases}$$

1. 10 points Suppose that  $X$  is a random variable with moment generating function

$$\mathbb{E}[e^{\theta X}] = \exp \left[ \frac{9}{2}\theta^2 + 5\theta \right]$$

- (a) 5 points Define  $Y \stackrel{\text{def}}{=} X/10$ . What is the moment generating function of  $Y$ ?
- (b) 5 points Let  $Y_n \stackrel{\text{def}}{=} X/n$ , and let  $\varphi_{Y_n}$  be the moment generating function of  $Y_n$ . Compute  $\lim_{n \rightarrow \infty} \varphi_{Y_n}(\theta)$  for each  $\theta \in \mathbb{R}$ .

## ANSWERS

1. (a)

$$\mathbb{E}[e^{\theta Y}] = \exp \left[ \frac{9}{200} \theta^2 + \frac{5}{10} \theta \right]$$

(b)

$$\lim_{n \nearrow \infty} \varphi_{Y_n}(\theta) = \lim_{n \nearrow \infty} \exp \left[ \frac{9}{2n^2} \theta^2 + \frac{5}{n} \theta \right] = e^0 = 1$$

for all  $\theta \in \mathbb{R}$ .

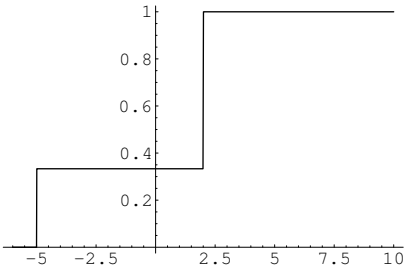
1. 10 points Suppose that  $X$  is a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < -5 \\ \frac{1}{3} & \text{if } -5 \leq t < 2 \\ 1 - e^{-30t} & \text{if } t \geq 2 \end{cases}$$

- (a) 2 points Graph  $F_X$ .
- (b) 4 points Compute  $\mathbb{P}\{X = 2\}$
- (c) 4 points Compute  $\mathbb{P}\{-5 < X \leq 10\}$

## ANSWERS

1. (a) See figure.



(b)  $\mathbb{P}\{X = 2\} = F_X(2) - F_X(2-) = 1 - e^{-60} - \frac{1}{3} = \frac{2}{3} - e^{-60}$ .

(c)  $\mathbb{P}\{-5 < X \leq 10\} = F_X(10) - F_X(-5) = 1 - e^{-300} - \frac{1}{3} = \frac{2}{3} - e^{-300}$ .

1. 10 points Fix  $\lambda > 0$ . Suppose that  $X$  and  $Y$  are continuous random variables with joint density

$$f_{X,Y}(s, t) = \begin{cases} \lambda^2 e^{-\lambda(s+t)} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Compute  $\mathbb{P}\{X + Y \leq 5\}$ .

## ANSWERS

1.

$$\mathbb{P}\{X + Y \leq 5\} = \int_{s=0}^5 \int_{t=0}^{5-s} \lambda^2 e^{-\lambda(s+t)} ds dt = 1 - e^{-5\lambda} - 5\lambda e^{-5\lambda}.$$

**Math 361, Section X1, Fall 2002**  
**Exam 1, October 2**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 25 points Consider a box containing 6 white balls and 8 black balls. Pick the balls from the box one by one.
  - (a) 10 points What is the probability that the fourth ball is white and the fifth ball is black?
  - (b) 10 points What is the probability that the fourth ball is white?
  - (c) 5 points What is the probability that the fifth ball is black given that the fourth ball is white?

Hint: you may want to “reserve” balls.

2. 15 points An instructor gives her class 15 problems, telling them that the exam will consist of a random selection of 7 of them. By the time of the exam, a student knows how to do 12 out of the 15.
  - (a) 8 points What is the probability that the student knows how to do all of the questions on the exam?
  - (b) 7 points What is the probability that the student knows how to do at least 6 of the questions on the exam?

Hint: compare this problem to a lottery problem.

3. 15 points There are 7 restaurants in a town. Suppose three students are hungry, and each student randomly chooses a restaurant.
  - (a) 7 points What is the probability that no two students go to the same restaurant?
  - (b) 8 points What is the probability that exactly two students go to the same restaurant?
4. 30 points Suppose that each child born to Jim and Jane is equally likely to be a boy or girl, independently of the gender of all other children. Jim and Jane have 5 children. Compute the following probabilities:
  - (a) 10 points All children are of the same sex.
  - (b) 5 points Exactly 3 are boys
  - (c) 10 points youngest and oldest are girls
  - (d) 5 points At least one boy.

5. 15 points Suppose that

$$\mathbb{P}(A \cap C|B) = 0.4 \quad \mathbb{P}(B) = 0.1 \quad \mathbb{P}(A \cap B \setminus C) = 0.02 \quad \text{and} \quad \mathbb{P}(A \setminus B) = 0.2$$

Compute

(a) 5 points  $\mathbb{P}(A \cap B)$

(b) 5 points  $\mathbb{P}(A)$

(c) 5 points  $\mathbb{P}(A \cup B)$

ANSWERS

1. (a) Reserve a black ball and a white ball.

$$\frac{8 \times 6 \times (12)_3}{(14)_5}$$

- (b) Reserve a white ball.

$$\frac{6 \times (13)_3}{(14)_4}$$

- (c) Note that

$$\frac{6 \times (13)_3}{(14)_4} = \frac{6 \times (13)_4}{(14)_5}.$$

Answer is

$$\frac{8 \times 6 \times (12)_3}{6 \times (13)_4} = \frac{8}{13}.$$

2. Define  $q \stackrel{\text{def}}{=} 1/\binom{15}{7}$ .

(a)  $\binom{12}{7}q$ .

(b)  $((\binom{12}{7} + \binom{12}{6}\binom{3}{1}))q$ .

3. Define  $q \stackrel{\text{def}}{=} 1/7^3$ .

(a)  $\binom{7}{3}3!q$ .

(b)  $\binom{3}{2}\binom{7}{2}2!q$ .

4. Set  $q \stackrel{\text{def}}{=} (1/2)^5$ .

(a)  $2q$

(b)  $\binom{5}{3}q$

(c)  $2^3q$

(d)  $1 - q$ .

5. (a)

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap B \setminus C) \\ &= \mathbb{P}(A \cap C|B)\mathbb{P}(B) + \mathbb{P}(A \cap B \setminus C) = 0.4 \times 0.1 + 0.02 = 0.06. \end{aligned}$$

(b)  $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B) = 0.06 + 0.2 = 0.26$ .

(c)  $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \setminus B) = 0.1 + 0.2 = 0.3$ .

**Math 361, Section X1, Fall 2002**  
**Exam 2, November 1**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 30 points Suppose that  $X$  and  $Y$  are independent random variables with densities

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} (j+1)p^2(1-p)^j & \text{if } j \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$
$$f_Y(j) \stackrel{\text{def}}{=} \begin{cases} p(1-p)^j & \text{if } j \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

Define  $Z \stackrel{\text{def}}{=} X + Y$ .

- (a) 15 points Compute the density of  $Z$ .
- (b) 15 points Compute  $f_{X|Z}(j|k)$  when  $0 \leq j \leq k$ .

Hint: Remember that  $\sum_{j=0}^N j = N(N+1)/2$ .

2. 40 points Let  $X$  be a random variable with density

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} p(1-p)^j & \text{if } j \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

Define the transformation

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 10 - u & \text{if } u \leq 10 \\ u - 5 & \text{if } u > 10 \end{cases}$$

Define  $Y \stackrel{\text{def}}{=} \varphi(X)$ .

- (a) 10 points Compute  $\mathbb{P}\{Y = 6\}$
- (b) 10 points Compute  $\mathbb{P}\{Y = 4\}$
- (c) 10 points Compute  $\mathbb{P}\{Y = 22\}$
- (d) 10 points Compute the density of  $Y$ .

Hint: Note that  $\varphi$  is discontinuous and that  $\varphi(10) = 0$ . You may want to *carefully* graph it.

3. 20 points Let  $X$  and  $Y$  be random variables with joint density

$$f_{X,Y}(j, k) \stackrel{\text{def}}{=} \begin{cases} \frac{p(1-p)^j}{j+1} & \text{if } 0 \leq k \leq j \\ 0 & \text{else} \end{cases}$$

(a) 10 points Compute  $\mathbb{P}\{X = 5\}$ .

(b) 10 points Compute  $\mathbb{P}\{Y = 3|X = 5\}$ .

4. 10 points Suppose that  $X$  is a random variable with

$$\mathbb{P}\{X = -2\} = \frac{1}{4}, \quad \mathbb{P}\{X = 0\} = \frac{1}{4}, \quad \text{and} \quad \mathbb{P}\{X = 1\} = \frac{2}{4}.$$

Compute  $\mathbb{E}[X^3]$ .

ANSWERS

1. (a)

$$f_Z(j) \stackrel{\text{def}}{=} \begin{cases} \binom{j+2}{2} p^2 (1-p)^j & \text{if } j \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

(note that  $\binom{j+2}{2} = (j+1)(j+2)/2$ ; the given representation is a more standard representation for negative binomials).

(b)

$$f_{X|Z}(j|k) = \frac{j+1}{(k+1)(k+2)/2} = \frac{\binom{j+1}{1}}{\binom{k+2}{2}}$$

if  $0 \leq j \leq k$ .

2. (a)  $f_Y(6) = f_X(4) + f_X(11) = p\{(1-p)^4 + (1-p)^{11}\}$ .

(b)  $f_Y(4) = f_X(6) = p(1-p)^6$ .

(c)  $f_Y(22) = f_X(27) = p(1-p)^{27}$ .

(d)

$$f_Y(j) = \begin{cases} p(1-p)^{10-j} & \text{if } j \in \{0, 1, 2, 3, 4, 5\} \\ p\{(1-p)^{10-j} + (1-p)^{j+5}\} & \text{if } j \in \{6, 7, 8, 9, 10\} \\ p(1-p)^{j+5} & \text{if } j \in \{11, 12, \dots\} \\ 0 & \text{else} \end{cases}$$

3. (a)  $p(1-p)^5$

(b)  $\frac{1}{6}$

4.

$$(-2)^3 \frac{1}{4} + 0^3 \frac{1}{4} + 1^3 \frac{2}{4} = \frac{-8+2}{4} = -\frac{6}{4}.$$

**Math 361, Section X1, Fall 2002**  
**Exam 3, December 6**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 35 points Let  $X$  be a continuous random variable with density

$$f_X(t) = \begin{cases} 1 & \text{if } t \in (0, 1] \\ 0 & \text{else} \end{cases}$$

(i.e.,  $X$  is  $U(0, 1)$ ). Define the transformation

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 4 & \text{if } u \in (0, \frac{1}{10}] \\ -2 & \text{if } u \in (\frac{1}{10}, \frac{4}{10}] \\ 3 & \text{if } u \in (\frac{4}{10}, 1] \end{cases}$$

Define  $Y \stackrel{\text{def}}{=} \varphi(X)$ .

- (a) 10 points Compute  $F_Y$ , the cumulative distribution function of  $Y$ .
- (b) 5 points Is  $Y$  continuous, or is it discrete (Hint: in this case, it is one or the other).
- (c) 10 points Compute the density (i.e., the discrete density if  $Y$  is discrete, and the continuous density if  $Y$  is continuous).
- (d) 10 points Compute  $\mathbb{E}[Y]$ .
2. 31 points Suppose that  $X$  and  $Y$  are independent continuous random variables with densities

$$f_X(s) \stackrel{\text{def}}{=} \begin{cases} \lambda e^{-\lambda s} & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$
$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \nu e^{-\nu t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

where  $\lambda$  and  $\nu$  are positive parameters. Define  $Z \stackrel{\text{def}}{=} \min\{X, Y\}$ .

- (a) 10 points Compute  $\mathbb{P}\{X \geq Y\}$ .
- (b) 10 points Compute  $\mathbb{P}\{Z > 3\}$ .
- (c) 3 points Compute  $\mathbb{P}\{Z \leq 3\}$ .
- (d) 8 points Compute  $F_Z$ , the cumulative distribution function of  $Z$ .
3. 34 points Suppose that  $X$  and  $Y$  are continuous random variables with joint density

$$f_{X,Y}(s, t) \stackrel{\text{def}}{=} \begin{cases} 6t & \text{if } s \geq 0, t \geq 0, \text{ and } s + t \leq 1 \\ 0 & \text{else} \end{cases}$$

Define  $Z \stackrel{\text{def}}{=} X + Y$ .

- (a) 10 points Compute  $f_X$ , the first marginal of  $f_{X,Y}$ .
- (b) 10 points Compute  $\mathbb{P}\{Z \leq \frac{2}{3}\}$ .
- (c) 7 points Compute  $F_Z$ , the cumulative distribution function of  $Z$ . Hint:  $F_Z$  should be continuous.
- (d) 7 points Compute  $f_Z$ , the density of  $Z$ .

ANSWERS

1. (a)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < -2 \\ \frac{3}{10} & \text{if } -2 \leq t < 3 \\ \frac{9}{10} & \text{if } 3 \leq t < 4 \\ 1 & \text{if } t \geq 4 \end{cases}$$

(b)  $Y$  is discrete.

(c)

$$f_Y(j) = \begin{cases} \frac{3}{10} & \text{if } j = -2 \\ \frac{6}{10} & \text{if } j = 3 \\ \frac{1}{10} & \text{if } j = 4 \end{cases}$$

(d)

$$\mathbb{E}[Y] = (-3)\frac{1}{10} + 3\frac{6}{10} + 4\frac{1}{10} = 1.6.$$

2. (a)

$$\mathbb{P}\{X \geq Y\} = \int_{t=0}^{\infty} \int_{s=t}^{\infty} \lambda e^{-\lambda s} \nu e^{-\nu t} dt ds = \int_{t=0}^{\infty} \nu e^{-\lambda s - \nu s} dt = \frac{\nu}{\lambda + \nu}.$$

(b)

$$\mathbb{P}\{Z > 3\} = \mathbb{P}\{X > 3\}\mathbb{P}\{Y > 3\} = \left( \int_{s=3}^{\infty} \lambda e^{-\lambda s} ds \right) \left( \int_{t=3}^{\infty} \nu e^{-\nu t} dt \right) = e^{-3(\lambda + \nu)}.$$

(c)

$$\mathbb{P}\{Z \leq 3\} = 1 - e^{-3(\lambda + \nu)}.$$

(d)

$$F_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-(\lambda + \nu)t} & \text{if } t \geq 0 \end{cases}$$

3. (a)

$$f_X(s) = \begin{cases} 3(1-s)^2 & \text{if } s \in [0, 1] \\ 0 & \text{else} \end{cases}$$

(b)

$$\mathbb{P}\left\{Z \leq \frac{2}{3}\right\} = \int_{s=0}^{2/3} \int_{t=0}^{2/3-s} (6t) dt ds = 3 \int_{s=0}^{2/3} \left(\frac{2}{3} - s\right)^2 ds = \left(\frac{2}{3}\right)^3.$$

(c)

$$F_Z(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^3 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

(d)

$$f_Z(t) = \begin{cases} 3t^2 & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases}$$

**Math 361, Section X1, Fall 2002**  
**Final, December 16**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 150 Points

1. 30 points Suppose that there are 10 married couples in a room. Suppose for specificity that one of the married couples is Don and Donna. Randomly pick four people to sit at a table.
  - (a) 10 points What is the probability that you pick Don and Donna?
  - (b) 10 points What is the probability that you pick Don and Donna and another married couple?
  - (c) 10 points What is the probability that you pick two married couples?
2. 20 points Suppose that  $X$  and  $Y$  have joint density

		Y			
		0	1	2	3
	-1	3/27	1/27	1/27	3/27
X	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words,  $f_{X,Y}(0, 1) = 0$ .

- (a) 5 points Compute  $f_X$ .
- (b) 10 points Compute  $\mathbb{E}[X^2Y]$ .

Now define  $Z \stackrel{\text{def}}{=} X + Y$ .

- (c) 5 points Compute  $\mathbb{P}\{Z = 1\}$ .
3. 40 points Fix  $\lambda > 0$ . Let  $X$  and  $Y$  be continuous random variables with joint density

$$f_{X,Y}(s, t) = \begin{cases} \lambda e^{-\lambda(t-s)} & \text{if } t \geq s \text{ and } 0 \leq s \leq 1 \\ 0 & \text{else} \end{cases}$$

- (a) 10 points Compute  $f_Y(4)$
- (b) 10 points Compute  $f_Y(\frac{1}{3})$
- (c) 10 points Compute  $f_Y$ .
- (d) 10 points Compute  $f_{X|Y}(s|4)$  for  $0 < s < 1$ .

4. 50 points Let  $X$  be a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - e^{-3t} & \text{if } 1 \leq t < 10 \\ 1 & \text{if } t \geq 10 \end{cases}$$

- (a) 5 points Compute  $\mathbb{P}\{X < 10\}$ .  
(b) 5 points Compute  $\mathbb{P}\{X = 1\}$ .

Define now

$$\tilde{F}(t) \stackrel{\text{def}}{=} \mathbb{P}\{X \leq t | X \in (1, 10)\}$$

for all  $t \in \mathbb{R}$  (this is the cumulative distribution of  $X$  given that  $X \in (1, 10)$ ).

- (c) 5 points Compute  $\mathbb{P}\{X \in (1, 10)\}$   
(d) 5 points Compute  $\mathbb{P}\{X \leq 5 \text{ and } X \in (1, 10)\}$ .  
(e) 3 points Compute  $\tilde{F}(5)$ .  
(f) 5 points Compute  $\mathbb{P}\{X \leq 12 \text{ and } X \in (1, 10)\}$ .  
(g) 3 points Compute  $\tilde{F}(12)$ .  
(h) 5 points Compute  $\mathbb{P}\{X \leq -2 \text{ and } X \in (1, 10)\}$ .  
(i) 3 points Compute  $\tilde{F}(-2)$ .  
(j) 5 points Compute  $\mathbb{P}\{X \leq 1 \text{ and } X \in (1, 10)\}$ .  
(k) 3 points Compute  $\tilde{F}(1)$ .  
(l) 3 points Compute  $\tilde{F}(t)$  for all  $t$ .
5. 10 points Let  $X$  and  $Y$  be independent Gaussian random variables with mean 0 and variance 1; i.e.,

$$f_X(t) = f_Y(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \quad t \in \mathbb{R}$$

Define  $Z_1 \stackrel{\text{def}}{=} 7X + 3$  and  $Z_2 \stackrel{\text{def}}{=} 5X + 4Y$ .

- (a) 5 points Compute the moment generating function of  $Z_1$ .  
(b) 5 points Compute the moment generating function of  $Z_2$ .

ANSWERS

1. (a)  $\frac{\binom{18}{2}}{\binom{20}{4}}$ .  
 (b)  $\frac{9}{\binom{20}{4}}$ .  
 (c)  $\frac{\binom{10}{2}}{\binom{20}{4}}$ .

2. (a)

$$f_X(j) = \begin{cases} \frac{8}{27} & \text{if } j = -1 \\ \frac{12}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 3 \end{cases}$$

- (b)

$$\mathbb{E}[X^2Y] = (-1)^2(1)\frac{1}{27} + (-1)^2(2)\frac{1}{27} + (-1)^2(3)\frac{3}{27} + (3)^2(3)\frac{4}{27} = \frac{120}{27}.$$

(c)  $\mathbb{P}\{Z = 1\} = f_{X,Y}(-1, 2) + f_{X,Y}(0, 1) = \frac{1}{27}.$

3. (a)  $f_Y(4) = \int_{s=0}^1 \lambda e^{-\lambda(4-s)} ds = e^{-3\lambda} - e^{-4\lambda}.$

(b)  $f_Y(\frac{1}{3}) = \int_{s=0}^{1/3} \lambda e^{-\lambda((1/3)-s)} ds = 1 - e^{-(1/3)\lambda}.$

- (c)

$$f_Y(t) = \begin{cases} e^{-\lambda(t-1)} - e^{-\lambda t} & \text{if } t \geq 1 \\ 1 - e^{-\lambda t} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

- (d)

$$f_{X|Y}(s|4) = \frac{f_{X,Y}(s, 4)}{f_Y(4)} = \frac{\lambda e^{-\lambda(4-s)}}{e^{-3\lambda} - e^{-4\lambda}}.$$

4. (a)  $\mathbb{P}\{X < 10\} = 1 - e^{-30}.$

(b)  $\mathbb{P}\{X = 1\} = 1 - e^{-3}.$

(c)  $\mathbb{P}\{X \in (1, 10)\} = F_X(10-) - F_X(1) = e^{-3} - e^{-30}.$

(d)  $\mathbb{P}\{X \leq 5 \text{ and } X \in (1, 10)\} = \mathbb{P}\{X \in (1, 5]\} = F_X(5) - F_X(1) = e^{-3} - e^{-15}.$

(e)  $\tilde{F}(5) = (e^{-3} - e^{-15}) / (e^{-3} - e^{-30}).$

(f)  $\mathbb{P}\{X \leq 12 \text{ and } X \in (1, 10)\} = \mathbb{P}\{X \in (1, 10)\} = e^{-3} - e^{-30}.$

(g)  $\tilde{F}(12) = 1.$

(h)  $\mathbb{P}\{X \leq -2 \text{ and } X \in (1, 10)\} = \mathbb{P}(\emptyset) = 0.$

(i)  $\tilde{F}(-2) = 0.$

(j)  $\mathbb{P}\{X \leq 1 \text{ and } X \in (1, 10)\} = \mathbb{P}(\emptyset) = 0.$

(k)  $\tilde{F}(1) = 0.$

(1)

$$\tilde{F}(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{e^{-3} - e^{-3t}}{e^{-3} - e^{-30}} & \text{if } 1 \leq t < 10 \\ 1 & \text{if } t \geq 10 \end{cases}$$

5. (a)  $\mathbb{E}[e^{\theta Z_1}] = \exp\left[\frac{49}{2}\theta^2 + 3\theta\right]$ .  
(b)  $\mathbb{E}[e^{\theta Z_2}] = \exp\left[\frac{41}{2}\theta^2\right]$ .