

Math 361, Section A1, Summer 1996
Exam 1, June 25

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 15 points Evaluate the following:

(a) 5 points $3!$

(b) 5 points $\binom{6}{4}$

(c) 5 points $\frac{6!}{4!}$

2. 40 points Suppose we have two sets A and B . Suppose we know that

$$\mathbb{P}(A) = 0.3 \quad \mathbb{P}(B^c) = 0.5 \quad \text{and} \quad \mathbb{P}(A \setminus B) = 0.2.$$

(a) 5 points $\mathbb{P}(A \cap B)$

(b) 5 points $\mathbb{P}(B)$

(c) 5 points $\mathbb{P}(B \setminus A)$

(d) 5 points $\mathbb{P}(A^c \cup B)$

(e) 5 points $\mathbb{P}(B^c|A)$

(f) 5 points $\mathbb{P}(A \cup B)$

(g) 5 points $\mathbb{P}(B^c|A \cup B)$

(h) 5 points Are the sets A and B independent? Why or why not?

3. 30 points A janitor has 10 keys, exactly one of which fits a certain lock. She tries the keys one at a time, at each trial choosing at random from the keys that were not tried earlier.

(a) 15 points Find the probability that the 6th key tried is the correct key.

(b) 15 points Find the probability that the 6th key tried is the correct key, given that the first two keys don't work.

4. 15 points A box contains 80 good fuses and 20 bad fuses. Select 5 fuses from the box. What is the probability that 4 of these are good and one is bad?

ANSWERS

1. (a) 6

(b) $\frac{6 \cdot 5}{2} = 15$.

(c) 30.

2. (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \setminus B) = 0.3 - 0.2 = 0.1$.

(b) $\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - 0.5 = 0.5$.

(c) $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = 1 - \mathbb{P}(B^c) - \mathbb{P}(A \cap B) = 1 - 0.5 - 0.1 = 0.4$.

(d) $\mathbb{P}(A^c \cup B) = 1 - \mathbb{P}(A \setminus B) = 1 - 0.2 = 0.8$.

(e) $\mathbb{P}(B^c | A) = \frac{\mathbb{P}(A \setminus B)}{\mathbb{P}(A)} = \frac{0.2}{0.3} = \frac{2}{3}$.

(f) $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \setminus B) = 1 - \mathbb{P}(B^c) + \mathbb{P}(A \setminus B) = 1 - 0.5 + 0.2 = 0.7$.

(g) $\mathbb{P}(B^c | A \cup B) = \frac{\mathbb{P}(B^c \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A \setminus B)}{\mathbb{P}(A \cup B)} = \frac{0.2}{0.7} = \frac{2}{7}$.

(h) No, since $\mathbb{P}(B^c | A) = \frac{2}{3} \neq 0.5 = \mathbb{P}(B^c)$.

3.

$$A \stackrel{\text{def}}{=} \{\text{6th key works}\} \quad B \stackrel{\text{def}}{=} \{\text{first two keys do not work}\}.$$

(a)

$$\mathbb{P}(A) = \frac{(9)_5 \cdot 1}{(10)_6} = \frac{1}{10}.$$

(b) Note that $A \subset B$ and that $\mathbb{P}(B) = \frac{(9)_2}{(10)_2} = \frac{8}{10}$. Thus $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{1}{8}$.

4.

$$\frac{20 \binom{60}{4}}{\binom{80}{5}}.$$

Math 361, Section A1, Summer 1996
Exam 2, July 11

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 20 points A random variable X has moment generating function

$$\Phi_X(t) \stackrel{\text{def}}{=} \mathbb{E}[t^X] = \exp[\lambda(t^2 - 1)].$$

- (a) 5 points Find $\mathbb{E}[X]$.
 (b) 5 points Find $\mathbb{E}[X^2]$.
 (c) 5 points Find $\text{Var}(X)$.
 (d) 5 points Find $\mathbb{E}[X^3]$ (Hint: what is the third derivative of $\Phi_X(1)$?).

Note: This is not a good problem

2. 15 points Suppose that X and Y are independent random variables such that

$$\mathbb{E}[X^2] = 4 \quad \mathbb{E}[Y^2] = 2 \quad \mathbb{E}[X^2] = 1 \quad \text{and} \quad \mathbb{E}[Y] = 0.$$

Compute the variance of $Z \stackrel{\text{def}}{=} X^2Y$.

3. 15 points Suppose a box has 20 balls labelled $1, 2, \dots, 20$. Two balls are selected from the box (without replacement). Let X denote the larger of the two numbers on the balls.

- (a) 10 points Compute the density of X .
 (b) 5 points Compute $\mathbb{E}[X]$.

4. 50 points Let X and Y be two random variables having the joint density given by the following table:

		Y			
		-1	0	2	6
	-2	1/9	1/27	1/27	1/9
X	1	2/9	0	1/9	1/9
	3	0	0	1/9	4/27

- (a) 2 points Compute $\mathbb{P}\{XY \text{ is odd}\}$.
 (b) 2 points Compute $\mathbb{P}\{X \geq 0 \text{ and } Y < 0\}$.
 (c) 5 points Compute the density of X .
 (d) 5 points Compute the density of Y .
 (e) 5 points Compute $\mathbb{E}[X]$.

- (f) 3 points Compute $\mathbb{E}[Y]$.
- (g) 5 points Find $\mathbb{E}[X^2]$.
- (h) 3 points Find $\mathbb{E}[Y^2]$.
- (i) 5 points Find $\text{Var}(X)$.
- (j) 5 points Find $\text{Var}(Y)$.
- (k) 5 points Find $\text{Cov}(X, Y)$.
- (l) 5 points Find $\rho(X, Y)$.

ANSWERS

1. Note that $\Phi'_X(t) = 2\lambda t\Phi_X(t)$; thus

$$\begin{aligned}\Phi''_X(t) &= \{2\lambda + 4\lambda^2 t^2\}\Phi_X(t) \\ \Phi'''_X(t) &= \{8\lambda^2 t + (2\lambda + 4\lambda^2 t^2)(2\lambda t)\}\Phi_X(t)\end{aligned}$$

- (a) $\mathbb{E}[X] = \Phi'_X(1) = 2\lambda$.
 (b) $\mathbb{E}[X(X-1)] = \Phi''_X(1) = 2\lambda + 4\lambda^2$, so $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = 2\lambda + 4\lambda^2 - 2\lambda = 4\lambda^2$.
 (c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = 4\lambda^2 - 4\lambda^2 = 0$. This indicates that the problem is poorly written.
 (d) $\mathbb{E}[X(X-1)(X-2)] = \Phi'''_X(1) = 8\lambda^2 + 4\lambda^2 + 8\lambda^3 = 12\lambda^2 + 8\lambda^3$, so

$$\begin{aligned}\mathbb{E}[X^3] &= \mathbb{E}[X(X-1)(X-2)] + 3\mathbb{E}[X^2] - 2\mathbb{E}[X] \\ &= 12\lambda^2 + 8\lambda^3 + 12\lambda^2 - 4\lambda = 8\lambda^3 + 24\lambda^2 - 4\lambda.\end{aligned}$$

2.

$$\mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \mathbb{E}[X^4 Y^2] - \mathbb{E}[X^2 Y]^2 = \mathbb{E}[X^4]\mathbb{E}[Y^2] - \mathbb{E}[X^2]^2\mathbb{E}[Y]^2 = 4 \cdot 2 - 1^2 \cdot 0 = 8.$$

3. (a) Note that $2 \leq X \leq 20$. For $j \in \{2, 3, \dots, 20\}$,

$$\mathbb{P}\{X \leq j\} = \frac{\binom{j}{2}}{\binom{20}{2}}.$$

Thus, for $3 \leq X \leq 20$,

$$p_X(j) = \mathbb{P}\{X \leq j\} - \mathbb{P}\{X \leq j-1\} = \frac{\binom{j}{2} - \binom{j-1}{2}}{\binom{20}{2}} = \frac{j-1}{190}$$

(check by direct computation that $\binom{j}{2} - \binom{j-1}{2} = j-1$). We also have that

$$p_X(2) = \mathbb{P}\{X \leq 2\} = \frac{1}{190}.$$

For all other j , $p_X(j) = 0$.

(b)

$$\mathbb{E}[X] = \sum_{j=2}^{20} j \frac{j-1}{190}.$$

4. (a) $2/9+0=2/9$.
 (b) $2/9+0=2/9$.

(c)

$$p_X(j) = \begin{cases} \frac{8}{27} & \text{if } j = -2 \\ \frac{12}{27} & \text{if } j = 1 \\ \frac{7}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$

(d)

$$p_Y(j) = \begin{cases} \frac{9}{27} & \text{if } j = -1 \\ \frac{1}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 2 \\ \frac{10}{27} & \text{if } j = 6 \\ 0 & \text{else} \end{cases}$$

(e)

$$\mathbb{E}[X] = \frac{1}{27}\{(-2)8 + (1)12 + 3(7)\} = \frac{1}{27}\{-16 + 12 + 21\} = \frac{17}{27}.$$

(f)

$$\mathbb{E}[Y] = \frac{1}{27}\{(-1)9 + (0)1 + (2)7 + (6)10\} = \frac{-9 + 14 + 60}{27} = \frac{65}{27}.$$

(g)

$$\mathbb{E}[X^2] = \frac{1}{27}\{(-2)^2 8 + (1)^2 12 + (3)^2 7\} = \frac{32 + 12 + 63}{27} = \frac{107}{27}.$$

(h)

$$\mathbb{E}[Y^2] = \frac{1}{27}\{(-1)^2 9 + (0)^2 1 + (2)^2 7 + (6)^2 10\} = \frac{9 + 28 + 360}{27} = \frac{397}{27}.$$

(i) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{107}{27} - \frac{289}{729} = \frac{2600}{729}.$

(j) $\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{397}{27} - \frac{4225}{729} = \frac{6494}{729}.$

(k)

$$\begin{aligned} \mathbb{E}[XY] &= \frac{1}{27}\{(-2)(-1)3 + (1)(-1)6 + (-2)(2)1 + (1)(2)3 \\ &\quad + (3)(2)3 + (-2)(6)3 + (1)(6)3 + (3)(6)4\} = \frac{6 - 6 - 4 + 6 + 18 - 36 + 18 + 72}{27} \\ &= \frac{74}{27}. \end{aligned}$$

Thus

$$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{74}{27} - \left(\frac{17}{27}\right)\left(\frac{65}{27}\right) = \frac{74}{27} - \frac{1105}{729} = \frac{893}{729}.$$

(l)

$$\frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{893}{\sqrt{(2600)(6494)}}.$$

Math 361, Section A1, Summer 1996
Exam 3, July 24

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 38 points Fix $a > 0$ and let X be a random variable uniformly distributed on $[0, a]$. Define a random variable Y by

$$Y = \begin{cases} a/3 & \text{if } X \leq a/3 \\ X & \text{if } a/3 < X < 2a/3 \\ 2a/3 & \text{if } X \geq 2a/3. \end{cases}$$

- (a) 10 points Find the distribution function of Y .
- (b) 4 points What is $\mathbb{P}\{Y \leq a/2\}$?
- (c) 5 points What is $\mathbb{P}\{Y < a/2\}$?
- (d) 5 points What is $\mathbb{P}\{Y = a/2\}$?
- (e) 4 points What is $\mathbb{P}\{Y \leq a/3\}$?
- (f) 5 points What is $\mathbb{P}\{Y < a/3\}$?
- (g) 5 points What is $\mathbb{P}\{Y = a/3\}$?
2. 32 points Let X be a random variable with distribution function defined by

$$F_X(t) = \frac{1}{2} + \frac{t}{2(|t| + 1)}. \quad t \in \mathbb{R}$$

- (a) 7 points Express $\mathbb{P}\{|X - 1| \geq 2\}$ in terms of F .
- (b) 5 points Does X have a density? If so, find it.
- (c) 10 points Find the distribution function of the random variable $Y = |X|$.
- (d) 10 points Does Y have a density? If so, find it.
3. 30 points Let

$$f_{X,Y}(s, t) = \begin{cases} c(t - s)^\alpha & \text{if } 0 \leq s < t \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

- (a) 10 points For what values of α and c will f be a density function?
- (b) 15 points For such a choice of α and c , what is f_X and f_Y ?
- (c) 5 points Are X and Y independent?

ANSWERS

1. (a)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < a/3 \\ \frac{t}{a} & \text{if } a/3 \leq t < 2a/3 \\ 1 & \text{if } t \geq 2a/3. \end{cases}$$

(b) 1/2

(c) 1/2

(d) 0

(e) 1/3

(f) 0

(g) 1/3.

2. (a)

$$\begin{aligned} \mathbb{P}\{|X - 1| \geq 2\} &= \mathbb{P}\{X \in (-\infty, -1] \cup [3, \infty)\} = F_X(-1) + 1 - F_X(3^-) \\ &= \frac{1}{4} + 1 - \frac{7}{8} = \frac{3}{8}. \end{aligned}$$

(b)

$$f_X(t) = \begin{cases} \frac{(t+1)-t}{2(t+1)^2} & \text{if } t > 0 \\ \frac{(1-t)+t}{2(1-t)} & \text{if } t < 0 \end{cases} = \begin{cases} \frac{1}{2(t+1)^2} & \text{if } t > 0 \\ \frac{1}{2(1-t)} & \text{if } t < 0 \end{cases} = \frac{1}{2(|t| + 1)}.$$

(c) $F_Y(t) = \mathbb{P}\{|X| \leq t\}$. If $t < 0$, $F_Y(t) = 0$. If $t \geq 0$,

$$\begin{aligned} F_Y(t) &= \mathbb{P}\{-t \leq X \leq t\} F_X(t) - F_X(-t^-) \\ &= \frac{t}{2(|t| + 1)} - \frac{|-t|}{2(|-t| + 1)} = \frac{t}{2(t+1)} + \frac{t}{2(t+1)} = \frac{t}{t+1}. \end{aligned}$$

(d)

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2(t+1)^2} & \text{if } t > 0 \end{cases}$$

3. (a)

$$\begin{aligned} 1 &= \int_{t \in \mathbb{R}} \int_{s \in \mathbb{R}} f_{X,Y}(s, t) ds dt = c \int_{t=0}^1 \int_{s=0}^t (t-s)^\alpha ds dt \\ &= \frac{c}{\alpha+1} \int_{t=0}^1 t^{\alpha+1} dt = \frac{c}{(\alpha+1)(\alpha+2)}; \end{aligned}$$

for this to hold, we need that $\alpha > -1$; then $c = (\alpha+1)(\alpha+2)$.

(b)

$$\begin{aligned} f_X(s) &= \int_{t \in \mathbb{R}} f_{X,Y}(s,t) dt = \begin{cases} c \int_{t=s}^1 (t-s)^\alpha dt & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{c(1-s)^{\alpha+1}}{\alpha+1} & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases} \\ f_Y(t) &= \int_{s \in \mathbb{R}} f_{X,Y}(s,t) ds = \begin{cases} c \int_{s=0}^t (t-s)^\alpha dt & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{ct^{\alpha+1}}{\alpha+1} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

(c) They are dependent, since if $0 < t < s < 1$, $f_{X,Y}(s,t) = 0 \neq f_X(s)f_Y(t)$.

Math 361, Section A1, Summer 1996
Final, August 3

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 200 Points

1. 10 points Suppose that we have n balls and that we distribute them into r boxes. Let X be the number of balls which are in box 1. Find the density of X .
2. 10 points Choose 4 cards from a standard deck of cards. Let X be the number of red cards which you have chosen. Find the density of X .
3. 20 points Let X be geometrically distributed with parameter p .
 - (a) 5 points Compute $\mathbb{E}[X]$.
 - (b) 5 points Compute $\mathbb{E}[X^2]$.
 - (c) 5 points Compute the variance of X .
 - (d) 5 points Compute $\mathbb{E}[t^X]$.
4. 10 points Let X_1, X_2, \dots be random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 4^{-i}$. Compute

$$\mathbb{E} \left[\sum_{i=1}^{\infty} 2^i X_i^2 \right].$$

5. 55 points Let X and Y be independent geometrically distributed random variables with parameters p_1 and p_2 (respectively). Set $Z = X + Y$.
 - (a) 10 points Compute the density of Z .
 - (b) 20 points Compute the joint density of X and Z .
 - (c) 10 points Compute $p_{X|Z}(j|k)$, the conditional density of X given Z , where $k \geq 0$.
 - (d) 15 points Compute $\mathbb{E}[X|Z = k]$.
6. 50 points Let X be a continuous random variable with density

$$f_X(t) = ce^{-|t|}, \quad t \in \mathbb{R}$$

Set $Y = X^3$.

- (a) 5 points Compute what c must be.
- (b) 5 points Compute $\mathbb{E}[X]$.
- (c) 10 points Compute the cumulative distribution function of X .
- (d) 15 points Compute the cumulative distribution function of Y .
- (e) 10 points Compute the density of Y .

(f) 5 points Compute $\mathbb{E}[Y]$.

7. 45 points Let

$$f(x, y) = \begin{cases} c(y - x)^\alpha & \text{if } 0 \leq x < y \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) 10 points For what values of α and c will f be a density function?

(b) 20 points For such a choice of α and c , what is f_X and f_Y ?

(c) 5 points Compute $\mathbb{E}[X]$.

(d) 5 points Compute $\mathbb{E}[Y]$.

(e) 15 points Compute $\mathbb{E}[XY]$.

ANSWERS

1. For $1 \leq j \leq n$,

$$p_X(j) = \frac{\binom{n}{j}(r-1)^{n-j}}{r^n}.$$

For all other j , $p_X(j) = 0$.

2. For $1 \leq j \leq n$,

$$p_X(j) = \frac{\binom{26}{j}\binom{26}{4-j}}{\binom{52}{4}}.$$

For all other j , $p_X(j) = 0$.

- 3.

$$\begin{aligned} \Phi_X(t) &\stackrel{\text{def}}{=} \mathbb{E}[t^X] = \sum_{k=1}^{\infty} p(1-p)^{k-1}t^k = pt \sum_{k=1}^{\infty} ((1-p)t)^{k-1} \\ &= pt \sum_{k=0}^{\infty} ((1-p)t)^k = \begin{cases} \frac{pt}{1-t(1-p)} & \text{if } t < 1/(1-p) \\ \infty & \text{else} \end{cases} \end{aligned}$$

For $t < 1/(1-p)$,

$$\begin{aligned} \Phi'_X(t) &= \frac{p\{1-t(1-p)\} + (1-p)pt}{(1-t(1-p))^2} = \frac{p}{(1-t(1-p))^2} \\ \Phi''_X(t) &= \frac{2p(1-p)}{(1-t(1-p))^3}. \end{aligned}$$

(a) $\mathbb{E}[X] = \Phi'_X(1) = \frac{p}{p^2} = \frac{1}{p}$.

(b) $\mathbb{E}[X(X-1)] = \Phi''_X(1) = \frac{2p(1-p)}{p^3} = \frac{2(1-p)}{p^2}$, so $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2}{p^2} - \frac{1}{p}$.

(c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$.

- (d)

$$\mathbb{E}[t^X] = \begin{cases} \frac{pt}{1-t(1-p)} & \text{if } t < 1/(1-p) \\ \infty & \text{else} \end{cases}$$

- 4.

$$\mathbb{E}\left[\sum_{i=1}^{\infty} 2^i X_i^2\right] = \sum_{i=1}^{\infty} 2^i \mathbb{E}[X_i^2] = \sum_{i=1}^{\infty} 2^i 4^{-i} = \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

5. (a) For $j \in \{2, 3, \dots\}$,

$$\begin{aligned}
p_Z(k) &= \sum_{j=-\infty}^{\infty} \mathbb{P}\{X = j\} \mathbb{P}\{Y = k - j\} = p_1 p_2 \sum_{j=1}^{k-1} (1 - p_1)^{j-1} (1 - p_2)^{k-j-1} \\
&= p_1 p_2 (1 - p_2)^{k-2} \sum_{j=1}^{k-1} \left(\frac{1 - p_1}{1 - p_2} \right)^{j-1} = p_1 p_2 (1 - p_2)^{k-2} \sum_{j=0}^{k-2} \left(\frac{1 - p_1}{1 - p_2} \right)^j \\
&= p_1 p_2 (1 - p_2)^{k-2} \left\{ \frac{1 - \left(\frac{1 - p_1}{1 - p_2} \right)^{k-1}}{1 - \left(\frac{1 - p_1}{1 - p_2} \right)} \right\} = p_1 p_2 \left\{ \frac{(1 - p_2)^{k-1} - (1 - p_1)^{k-1}}{(1 - p_2) - (1 - p_1)} \right\} \\
&= \frac{p_1 p_2}{p_1 - p_2} \left\{ (1 - p_2)^{k-1} - (1 - p_1)^{k-1} \right\}
\end{aligned}$$

if $p_1 \neq p_2$; if $p_1 = p_2 = q$, then

$$p_Z(k) = q^2 \sum_{j=1}^{k-1} (1 - q)^{j-1} (1 - q)^{k-j-1} = q^2 (1 - q)^{k-2} \sum_{j=1}^{k-1} 1 = q^2 (k - 1) (1 - q)^{k-2}.$$

(b)

$$\begin{aligned}
p_{X,Z}(j, k) &= \mathbb{P}\{X = j, Z = k\} = \mathbb{P}\{X = j, Y = k - j\} \\
&= \begin{cases} p_1 p_2 (1 - p_1)^{j-1} (1 - p_2)^{k-j-1} & \text{if } j \geq 1 \text{ and } k - j \geq 1 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} p_1 p_2 (1 - p_1)^{j-1} (1 - p_2)^{k-j-1} & \text{if } 1 \leq j \leq k - 1 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

(c)

$$p_{X|Z}(j|k) = \frac{p_{X,Z}(j, k)}{p_Z(k)}$$

If $1 \leq j \leq k - 1$,

$$p_{X|Z}(j|k) = \frac{(p_1 - p_2)(1 - p_1)^{j-1}(1 - p_2)^{k-j-1}}{(1 - p_2)^{k-1} - (1 - p_1)^{k-1}}$$

if $p_1 \neq p_2$, and

$$p_{X|Z}(j|k) = \frac{1}{k - 1}$$

if $p_1 = p_2 = q$.

(d) For $k \geq 0$,

$$\mathbb{E}[X|Z = k] = \sum_{j \in \mathbb{Z}} j p_{X|Z}(j|k).$$

Thus

$$\mathbb{E}[X|Z = k] = \sum_{j=1}^{k-1} j \frac{(p_1 - p_2)(1 - p_1)^{j-1}(1 - p_2)^{k-j-1}}{(1 - p_2)^{k-1} - (1 - p_1)^{k-1}}$$

if $p_1 \neq p_2$, and

$$\mathbb{E}[X|Z = k] = \sum_{j=1}^{k-1} \frac{j}{k-1} = \frac{(k-1)k}{2(k-1)} = \frac{k}{2}$$

if $p_1 = p_2 = q$.

6.

$$1 = \int_{t \in \mathbb{R}} f_X(t) dt = 2c \int_{t=0}^{\infty} = 2c;$$

$$c = \frac{1}{2}.$$

$$\mathbb{E}[X] = \frac{1}{2} \int_{t \in \mathbb{R}} te^{-|t|} dt = 0$$

(note that f_X is even).

(b)

$$\begin{aligned} F_X(t) &= \int_{s=-\infty}^t f_X(s) ds = \begin{cases} \frac{1}{2} \int_{s=-\infty}^t e^s ds & \text{if } t < 0 \\ \frac{1}{2} \int_{s=-\infty}^0 e^s ds + \frac{1}{2} \int_{s=0}^t e^{-s} ds & \text{if } t \geq 0 \end{cases} \\ &= \begin{cases} \frac{e^t}{2} & \text{if } t < 0 \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-t}) & \text{if } t \geq 0 \end{cases} = \begin{cases} \frac{e^t}{2} & \text{if } t < 0 \\ 1 - \frac{e^{-t}}{2} & \text{if } t \geq 0 \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} F_Y(t) &= \mathbb{P}\{X^3 \leq t\} = F_X(t^{1/3}) = \begin{cases} \frac{e^{t^{1/3}}}{2} & \text{if } t^{1/3} < 0 \\ 1 - \frac{e^{-t^{1/3}}}{2} & \text{if } t^{1/3} \geq 0 \end{cases} \\ &= \begin{cases} \frac{e^{t^{1/3}}}{2} & \text{if } t < 0 \\ 1 - \frac{e^{-t^{1/3}}}{2} & \text{if } t \geq 0 \end{cases} \end{aligned}$$

(d)

$$f_Y(t) = \begin{cases} \frac{|t|^{-2/3}}{6} e^{t^{1/3}} & \text{if } t < 0 \\ \frac{|t|^{-2/3}}{6} e^{-t^{1/3}} & \text{if } t \geq 0 \end{cases} = \frac{|t|^{-2/3}}{6} e^{-|t|^{1/3}}$$

(e)

$$\mathbb{E}[Y] = \frac{1}{6} \int_{t \in \mathbb{R}} t |t|^{-2/3} e^{-|t|^{1/3}} dt = 0$$

(note that f_Y is even).

7. (a)

$$\begin{aligned} 1 &= \int_{t \in \mathbb{R}} \int_{s \in \mathbb{R}} f_{X,Y}(s,t) ds dt = c \int_{t=0}^1 \int_{s=0}^t (t-s)^\alpha ds dt \\ &= \frac{c}{\alpha+1} \int_{t=0}^1 t^{\alpha+1} dt = \frac{c}{(\alpha+1)(\alpha+2)}; \end{aligned}$$

for this to hold, we need that $\alpha > -1$; then $c = (\alpha+1)(\alpha+2)$.

(b)

$$\begin{aligned} f_X(s) &= \int_{t \in \mathbb{R}} f_{X,Y}(s,t) dt = \begin{cases} c \int_{t=s}^1 (t-s)^\alpha dt & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{c(1-s)^{\alpha+1}}{\alpha+1} & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases} \\ f_Y(t) &= \int_{s \in \mathbb{R}} f_{X,Y}(s,t) ds = \begin{cases} c \int_{s=0}^t (t-s)^\alpha ds & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{ct^{\alpha+1}}{\alpha+1} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

(c)

$$\mathbb{E}[X] = \int_{s \in \mathbb{R}} s f_X(s) ds = \frac{c}{\alpha+1} \int_{s=0}^1 s(1-s)^{\alpha+1} ds.$$

(d)

$$\mathbb{E}[Y] = \int_{t \in \mathbb{R}} t f_Y(t) dt = \frac{c}{\alpha+1} \int_{t=0}^1 t^{\alpha+2} ds = \frac{c}{(\alpha+1)(\alpha+2)} = \frac{\alpha+2}{\alpha+3}.$$

(e)

$$\mathbb{E}[XY] = \int_{s \in \mathbb{R}} \int_{t \in \mathbb{R}} st f_{X,Y}(s,t) ds dt = c \int_{t=0}^1 \int_{s=0}^t st(t-s)^\alpha ds dt.$$