Math 361, Section F1, Spring 2002
Quiz 1, January 18 **Name:** ______________________

1. [10 points] Assume that

$$\mathbb{P}(A) = 0.5, \qquad \mathbb{P}(B) = 0.6 \qquad \text{and} \qquad \mathbb{P}(B \setminus A) = 0.1.$$

   (a) [5 points] Compute $\mathbb{P}(A \cup B)$

   (b) [5 points] Compute $\mathbb{P}(A \setminus B)$.

Answers

1. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) = 0.5 + 0.1 = 0.6.$
2. $\mathbb{P}(A \setminus B) = \mathbb{P}(A \cup B) + \mathbb{P}(B) = 0.6 - 0.6 = 0.$

Math 361, Section F1, Spring 2002
Quiz 2, January 25 **Name:** ____________________

1. 10 points Assume that

$$\mathbb{P}(A) = 0.5, \qquad \mathbb{P}(B) = 0.6 \qquad \text{and} \qquad \mathbb{P}(B \setminus A) = 0.1.$$

   (a) 7 points Compute $\mathbb{P}(A|B)$

   (b) 3 points Are A and B independent?

Answers

1.
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A) + \mathbb{P}(B \setminus A)}{\mathbb{P}(B)} = \frac{0.5 + 0.1}{0.6} = 1.$$

2. No, since $\mathbb{P}(A|B) = 1 \neq 0.5 = \mathbb{P}(A)$.

Math 361, Section F1, Spring 2002
Quiz 3, January 30 **Name:** ____________________

1. 10 points Suppose that cards are dealt from an ordinary deck of playing cards on at a time until the first king appears. Find the probability that this occurs on the 7th card dealt. (Remember that the deck has 52 cards, 4 of which are kings.)

Answers

1.

$$\frac{(48)_6 \cdot 4}{(52)_7}.$$

Math 361, Section F1, Spring 2002
Quiz 4, February 8 **Name:** ______________________

1. [10 points] Suppose a random variable X has density f_X, where

$$f_X(0) = 0.1, \quad f_X(1) = 0.5, \quad f_X(2) = 0.05, \quad f_X(3) = .3, \quad \text{and} \quad f_X(4) = 0.05$$

   (a) [5 points] Compute $\mathbb{P}\{X \text{ is odd}\}$.

   (b) [5 points] Compute $\mathbb{P}\{X \leq 2.5\}$.

Answers

1. $\mathbb{P}\{X \text{ is odd}\} = f_X(1) + f_X(3) = 0.5 + 0.3 = 0.8$.
2. Compute $\mathbb{P}\{X \le 2.5\} = f_X(0) + f_X(1) + f_X(2) = 0.1 + 0.5 + 0.05 = 0.65$.

Math 361, Section F1, Spring 2002
Quiz 5, February 13 **Name:** ____________________

1. 10 points Explicitly evaluate

$$\sum_{n=6}^{\infty} \left(\frac{1}{3}\right)^{n-4} \left(\frac{4}{10}\right).$$

1.

$$\sum_{n=6}^{\infty}\left(\frac{1}{3}\right)^{n-4}\left(\frac{4}{10}\right)=\left(\frac{4}{10}\right)\left(\frac{1}{3}\right)^{2}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n}=\frac{\frac{4}{10}\cdot\frac{1}{3}}{1-\frac{1}{3}}.$$

Math 361, Section F1, Spring 2002
Quiz 6, February 25 **Name:** ____________________

1. 10 points Let X_1, X_2, X_3 and X_4 be independent random variables, all of which are geometrically distributed with parameter p. Define

$$Y \stackrel{\text{def}}{=} \min\{X_1, X_2, X_3, X_4\}.$$

Find the density of Y. Note: the fact that the X_i's are independent means that

$$\mathbb{P}\{X_1 = j_1, X_2 = j_2, X_3 = j_3, X_4 = j_4\} = \mathbb{P}\{X_1 = j_1\}\mathbb{P}\{X_2 = j_2\}\mathbb{P}\{X_3 = j_3\}\mathbb{P}\{X_4 = j_4\}$$

for all j_1, j_2, j_3 and j_4.

1. $\mathbb{P}\{Y = k\} = 0$ if $k \le 0$. For $k \ge 0$ an integer,

$$\mathbb{P}\{X_i \ge k\} = \sum_{j=k}^{\infty} p(1-p)^{j-1} = (1-p)^{j-1};$$

thus,

$$\mathbb{P}\{Y \ge k\} = \prod_{i=1}^{4} \mathbb{P}\{X_k \ge k\} = \{(1-p)^4\}^{j-1}$$

hence

$$\begin{aligned}\mathbb{P}\{Y = k\} = \mathbb{P}\{Y \ge k\} - \mathbb{P}\{Y \ge k+1\} &= \{(1-p)^4\}^{k-1} - \{(1-p)^4\}^k \\ &= \{(1-p)^4\}^{k-1}\{1 - \{(1-p)^4\}.\end{aligned}$$

Thus, Y is geometric with parameter $1-(1-p)^4$.

Math 361, Section F1, Spring 2002
Quiz 7, March 6 **Name:** ____________________

1. 10 points Let X and Y be independent geometric random variables with parameter p. Define $Z \overset{\text{def}}{=} X + Y$. Find the density of Z.

1.

$$f_Z(j) = \sum_k f_X(k) f_Y(j-k) = \sum_{k=-\infty}^{\infty} \chi_{\{k\ge 1\}} \chi_{\{j-k\ge 1\}} p^2 (1-p)^{j-1} (1-p)^{j-k-1}$$

$$= p^2 (1-p)^{k-2} \sum_{k=-\infty}^{\infty} \chi_{\{1\le k\le j-1\}} = \begin{cases} (j-1)p^2(1-p)^{k-2} & \text{if } j\ge 2 \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Spring 2002
Quiz 8, March 13 **Name:** ____________________

1. [10 points] Let X a random variable with density

$$f_X(0) = \frac{1}{8}, \quad f_X(1) = \frac{2}{8}, \quad f_X(2) = \frac{1}{8}, \quad f_X(3) = \frac{1}{8}, \quad f_X(4) = \frac{3}{8}$$

(a) [4 points] Compute $\mathbb{E}[X]$.

(b) [4 points] Compute $\mathbb{E}[X^2]$.

(c) [2 points] Compute the variance σ^2 of X.

Answers

1. (a) $\mathbb{E}[X] = 0 \cdot \frac{1}{8} + 1 \cdot \frac{2}{8} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{3}{8} = \frac{19}{8}$.

   (b) $\mathbb{E}[X] = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{2}{8} + 2^2 \cdot \frac{1}{8} + 3^2 \cdot \frac{1}{8} + 4^2 \cdot \frac{3}{8} = \frac{63}{8}$.

   (c) $\frac{63}{8} - \left(\frac{19}{8}\right)^2$.

Math 361, Section F1, Spring 2002
Quiz 9, March 27 Name: ______________________

1. 10 points Let X and Y be independent discrete random variables with densities

$$f_X(j) = \begin{cases} p(1-p)^j & \text{if } j \in \{0, 1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) = \begin{cases} (j+1)p^2(1-p)^j & \text{if } j \in \{0, 1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

Let $Z \stackrel{\text{def}}{=} X + Y$. Compute $f_Z(3)$.

Answers

1.

$$
\begin{aligned}
f_Z(3) = \sum_k f_X(3-k) f_Y(k) &= \sum_{k=-\infty}^{\infty} \chi_{\{3-k\geq 0\}}\chi_{\{k\geq 0\}} p(1-p)^{3-j}(j+1)p^2(1-p)^j \\
&= p^3(1-p)^3 \sum_{k=0}^{3}(j+1) = p^3(1-p)^3 \sum_{k=1}^{4} j \\
&= p^3(1-p)^3 \frac{4\cdot 5}{2} = 10p^3(1-p)^3
\end{aligned}
$$

Math 361, Section F1, Spring 2002
Quiz 10, April 8 **Name:** ______________________

1. [10 points] Let X be a continuous exponential random variable with parameter 5; i.e.,

$$f_X(t) = \begin{cases} 5e^{-5t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

   (a) [5 points] Compute $\mathbb{P}\{X \geq 10\}$.

   (b) [5 points] Compute $\mathbb{P}\{2 \leq X \leq 70 | X \geq 10\}$.

Answers

1. $\mathbb{P}\{X \geq 10\} = \int_{t=10}^{\infty} 5e^{-5t} dt = e^{-50}$.

2.

$$\mathbb{P}\{2 \leq X \leq 70 | X \geq 10\} = \frac{\mathbb{P}\{10 \leq X \leq 70\}}{\mathbb{P}\{X \geq 10\}}$$
$$= \frac{\int_{t=10}^{70} 5e^{-5t} dt}{e^{-50}} = \frac{e^{-50} - e^{-350}}{e^{-50}} = 1 - e^{-300}.$$

Math 361, Section F1, Spring 2002
Quiz 11, April 17 **Name:** ______________________

Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 5e^{-5t} & \text{if } 0 \le s \le 1 \text{ and } t \ge 0 \\ 0 & \text{else} \end{cases}$$

1. $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le X\}$.

1.

$$\mathbb{P}\{Y \le X\} = \int_{t=0}^{\infty}\int_{s=0}^{\infty} \chi_{\{t\le s\}} f_{X,Y}(s,t) ds\, dt = \int_{s=0}^{1}\int_{t=0}^{s} 5e^{-5t} dt\, ds$$
$$= \int_{s=0}^{1} \{1 - e^{-5s}\} ds = 1 - \frac{1}{5}\{1 - e^{-5}\}.$$

Math 361, Section F1, Spring 2002
Quiz 12, April 29 **Name:** ______________________

1. [10 points] Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \overset{\text{def}}{=} \begin{cases} 4st & \text{if } s \geq 1 \text{ and } 0 \leq t \leq \frac{1}{s^2} \\ 0 & \text{else.} \end{cases}$$

   (a) [5 points] Compute $f_X(2)$.

   (b) [5 points] Compute the conditional density $f_{Y|X}(t|2)$ for all t.

1. (a)

$$f_X(2) = \int_{t=-\infty}^{\infty} f_{X,Y}(2,t)dt = \int_{t=0}^{1/4} 8tdt = 4t^2\Big|_{t=0}^{1/4} = \frac{1}{4}.$$

(b)

$$f_{Y|X}(t|2) = \frac{f_{X,Y}(2,t)}{f_X(2)} = \begin{cases} 32t & \text{if } 0 \le t \le \frac{1}{4} \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Spring 2002
Exam 1, February 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] Toss a sequence of independent and identically distributed coins such that $\mathbb{P}\{H\} = 1/8$ for each coin. Let X be the position of the first heads.
   (a) [5 points] Compute the density of X.
   (b) [10 points] Compute $\mathbb{P}\{X \geq 7 | X \geq 3\}$.
   (c) [5 points] Are the sets $\{X \geq 7\}$ and $\{X \geq 3\}$ independent?

2. [10 points] Suppose that X is geometric with parameter p and Y is geometric with parameter q. Assume also that X and Y are independent. Compute $\mathbb{P}\{X = Y\}$.

3. [15 points] Suppose that X and Y have joint density

		Y: 0	1	2	3
X	-1	3/27	1/27	1/27	3/27
	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

   In other words, $f_{X,Y}(0,2) = \frac{3}{27}$.
   (a) [5 points] Compute $\mathbb{P}\{X = 3\}$
   (b) [5 points] Compute $\mathbb{P}\{X \geq 0\}$
   (c) [5 points] Compute $\mathbb{P}\{X = 0 | Y \text{ is even}\}$

4. [30 points] Suppose that a group of people contains 55 women and 45 men. Suppose that 10% of the men smoke and 15% of the women smoke. Randomly choose a person. If the person is a smoker, what is the probability that you have chosen a male? Please be clear about your calculations. If I don't see any calculations, I will not give you any points. Do not attempt to do any serious multiplication or division.

5. [25 points] A box consists of
   - 5 Alanis Morissette CD's
   - 3 Suzanne Vega CD's
   - 7 Led Zeppelin CD's

   You grab 7 CD's from the box.
   (a) [5 points] What is the probability that you grabbed 1 Alanis Morissette CD, 2 Suzanne Vega CD's, and 4 Led Zeppelin CD's?

(b) 10 points What is the probability that you grabbed at least one Alanis Morissette CD?

(c) 10 points What is the probability that you grabbed at least one Alanis Morissette CD given that you grabbed exactly 2 Suzanne Vega CD's?

1. (a)

$$\mathbb{P}\{X=k\}=\begin{cases}\frac{1}{8}\left(\frac{7}{8}\right)^{j-1} & \text{if } j\in\{1,2\dots\}\\ 0 & \text{else}\end{cases}$$

For $k\in\{1,2\dots\}$,

$$\mathbb{P}\{X\ge k\}=\sum_{j=k}\frac{1}{8}\left(\frac{7}{8}\right)^{j-1}=\left(\frac{7}{8}\right)^{k-1}.$$

(b)

$$\mathbb{P}\{X\ge 7|X\ge 3\}=\frac{\mathbb{P}\{X\ge 7 \text{ and } X\ge 3\}}{\mathbb{P}\{X\ge 3\}}=\frac{\mathbb{P}\{X\ge 7\}}{\mathbb{P}\{X\ge 3\}}=\frac{\left(\frac{7}{8}\right)^6}{\left(\frac{7}{8}\right)^2}=\left(\frac{7}{8}\right)^4.$$

(c) No, since $\mathbb{P}\{X\ge 7|X\ge 3\}=\left(\frac{7}{8}\right)^4\neq\left(\frac{7}{8}\right)^6=\mathbb{P}\{X\ge 7\}$.

2.

$$\mathbb{P}\{X=Y\}=\sum_{k=1}^{\infty}\mathbb{P}\{X=Y=k\}=\sum_{k=1}^{\infty}pq(1-p)^{k-1}(1-q)^{k-1}=pq\sum_{k=0}^{\infty}\{(1-p)(1-q)\}^k$$
$$=\frac{pq}{1-\{(1-p)(1-q)\}}=\frac{pq}{p+q-pq}$$

3. (a) $\mathbb{P}\{X=3\}=\frac{3+4+7}{27}$.
(b) $\mathbb{P}\{X\ge 0\}=\frac{6+3+3+3+4}{27}$.
(c) $\mathbb{P}\{X=0|Y \text{ is even}\}=\left(\frac{6+3}{27}\right)/\left(\frac{3+6+3+1+3}{27}\right)=\frac{9}{16}$

4.

$$\mathbb{P}\{M|S\}=\frac{\mathbb{P}\{S|M\}\mathbb{P}\{M\}}{\mathbb{P}\{S|M\}\mathbb{P}\{M\}+\mathbb{P}\{S|W\}\mathbb{P}\{W\}}=\frac{(0.1)(0.45)}{(0.1)(0.45)+(0.5)(0.55)}.$$

5. (a)

$$\binom{5}{1}\binom{3}{2}\binom{7}{4}\Big/\binom{15}{7}$$

(b)

$$1-\binom{10}{7}\Big/\binom{15}{7}$$

(c)

$$
\begin{aligned}
\mathbb{P}\{\text{at least one Morisette}|2\text{ Vega}\} &= 1-\mathbb{P}\{\text{no Morisette}|2\text{ Vega}\} \\
&= 1-\frac{\mathbb{P}\{\text{no Morisette and 2 Vega}\}}{\mathbb{P}\{2\text{ Vega}\}} \\
&= 1-\left\{\binom{3}{2}\binom{7}{5}\Big/\binom{15}{7}\right\}\Big/\left\{\binom{3}{2}\binom{12}{5}\Big/\binom{15}{7}\right\} \\
&= 1-\binom{3}{2}\binom{7}{5}\Big/\binom{3}{2}\binom{12}{5}.
\end{aligned}
$$

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [50 points] Let X and Y be independent random variables with densities

$$f_X(j) = \begin{cases} (1-p)^{j-1}p & \text{if } j \in \{1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) = \begin{cases} \frac{1}{4} & \text{if } j = 0 \\ \frac{3}{4} & \text{if } j = 2 \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$.

(a) [20 points] Compute the density f_Z of Z. Note. If you are so inclined, you might want to write the density of Y as $f_Y(j) = U(j)\chi_{\{0,2\}}(j)$, where $U(0) = \frac{1}{4}$ and $U(2) = \frac{3}{4}$.

(b) [10 points] Compute the moment generating function $\mathbb{E}[e^{\theta X}]$ of X.

(c) [10 points] Compute $\mathbb{E}[X]$.

(d) [5 points] Compute $\mathbb{E}[Y]$.

(e) [5 points] Compute $\mathbb{E}[10X - Y]$.

2. [20 points] Suppose that X and Y have joint density

		Y: 0	1	2	3
X	-1	3/27	1/27	1/27	3/27
	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words, $f_{X,Y}(0, 2) = \frac{3}{27}$.

(a) [7 points] Compute $\mathbb{E}[X]$

(b) [6 points] Compute $\mathbb{E}[X^2]$

(c) [7 points] Compute $\mathbb{E}[X^2Y]$

3. 30 points Let X be a geometric random variable with parameter p. Let φ be the function

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 10-u & \text{if } u \le 10 \\ 0 & \text{if } 10 < u \le 20 \\ u-20 & \text{if } u > 20 \end{cases}$$

Find the density of $Y \stackrel{\text{def}}{=} \varphi(X)$. Hint: You may want to first graph φ.

ANSWERS

1. (a)

$$f_Z(j) = \sum_{k=-\infty}^{\infty} f_X(j-k) f_Y(k) = \frac{1}{4} f_X(j) + \frac{3}{4} f_X(j-2)$$
$$= \begin{cases} 0 & \text{if } j \le 0 \\ \frac{1}{4}p(1-p)^{j-1} & \text{if } j \ge 1 \text{ and } j-2 \le 0 \\ \frac{1}{4}p(1-p)^{j-1} + \frac{3}{4}p(1-p)^{j-3} & \text{if } j-2 \ge 1 \end{cases}$$
$$= \begin{cases} 0 & \text{if } j \le 0 \\ \frac{1}{4}p(1-p)^{j-1} & \text{if } 1 \le j \le 2 \\ \frac{1}{4}p(1-p)^{j-1} + \frac{3}{4}p(1-p)^{j-3} & \text{if } j \ge 3 \end{cases}$$

(b)

$$\mathbb{E}\left[e^{\theta X}\right] = \sum_{j=1}^{\infty} e^{\theta j} p(1-p)^{j-1} = \frac{pe^{\theta j}}{1 - e^{\theta}(1-p)}.$$

(c) By differentiating moment generating function at $\theta = 0$, we get that

$$\mathbb{E}[X] = \frac{e^{\theta}p\{1 - e^{\theta}(1-p)\} + e^{\theta}(1-p)e^{\theta}p}{\{1 - e^{\theta}(1-p)\}^2} = \frac{1}{p}.$$

(d) $\mathbb{E}[Y] = 0\frac{1}{4} + 2\frac{3}{4} = \frac{6}{4} = \frac{3}{2}$.

(e) $\mathbb{E}[10X - Y] = 10\mathbb{E}[X] - \mathbb{E}[Y] = \frac{10}{p} - \frac{3}{2}$.

2. (a) $\mathbb{E}[X] = (-1)\frac{3+1+1+3}{27} + 3\frac{3+4}{27} = \frac{-8+21}{27} = \frac{13}{27}$.

(b) Compute $\mathbb{E}[X^2] = (-1)^2\frac{3+1+1+3}{27} + 3^2\frac{3+4}{27} = \frac{8+63}{27} = \frac{71}{27}$.

(c) Compute $\mathbb{E}[X^2Y] = 1\frac{1}{27} + 2\frac{1}{27} + 3\frac{3}{27} + 27\frac{4}{27} = \frac{120}{27}$.

3. If $j < 0$, $f_Y(j) = \mathbb{P}\{Y = j\} = 0$. If $j > 0$, then

$$f_Y(j) = \mathbb{P}\{Y = j\} = \mathbb{P}\{Y - 20 = j\} = \mathbb{P}\{Y = 20 + j\} = p(1-p)^{j+19}.$$

If $j = 0$, then

$$f_Y(0) = \mathbb{P}\{Y = 0\} = \mathbb{P}\{10 \le Y \le 20\} = \sum_{j=10}^{20} p(1-p)^{j-1} = p(1-p)^{10}\sum_{j=0}^{10}(1-p)^j$$
$$= p(1-p)^{10}\frac{1-(1-p)^{11}}{1-(1-p)} = (1-p)^{10}\{1-(1-p)^{11}\}.$$

If $1 \leq j \leq 9$, then

$$\begin{aligned} f_Y(j) = \mathbb{P}\{Y = j\} &= \mathbb{P}\{10 - X = j\} + \mathbb{P}\{X - 20 = j\} \\ &= \mathbb{P}\{X = 10 - j\} + \mathbb{P}\{X = 20 + j\} = p(1-p)^{9-j} + p(1-p)^{19-j}. \end{aligned}$$

Thus,

$$f_Y(j) = \begin{cases} 0 & \text{if } j < 0 \\ (1-p)^{10}\{1-(1-p)^{11}\} & \text{if } j = 0 \\ p(1-p)^{9-j} + p(1-p)^{19-j} & \text{if } 1 \leq j \leq 9 \\ p(1-p)^{j+19} & \text{if } j \geq 10 \end{cases}$$

Math 361, Section F1, Spring 2002
Exam 3, April 26

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. $\boxed{\text{30 points}}$ Let X be a continuous uniform random variable on $(0, 2)$; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{if } 0 < t < 2 \\ 0 & \text{else.} \end{cases}$$

Define the function

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{u} & \text{if } 0 < u \leq 1 \\ 3 & \text{if } 1 < u < 2. \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

(a) $\boxed{\text{4 points}}$ Graph φ.

(b) $\boxed{\text{9 points}}$ Compute $\mathbb{P}\{Y \leq 10\}$.

(c) $\boxed{\text{9 points}}$ Compute $\mathbb{P}\{Y \leq 2\}$.

(d) $\boxed{\text{8 points}}$ Compute F_Y, the cumulative distribution function of Y.

2. $\boxed{\text{65 points}}$ Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} 4st & \text{if } s \geq 1 \text{ and } 0 \leq t \leq \frac{1}{s^2} \\ 0 & \text{else.} \end{cases}$$

(a) $\boxed{\text{25 points}}$ Compute f_X.

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$.

(c) $\boxed{\text{30 points}}$ Compute $F_{X,Y}(s,t)$ for general s and t such that $s > 1$ and $t > 1$.

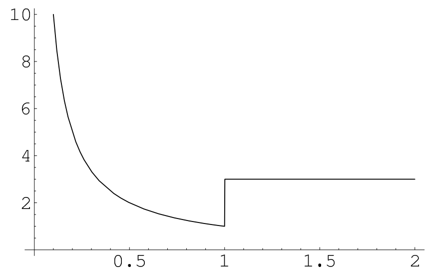
3. $\boxed{\text{5 points}}$ Suppose that X is a random variable such that

$$\mathbb{E}[e^{\theta X}] = e^{8\theta^2 - 5\theta}$$

for all $\theta \in \mathbb{R}$. What is the density of X?

Answers

1. (a) See Figure.



(b) $\mathbb{P}\{Y \le 10\} = \mathbb{P}\{X \ge 1/10\} = 1 - \frac{1}{20}$.

(c) $\mathbb{P}\{Y \le 2\} = \mathbb{P}\{\frac{1}{2} \le X \le 1\} = \frac{1}{20}$.

(d)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2}(1 - \frac{1}{t}) & \text{if } 1 \le t < 3 \\ 1 - \frac{1}{2t} & \text{if } t \ge 3 \end{cases}$$

2. (a)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=0}^{1/s^2} 4stdt & \text{if } s \ge 1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{2}{s^3} & \text{if } s \ge 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$\mathbb{E}[X] = \int_{s=-\infty}^{\infty} s f_X(s) ds = \int_{s=1}^{\infty} \frac{2}{s^2} ds = 2.$$

(c)

$$F_{X,Y}(s,t) = \int_{u=-\infty}^{s} \left\{ \int_{v=-\infty}^{t} f_{X,Y}(u,v)dv \right\} dv$$

For $s \ge 1$ and $t < 1/s^2$,

$$F_{X,Y}(s,t) = \int_{u=1}^{s} \left\{ \int_{v=0}^{t} 4uvdu \right\} dv = \int_{u=1}^{s} \left\{ \int_{v=0}^{t} 4uvdu \right\} dv = t^2(s^2 - 1).$$

For $s \ge 1$ and $t \ge 1/s^2$,

$$F_{X,Y}(s,t) = \int_{u=1}^{s} \left\{ \int_{v=0}^{1/u^2} 4uvdu \right\} dv = 1 - \frac{1}{s^2}.$$

3. X is Gaussian with mean -5 and variance 16; i.e.,

$$f_X(t) = \frac{1}{\sqrt{32\pi}} \exp\left[-\frac{t^2}{32} \right]$$

for all $t \in \mathbb{R}$.

Math 361, Section F1, Spring 2002
Final, May 8

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [30 points] There 7 balls labelled A through G in a box. We pick them out of the box one by one.
   (a) [10 points] What is the probability that the fourth ball is C?
   (b) [10 points] What is the probability that the fourth ball is C and the fifth ball is D?
   (c) [10 points] What is the probability that the fourth ball is C given that the fifth ball is D?

2. [30 points] Suppose that X is a discrete random variable with density of the form

$$f_X(j) \overset{\text{def}}{=} \begin{cases} \frac{c}{2^{|j|}} & \text{if } j \text{ is an integer and } |j| \geq 1 \\ 0 & \text{else} \end{cases}$$

   for some constant c which we will determine. Define two new random variables

$$S \overset{\text{def}}{=} \frac{X}{|X|} \qquad \text{and} \qquad M \overset{\text{def}}{=} |X|;$$

   in other words, S, which takes values in $\{-1, 1\}$, is the sign of X and M is the magnitude of X).
   (a) [5 points] Determine the constant c.
   (b) [10 points] Compute the density of S
   (c) [10 points] Compute the density of M
   (d) [5 points] Are S and M independent?

3. [40 points] Suppose that X and Y are independent discrete random variables with

$$f_X(j) = \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2 \ldots 10\} \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) = \begin{cases} \frac{1}{20} & \text{if } j \in \{1, 2 \ldots 20\} \\ 0 & \text{else} \end{cases}$$

   Define $Z_1 \overset{\text{def}}{=} X + Y$ and $Z_2 \overset{\text{def}}{=} \max\{X, Y\}$.
   (a) [20 points] Compute f_{Z_1}, the density of Z_1.
   (b) [20 points] Compute $\mathbb{P}\{Z_2 \leq j\}$ for all integers j.

4. [20 points] Let X be a continuous uniform random variable on $(-1,1)$; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{if } -1 < t < 1 \\ 0 & \text{else.} \end{cases}$$

Define the function

$$\varphi(u) \stackrel{\text{def}}{=} 1 - u^2$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

(a) [2 points] Graph φ.

(b) [6 points] Compute $\mathbb{P}\{Y \le 1/2\}$.

(c) [6 points] Compute F_Y, the cumulative distribution function of Y.

(d) [6 points] If Y has a density compute it. If it does not have a density, tell me why.

5. [30 points] Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{s}e^{-t/s} & \text{if } t \ge s^2 \text{ and } s > 0 \\ 0 & \text{else} \end{cases}$$

(a) [10 points] Compute $\mathbb{P}\{Y \le X\}$ (Hint: you may first want to identify the set $\{(s,t) : f_{X,Y}(s,t) > 0 \text{ and } t \le s\}$).

(b) [10 points] Compute f_X, the density of X.

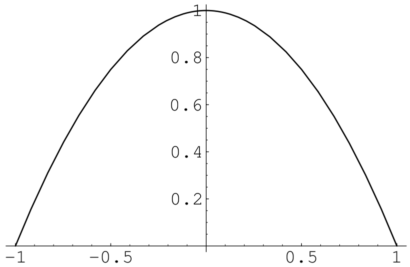
(c) [10 points] Compute $f_{Y|X}$, the conditional density of Y given X.

ANSWERS

1. (a) $\frac{(6)_3 \cdot 1}{(7)_4} = \frac{1}{7}$.
   (b) $\frac{(5)_3 \cdot 1 \cdot 1}{(7)_4} = \frac{1}{42}$.
   (c) $\left(\frac{1}{42}\right) / \left(\frac{(6)_4 \cdot 1}{(7)_5}\right) = \frac{1}{6}$.
2. (a)
$$1 = \sum_{k=-\infty}^{\infty} f_X(j) = 2c \sum_{j=1}^{\infty} \frac{1}{2^j} = c \sum_{j=0}^{\infty} \frac{1}{2^j} = 2c;$$
   $c = \frac{1}{2}$.
   (b) We have
$$f_S(1) = \mathbb{P}\{X > 0\} = \sum_{j=1}^{\infty} \frac{1}{2^{|j|+1}} = \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{2}$$
$$f_S(-1) = \mathbb{P}\{X < 0\} = \sum_{j=1}^{\infty} \frac{1}{2^{|j|+1}} = \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{2}.$$
   (c) For j a positive integer,
$$f_M(j) = \mathbb{P}\{S = j\} = \mathbb{P}\{X = j\} + \mathbb{P}\{X = -j\} = 2\frac{1}{2^{|j|+1}} = \frac{1}{2^{|j|}}.$$
   f_S is zero elsewhere.
   (d) Yes; for any positive integer j,
$$f_{M,S}(j, 1) = \mathbb{P}\{M = j \text{ and } S = 1\} = \mathbb{P}\{X = j\} = \frac{1}{2^{|j|+1}} = f_M(j) f_S(1)$$
$$f_{M,S}(j, -1) = \mathbb{P}\{M = j \text{ and } S = -1\} = \mathbb{P}\{X = -j\} = \frac{1}{2^{|j|+1}} = f_M(j) f_S(-1);$$
   both $f_{M,S}(j, s)$ and $f_M(j) f_S(s)$ are zero unless j is a positive integer and $s \in \{1, -1\}$.
3. (a)
$$f_{Z_1}(k) = \sum_{k=-\infty}^{\infty} f_X(j-k) f_Y(k) = \frac{1}{200} \sum_{k=-\infty}^{\infty} \chi_{[1,10]}(j-k) \chi_{[1,20]}(k)$$
$$= \frac{1}{200} \sum_{k=-\infty}^{\infty} \chi_{[1,20] \cap [j-10, j-1]}(k) = \begin{cases} \frac{j-1-1+1}{200} & \text{if } j-1 \geq 1 \text{ and } j-10 \leq 1 \\ \frac{(j-1)-(j-10)+1}{200} & \text{if } j-1 > 10 \text{ and } j-1 < 20 \\ \frac{20-(j-10)+1}{200} & \text{if } j-10 \leq 20 \text{ and } j-1 > 20 \\ 0 & \text{else} \end{cases}$$
$$= \begin{cases} \frac{j-1}{200} & \text{if } 1 \leq j \leq 10 \\ \frac{1}{20} & \text{if } 10 < j < 21 \\ \frac{31-j}{20} & \text{if } 21 \leq j \leq 30 \\ 0 & \text{else} \end{cases}$$

(b) For all j, $\mathbb{P}\{Z_2 \le j\} = \mathbb{P}\{X \le j\}\mathbb{P}\{Y \le j\}$. For $j \le 0$, $\mathbb{P}\{Z_2 \le j\} = 0$. For $j \ge 20$, $\mathbb{P}\{Z_2 \le j\} = 1$. For integers $j \in [1, 10]$, $\mathbb{P}\{Z_2 \le j\} = \frac{j^2}{200}$, and for integers $j \in (11, 20)$, $\mathbb{P}\{Z_2 \le j\} = \frac{j}{20}$.

4. (a) See figure.



(b)

$$\mathbb{P}\{Y \le 1/2\} = \mathbb{P}\left\{1 - X^2 \le \frac{1}{2}\right\} = \mathbb{P}\left\{X^2 \ge \frac{1}{2}\right\} = \mathbb{P}\left\{|X| \le \frac{1}{\sqrt{2}}\right\} = 1 - \frac{1}{\sqrt{2}}.$$

(c)

$$F_Y(t) = \mathbb{P}\{1 - X^2 \le t\} = \mathbb{P}\{X^2 \ge 1 - t\} = \begin{cases} 0 & \text{if } 1 - t \ge 1 \\ \mathbb{P}\{|X| \ge \sqrt{1-t}\} & \text{if } 0 < 1 - t < 1 \\ 1 & \text{if } 1 - t \le 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t \le 0 \\ 1 - \sqrt{1-t} & \text{if } 0 < t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 0 \\ 1 - \sqrt{1-t} & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(d)

$$f_Y(t) = \begin{cases} \frac{1}{2\sqrt{1-t}} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

5. (a)

$$\mathbb{P}\{Y \le X\} = \mathbb{P}\{X \ge Y\} = \iint_{s>t} f_{X,Y}(s,t)ds\,dt = \int_{s=0}^{1}\int_{t=s^2}^{s} \frac{1}{s}e^{-t/s}dt\,ds$$

$$= \int_{s=0}^{1} \{e^{-s} - e^{-1}\}ds = 1 - e^{-1} - e^{-1} = 1 - 2e^{-1}.$$

(b) For $s < 0$, $f_X(s) = 0$. For $s > 0$,

$$f_X(s) = \int_{t=s^2}^{\infty} \frac{1}{s} e^{-t/s} ds = e^{-s}.$$

(c)

$$f_{Y|X}(t|s) = \frac{f_{X,Y}(s,t)}{f_X(s)} = \begin{cases} \frac{1}{s} e^{-t/s+s} & \text{if } t \geq s^2 \text{ and } s > 0 \\ 0 & \text{else} \end{cases}$$