

1. 10 points Suppose that we want to arrange 15 textbooks on a shelf. There are
 - 6 math books
 - 5 history books
 - 4 chemistry books.
- (a) 5 points How many ways can we arrange the textbooks if we want to keep each subject together?
- (b) 5 points How many ways can we arrange the textbooks if we want a math book at each end?

ANSWERS

1. (a) $6!5!4!3!$
(b) $(6)_2 13!$

1. 10 points This is problem 36 from the book. Suppose that two cards are randomly chosen from a deck of 52 cards.
 - (a) 5 points What is the probability that both are aces?
 - (b) 5 points What is the probability that both have the same face value?

ANSWERS

1. Set $p \stackrel{\text{def}}{=} 1/\binom{52}{2}$.

(a) $\binom{4}{2}p$.

(b) $13\binom{4}{2}p$.

1. 10 points This is problem 18 from the book. Suppose that two cards are randomly chosen from a deck of 52 cards. What is the probability that one of the cards is an ace and the other card is either a 10, jack, queen or king (i.e., what is the probability of a blackjack)?

ANSWERS

1.

$$4 \times 16 / \binom{52}{2}.$$

1. 10 points This is essentially problem 35 from the book. It is your birthday! Time to do some math. Mom or Dad have hidden a present for you, either upstairs or downstairs. With probability $.6$, the present was hidden by Mom. If Mom has hidden the present, it is upstairs with probability $.7$. Dad is equally likely to have hidden it upstairs or downstairs.
 - (a) 5 points What is the probability that the present is upstairs?
 - (b) 5 points If it is upstairs, what is the probability that it was hidden by Dad?

ANSWERS

1.

$$M = \{\text{Mom hid the present}\} \quad \text{and} \quad U = \{\text{present is upstairs}\}.$$

We know that

$$\mathbb{P}(M) = .6, \quad \mathbb{P}(U|M) = .7, \quad \text{and} \quad \mathbb{P}(U|D) = .5.$$

We then compute that

$$\mathbb{P}(U \cap M) = \mathbb{P}(U|M)\mathbb{P}(M) = .7 \times .6 = .42$$

$$\mathbb{P}(U \cap D) = \mathbb{P}(U|D)\mathbb{P}(D) = .5 \times .4 = .20$$

$$(a) \quad \mathbb{P}(U) = \mathbb{P}(U \cap M) + \mathbb{P}(U \cap D) = .42 + .2 = .62.$$

$$(b) \quad \mathbb{P}(D|U) = \mathbb{P}(U \cap D)/\mathbb{P}(U) = .2/.62.$$

1. 10 points Suppose we have an unfair coin with $\mathbb{P}\{H\} = p$ and $\mathbb{P}\{T\} = 1 - p$. We flip the coin a countable sequence of times. We are interested in the statistics of the third heads.
 - (a) 2 points Pick a specific configuration where the third heads is on the 9th toss.
 - (b) 2 points What is the probability of the configuration chosen in part a)?
 - (c) 3 points What is the probability that the third heads is on the 9th toss?
 - (d) 2 points For any integer $n \geq 3$, what is the probability that the third heads is on the n -th toss.

ANSWERS

1. (a) $(TTTHTHTTH)$
- (b) $p^3(1-p)^6$.
- (c) $\binom{8}{2}p^3(1-p)^6$.
- (d) $\binom{n-1}{2}p^3(1-p)^{n-3}$.

1. 10 points A box has balls labelled B_1 through B_{20} . Pick three balls. Let X be the number of the middle ball. In other words, if we pick balls B_5 , B_3 and B_{17} , we order them and the middle ball is B_5 , so $X = 5$.
 - (a) 2 points Verbally describe what it means that $X = 5$.
 - (b) 2 points Find $\mathbb{P}\{X = 5\}$.
 - (c) 2 points Find $\mathbb{P}\{X = 9\}$.
 - (d) 2 points Find the possible values of X .
 - (e) 2 points Find the probability mass function of X ; i.e., find $\mathbb{P}\{X = n\}$.

ANSWERS

1. (a) There is one ball among $\{B_1, B_2 \dots B_4\}$, we picked B_5 , and there is one ball among $\{B_5 \dots B_{20}\}$.
- (b) $4 \times (20 - 5) / \binom{20}{3}$.
- (c) $8 \times (20 - 9) / \binom{20}{3}$.
- (d) $\{2, 3 \dots 19\}$.
- (e)

$$p_X(n) = \begin{cases} \frac{(n-1)(20-n)}{\binom{20}{3}} & \text{if } n \in \{2, 3 \dots 19\} \\ 0 & \text{else} \end{cases}$$

1. 10 points Let X be geometric with parameter p ; i.e.,

$$p_X(n) = \begin{cases} (1-p)^{n-1}p & \text{if } n \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

Define the function $f(x) \stackrel{\text{def}}{=} \max\{x, 10\}$ for all $x \in \mathbb{R}$. Define $Y \stackrel{\text{def}}{=} f(X)$.

- (a) 2 points Graph f .
- (b) 2 points Compute $\mathbb{P}\{X = 5\}$.
- (c) 2 points Compute $\mathbb{P}\{X = 15\}$.
- (d) 2 points Compute $\mathbb{P}\{X = 10\}$.
- (e) 2 points Compute the probability mass function of Y .

ANSWERS

1. (a) Flat line at height 10, then up with slope 1.

(b) $\mathbb{P}\{Y = 5\} = 0$.

(c) $\mathbb{P}\{Y = 15\} = \mathbb{P}\{X = 15\} = (1 - p)^{14}p$.

(d)

$$\mathbb{P}\{Y = 10\} = \mathbb{P}\{X \leq 10\} = 1 - \mathbb{P}\{X \geq 11\} = 1 - \sum_{n=11}^{\infty} (1 - p)^{n-1}p = 1 - (1 - p)^{10}.$$

(e)

$$p_Y(n) = \begin{cases} 1 - (1 - p)^{10} & \text{if } n = 10 \\ (1 - p)^{n-1}p & \text{if } n \in \{11, 12, \dots\} \\ 0 & \text{else} \end{cases}$$

1. 10 points Let X be a random variable with probability mass function

$$p_X(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{12} & \text{if } n = 2 \\ \frac{2}{12} & \text{if } n = 1 \\ \frac{3}{12} & \text{if } n = 0 \\ \frac{5}{12} & \text{if } n = -1 \\ \frac{1}{12} & \text{if } n = -2 \end{cases}$$

- (a) 2 points Compute $\mathbb{E}[X]$
- (b) 2 points Compute $\mathbb{E}[X^2]$
- (c) 3 points Compute the variance of X .
- (d) 3 points Compute $\mathbb{E}[e^{5X}]$. I want to see that you know what this means; I know that you can't explicitly evaluate exponentials without a calculator.

ANSWERS

1. (a)

$$\mathbb{E}[X] = \frac{2 \times 1 + 1 \times 2 + 0 \times 3 + (-1) \times 5 + (-2) \times 1}{12} = \frac{-3}{12}.$$

(b)

$$\mathbb{E}[X^2] = \frac{2^2 \times 1 + 1^2 \times 2 + 0^2 \times 3 + (-1)^2 \times 5 + (-2)^2 \times 1}{12} = \frac{15}{12}.$$

(c)

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{15}{12} - \left(\frac{-3}{12}\right)^2 = \frac{15 \times 12 - 9}{144} = \frac{180 - 9}{144} = \frac{171}{144}.$$

(d)

$$\begin{aligned} \mathbb{E}[e^{5X}] &= \frac{e^{10} \times 1 + e^5 \times 2 + 1 \times 3 + e^{-5} \times 5 + e^{-10} \times 1}{12} \\ &= \frac{e^{10} + 2e^5 + 3 + 5e^{-5} + e^{-10}}{12}. \end{aligned}$$

1. 10 points Let X be a continuous random variable with density

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

Compute the moment generating function $\Phi(\theta) \stackrel{\text{def}}{=} \mathbb{E}[e^{\theta X}]$ for $\theta < 5$.

ANSWERS

1. For $\theta < 5$,

$$\Phi(\theta) = 5 \int_{t=0}^{\infty} \exp [-(5 - \theta)t] dt = \frac{5}{5 - \theta}$$

1. 10 points Let U be a Uniform(0, 1) continuous random variable; i.e., it has density

$$f_U(t) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Define the transformation

$$g(u) \stackrel{\text{def}}{=} \begin{cases} 1 - u & \text{if } u \leq \frac{2}{3} \\ u & \text{if } u > \frac{2}{3} \end{cases}$$

and define $X \stackrel{\text{def}}{=} g(U)$.

- (a) 5 points Compute F_X , the cumulative distribution function of X .
- (b) 5 points Compute f_X , the density of X .

ANSWERS

1. (a) $F_X(t) = 1$ if $t \geq 1$ and $F_X(t) = 0$ for $t < 0$. For $\frac{1}{3} \leq t < \frac{2}{3}$, we have that

$$F_X(t) = \mathbb{P}\left\{1 - t \leq U \leq \frac{2}{3}\right\} = \frac{2}{3} - (1 - t) = t - \frac{1}{3}.$$

For $\frac{2}{3} \leq t < 1$, we have that

$$F_X(t) = \mathbb{P}\{1 - t \leq U \leq t\} = t - (1 - t) = 2t - 1.$$

Thus

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t - \frac{1}{3} & \text{if } \frac{1}{3} \leq t < \frac{2}{3} \\ 2t - 1 & \text{if } \frac{2}{3} \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

- (b) Differentiate F_X to get that

$$f_X(t) = \begin{cases} 1 & \text{if } \frac{1}{3} < t < \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} < t < 1 \\ 0 & \text{else} \end{cases}$$

1. 10 points X be an $N(0, 1)$ Gaussian; i.e., a continuous random variable with density

$$f_X(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \quad t \in \mathbb{R}$$

Set $Y \stackrel{\text{def}}{=} X + 7$.

- (a) 5 points Compute $\mathbb{P}\{Y \leq 13\}$ in terms of an integral of f_X .
- (b) 5 points Compute the cumulative distribution function of Y in terms of an integral of f_X .

ANSWERS

1. (a)

$$\mathbb{P}\{Y \leq 13\} = \mathbb{P}\{X + 7 \leq 13\} = \mathbb{P}\{X \leq 6\} = \int_{s=-\infty}^6 f_X(s) ds$$

(b) For $t \in \mathbb{R}$

$$F_Y(t) = \mathbb{P}\{X + 7 \leq t\} = \mathbb{P}\{X \leq t - 7\} = \int_{s=-\infty}^{t-7} f_X(s) ds.$$

1. 10 points Evaluate the integral

$$\int_{s=0}^5 0 ds.$$

ANSWERS

1. 0.

Math 461, Section X13 and X14, Fall 2009
Exam 1, September 25

Show all work; partial credit will be given

No calculators allowed

Remember that I have to grade this; don't over-reduce your answers

Remember that I have to grade this; be neat

1. 45 points This is an extension of Wednesday's quiz. Suppose we have an unfair coin with $\mathbb{P}\{H\} = p$ and $\mathbb{P}\{T\} = 1 - p$. We flip the coin a countable sequence of times. We are interested in the statistics of the first and third heads.
 - (a) 5 points Pick a specific configuration where the third heads is on the 9th toss.
 - (b) 5 points What is the probability of the configuration chosen in part a)?
 - (c) 5 points What is the probability that the third heads is on the 9th toss?
 - (d) 5 points For any integer $n \geq 3$, what is the probability that the third heads is on the n -th toss.
 - (e) 5 points Pick a specific configuration where the first heads is on the 4th toss and the third heads is on the 9th toss.
 - (f) 5 points What is the probability of the configuration chosen in part e)?
 - (g) 5 points What is the probability that the first heads is on the 4th toss and the third heads is on the 9th toss.
 - (h) 5 points Conditioned on the event that the third heads is on the 9th toss, what is the probability that the first heads is on the 4th toss?
 - (i) 5 points Conditioned on the event that the third heads is on the 9th toss, what is the probability that the first heads is on the n -th toss, for each $n \in \{1, 2 \dots 7\}$.

2. 20 points Suppose that

$$\mathbb{P}(A) = .3 \quad \mathbb{P}(B) = .6 \quad \mathbb{P}((A \cup B)^c) = .28.$$

- (a) 5 points Compute $\mathbb{P}(A \cup B)$.
 - (b) 5 points Compute $\mathbb{P}(A \cap B)$.
 - (c) 5 points Compute $\mathbb{P}(A \setminus B)$.
 - (d) 5 points Are A and B independent?
3. 21 points Suppose that Jack, Betty, and Rick are in a team of 30 people. Two committees will be formed; committee A contains 7 people and committee B contains of 9 people. Assume that all committees are equally likely. Assume also that the committees can overlap.
 - (a) 7 points Find the probability that Jack and Rick are both on committee A .
 - (b) 7 points Find the probability that Jack and Rick are both on committee A and Jack and Betty are on the same committee.

- (c) 7 points Find the probability that Jack and Betty are both on the same committee given that Jack and Rick are on committee A .
4. 14 points This question is motivated by question 3.19 in chapter 3. A “quit smoking” class is 40% male. A total of 30% of the women who took the class remained nonsmokers for a year. Similarly, a total of 20% of men who took the class remained nonsmokers for a year. A party was held at the end of the year for those who had quit for the year.
- (a) 7 points What percentage of the original class was at the party?
- (b) 7 points What percent of those attending the party were male?

ANSWERS

1. (a) $(HTTTTTHTH)$
- (b) $(1-p)^6 p^3$
- (c) $\binom{8}{2}(1-p)^6 p^3$.
- (d) $\binom{n-1}{2}(1-p)^{n-3} p^3$.
- (e) $(TTTHTHTTH)$
- (f) $(1-p)^6 p^3$.
- (g) $4(1-p)^6 p^3$.
- (h) $\frac{4}{\binom{8}{2}}$.
- (i) $\frac{8-n}{\binom{8}{2}}$.

2. (a) $\mathbb{P}(A \cup B) = 1 - \mathbb{P}((A \cup B)^c) = 1 - .28 = .72$.
- (b) $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = .3 + .6 - .72 = .9 - .72 = .18$.
- (c) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = .3 - .18 = .12$.
- (d) Yes; $\mathbb{P}(A \cap B) = .18 = .3 \times .6 = \mathbb{P}(A)\mathbb{P}(B)$.

3.

$$C = \{\text{Jack and Rick are on committee A}\}$$

$$D = \{\text{Jack and Betty are on the same committee}\}.$$

(a)

$$\mathbb{P}(C) = \frac{\binom{28}{5}}{\binom{30}{7}} = \frac{\binom{28}{5}\binom{30}{9}}{\binom{30}{7}\binom{30}{9}}$$

- (b) The challenge here is to write $C \cap D$ as a disjoint union. Let's partition the space based on whether or not Betty is on committee A. We have $C \cap D = E_1 \cup E_2$ where

$$E_1 = C \cap D \cap \{\text{Betty is on A}\}$$

$$E_2 = C \cap D \cap \{\text{Betty is not on A}\}.$$

Note that

$$C \cap \{\text{Betty is on A}\} \subset D$$

so $E_1 = \{\text{Jack, Rick and Betty are on A}\}$. Secondly, if Betty is not on committee A but she and Jack are on the same committee, then they must both be on B. In other words,

$$E_2 = \{\text{Jack and Rick are on committee A, but Betty is not}$$

$$\text{and Jack and Betty are on committee B}\}.$$

Then

$$\mathbb{P}(E_1) = \frac{\binom{27}{4}}{\binom{30}{7}}$$
$$\mathbb{P}(E_2) = \frac{\binom{27}{5} \binom{28}{7}}{\binom{30}{7} \binom{30}{9}}$$

so

$$\mathbb{P}(C \cap D) = \frac{\binom{27}{4}}{\binom{30}{7}} + \frac{\binom{27}{5} \binom{28}{7}}{\binom{30}{7} \binom{30}{9}}.$$

Alternately, $\mathbb{P}(C \cap D) = \mathbb{P}(C) - \mathbb{P}(C \setminus D)$ and

$C \setminus D = \{\text{Jack and Rick are on committee A and Betty is not on committee A}\}.$

Here

$$\mathbb{P}(C \cap D) = \frac{\binom{28}{5}}{\binom{30}{7}} - \frac{\binom{27}{5}}{\binom{30}{7}}$$

(c)

$$\mathbb{P}(D|C) = \frac{\binom{27}{4} \binom{30}{9} + \binom{27}{5} \binom{28}{7}}{\binom{28}{5} \binom{30}{9}}$$

4.

$M = \{\text{men who were in the class}\}$ $Q = \{\text{quit smoking for a year}\}.$

We have

$$\mathbb{P}(M) = .4 \quad \mathbb{P}(Q|M^c) = .3 \quad \mathbb{P}(Q|M) = .2.$$

We have that

$$\mathbb{P}(Q \cap M) = \mathbb{P}(Q|M)\mathbb{P}(M) = .2 \times .4 = .08$$

$$\mathbb{P}(Q \cap M^c) = \mathbb{P}(Q|M^c)\mathbb{P}(M^c) = .3 \times .6 = .18.$$

(a)

$$\mathbb{P}(Q) = \mathbb{P}(Q \cap M) + \mathbb{P}(Q \cap M^c) = .08 + .18 = .26.$$

(b)

$$\mathbb{P}(M|Q) = \frac{\mathbb{P}(Q \cap M)}{\mathbb{P}(Q)} = \frac{.08}{.26}.$$

Math 461, Section X13 and X14, Fall 2009
Exam 2, October 26

Show all work; partial credit will be given
No calculators allowed

Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. 28 points Let X be a random variable with probability mass function

$$p_X(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{15} & \text{if } n = 3 \\ \frac{2}{15} & \text{if } n = 2 \\ \frac{2}{15} & \text{if } n = 1 \\ \frac{3}{15} & \text{if } n = 0 \\ \frac{5}{15} & \text{if } n = -1 \\ \frac{2}{15} & \text{if } n = -2 \end{cases}$$

Define $Y = X^2$.

- (a) 6 points Compute $\mathbb{E}[X]$
(b) 6 points Compute $\mathbb{E}[X^2]$
(c) 8 points Compute $\mathbb{P}\{Y = 4\}$
(d) 8 points Compute p_Y , the probability mass function of Y .
2. 12 points Assume that we have a coin with bias $\mathbb{P}\{H\} = \frac{2}{10}$ (i.e., the probability of getting a heads is $\frac{2}{10}$). Flip the coin 10 times. Define the random variable

$$X = \begin{cases} 1 & \text{if the first, second, and third tosses come up heads} \\ 0 & \text{else} \end{cases}$$

- (a) 4 points Compute $\mathbb{E}[X]$
(b) 3 points Compute $\mathbb{E}[X^2]$
(c) 2 points Compute $\mathbb{E}[X^3]$
(d) 3 points Compute the variance of X .
3. 19 points Take a coin, and define $p \stackrel{\text{def}}{=} \mathbb{P}\{\text{heads}\}$. Flip the coin until the third heads appears; let X be the position of the third heads.
- (a) 4 points Write down a specific configuration such that $X = 7$.
(b) 4 points Compute the probability of the configuration in part a.
(c) 5 points Compute the probability that $X = 7$.
(d) 6 points Compute the probability mass function for X .

4. 17 points Suppose that $\mathbb{E}[X] = 2$ and the variance of X is 5. Define $Y = 7X + 1$
- (a) 5 points Compute the standard deviation of X .
 - (b) 5 points Compute $\mathbb{E}[X^2]$.
 - (c) 5 points Compute $\mathbb{E}[Y]$.
 - (d) 2 points Compute $\mathbb{E}[Y^2]$.

5. 21 points Let X be a random variable with moment generating function

$$\Phi(\theta) \stackrel{\text{def}}{=} \mathbb{E}[e^{\theta X}] = \begin{cases} \frac{5}{5-\theta} & \text{if } \theta < 5 \\ \infty & \text{else} \end{cases}$$

- (a) 6 points Compute $\mathbb{E}[X]$.
 - (b) 5 points Compute $\mathbb{E}[X^2]$.
 - (c) 5 points Compute $\mathbb{E}[X^3]$.
 - (d) 5 points Compute the variance of X .
6. 3 points Suppose that Timmy Typewriter is typesetting a book. On any given letter, he typically makes an error with probability .00007. We assume that the letters are independent, there are 400 letters on a page and 10 pages in a chapter. Use the Poisson approximation to approximate the probability that there will be at most 3 errors in the first chapter.

ANSWERS

1. (a) $\mathbb{E}[X] = \frac{3 \times 1 + 2 \times 2 + 1 \times 2 + 0 \times 3 + (-1) \times 5 + (-2) \times 2}{15} = 0.$
 (b) $\mathbb{E}[X^2] = \frac{3^2 \times 1 + 2^2 \times 2 + 1^2 \times 2 + 0^2 \times 3 + (-1)^2 \times 5 + (-2)^2 \times 2}{15} = \frac{32}{15}.$
 (c) $\mathbb{P}\{Y = 4\} = \mathbb{P}\{X = 2\} + \mathbb{P}\{X = -2\} = \frac{4}{15}.$
 (d)

$$p_Y(n) = \begin{cases} \frac{1}{15} & \text{if } n = 9 \\ \frac{4}{15} & \text{if } n = 4 \\ \frac{7}{15} & \text{if } n = 1 \\ \frac{3}{15} & \text{if } n = 0 \end{cases}$$

2. Note that $X = X^2 = X^3.$

- (a) $\mathbb{E}[X] = 1 \times \left(\frac{2}{10}\right)^3 + 0 \times \left(1 - \left(\frac{2}{10}\right)^3\right) = \frac{8}{1000}$
 (b) $\mathbb{E}[X^2] = \frac{8}{1000}$
 (c) $\mathbb{E}[X^3] = \frac{8}{1000}$
 (d) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{8}{1000} - \left(\frac{8}{1000}\right)^2 = \frac{8000-64}{10^6} = \frac{7936}{10^6}.$

3. (a) $(TTHHTTH)$

- (b) $p^3(1-p)^4.$
 (c) $\binom{6}{2}p^3(1-p)^4.$
 (d)

$$p_X(n) = \begin{cases} \binom{n-1}{2}p^3(1-p)^{n-3} & \text{if } n \in \{3, 4, \dots\} \\ 0 & \text{else} \end{cases}$$

4. (a) $\sqrt{5}.$

- (b) $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 5 + 4 = 9.$
 (c) $\mathbb{E}[Y] = 7\mathbb{E}[X] + 1 = 7 \times 2 + 1 = 15.$
 (d) $\mathbb{E}[Y^2] = \mathbb{E}[49X^2 + 14X + 1] = 49\mathbb{E}[X^2] + 14\mathbb{E}[X] + 1 = 49 \times 9 + 14 \times 2 + 1 = 470.$

5. For $\theta < 5,$

$$\Phi'(\theta) = \frac{5}{(5-\theta)^2} \quad \Phi^{(2)}(\theta) = \frac{2 \times 5}{(5-\theta)^3} \quad \Phi^{(3)}(\theta) = \frac{3! \times 5}{(5-\theta)^3}$$

- (a) $\mathbb{E}[X] = \Phi'(0) = \frac{1}{5}.$
 (b) $\mathbb{E}[X^2] = \Phi^{(2)}(0) = \frac{2}{25}.$
 (c) $\mathbb{E}[X^3] = \Phi^{(3)}(0) = \frac{6}{125}.$
 (d) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{25} - \frac{1}{25} = \frac{1}{25}.$

6. We assume that the errors are Poisson distributed with parameter $\lambda = .00007 \times 4000 = 2.8$. Thus

$$\mathbb{P}\{X \leq 3\} = \sum_{n=0}^3 e^{-2.8} \frac{(2.8)^n}{n!} = e^{-2.8} \left\{ 1 + 2.8 + \frac{(2.8)^2}{2} + \frac{(2.8)^3}{6} \right\}.$$

Math 461, Section X13 and X14, Fall 2009

Exam 3, November 18

Show all work; partial credit will be given

No calculators allowed

Remember that I have to grade this; don't over-reduce your answers

Remember that I have to grade this; be neat

For the purpose of this exam, we will define

$$\Phi(t) \stackrel{\text{def}}{=} \int_{s=-\infty}^t \frac{\exp\left[-\frac{s^2}{2}\right]}{\sqrt{2\pi}} ds \quad t \in \mathbb{R}$$

1. 15 points Let X be a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{5} & \text{if } 0 \leq t < 1 \\ \frac{1}{2} + \frac{t-1}{5} & \text{if } 1 \leq t < 2 \\ \frac{5}{6} & \text{if } 2 \leq t < 3 \\ 1 & \text{if } t \geq 3 \end{cases}$$

- (a) 5 points Compute $\mathbb{P}\{X = 1\}$.
(b) 5 points Compute $\mathbb{P}\{X = 2\}$
(c) 5 points Compute $\mathbb{P}\{X < 3\}$.

2. 31 points Let X be a continuous random variable with density function

$$f_X(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

- (a) 5 points Compute $\mathbb{P}\{X \geq 7\}$
(b) 5 points Compute $\mathbb{P}\{X \geq 3\}$
(c) 5 points Compute $\mathbb{P}\{X \geq 7 | X \geq 3\}$

Define $Y \stackrel{\text{def}}{=} X^2$

- (d) 5 points Compute $\mathbb{P}\{Y \leq 4\}$
(e) 5 points Compute the cumulative distribution function of Y .
(f) 6 points Is Y continuous? If so, compute its density. If not, state why it is not continuous.

3. 54 points Let X be a Gaussian with mean 1 and variance 4. Let N be a Gaussian with mean 0 and variance 1.

- (a) 5 points Write X in terms of N .
- (b) 5 points Compute $\mathbb{P}\{X \geq 3\}$ using the attached table.
- (c) 6 points Compute $\mathbb{P}\{|X| \geq 3\}$ using the attached table.
- (d) 5 points What is the density of X ?

Define a function

$$\varphi(z) \stackrel{\text{def}}{=} \begin{cases} -1 & \text{if } z \leq 0 \\ 2z & \text{if } 0 < z < 3 \\ 6 & \text{if } z \geq 3 \end{cases}$$

- (e) 5 points Graph φ .

Define $Y \stackrel{\text{def}}{=} \varphi(X)$. As necessary, answer the following questions in terms of Φ , not the table.

- (f) 5 points Compute $\mathbb{P}\{Y \leq -2\}$
- (g) 6 points Compute $\mathbb{P}\{Y \leq 3\}$
- (h) 5 points Compute $\mathbb{P}\{Y \leq 7\}$
- (i) 7 points What is the cumulative distribution function of Y (in terms of Φ as needed).
- (j) 5 points Is Y a continuous random variable? If not, why?

ANSWERS

1. (a)

$$\mathbb{P}\{X = 1\} = F_X(1) - F_X(1-) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

(b)

$$\mathbb{P}\{X = 2\} = F_X(2) - F_X(2-) = \frac{5}{6} - \frac{7}{10} = \frac{4}{30} = \frac{2}{15}.$$

(c) $\mathbb{P}\{X < 3\} = F_X(3-) = \frac{5}{6}.$

2. (a) $\mathbb{P}\{X \geq 7\} = \int_{t=7}^{\infty} f_X(t)dt = \int_{t=7}^{\infty} e^{-t}dt = e^{-7}.$

(b) $\mathbb{P}\{X \geq 3\} = \int_{t=3}^{\infty} f_X(t)dt = \int_{t=3}^{\infty} e^{-t}dt = e^{-3}.$

(c)

$$\mathbb{P}\{X \geq 7|X \geq 3\} = \frac{\mathbb{P}\{X \geq 7, X \geq 3\}}{\mathbb{P}\{X \geq 3\}} = \frac{\mathbb{P}\{X \geq 7\}}{\mathbb{P}\{X \geq 3\}} = \frac{e^{-7}}{e^{-3}} = e^{-4}.$$

(d) $\mathbb{P}\{Y \leq 4\} = \mathbb{P}\{|X| \leq 2\} = \int_{t=0}^2 e^{-t}dt = 1 - e^{-2}.$

(e)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\sqrt{t}} & \text{if } t \geq 0 \end{cases}$$

(f) Y is continuous and

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{e^{-\sqrt{t}}}{2\sqrt{t}} & \text{if } t > 0 \end{cases}$$

3. (a) $X = 2N + 1.$

(b)

$$\begin{aligned} \mathbb{P}\{X \geq 3\} &= \mathbb{P}\{2N + 1 \geq 3\} = \mathbb{P}\{N \geq 1\} \\ &= 1 - \mathbb{P}\{N < 1\} = 1 - \Phi(1) = 1 - .8413 = .1587 \end{aligned}$$

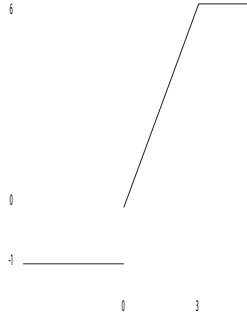
(c)

$$\begin{aligned} \mathbb{P}\{|X| \geq 3\} &= \mathbb{P}\{X \leq -3\} + \mathbb{P}\{X \geq 3\} = \mathbb{P}\{2N + 1 \leq -3\} + \mathbb{P}\{2N + 1 \geq 3\} \\ &= \mathbb{P}\{N \leq -2\} + \mathbb{P}\{N \geq 1\} = \mathbb{P}\{N \geq 2\} + \mathbb{P}\{N \geq 1\} \\ &= 1 - \mathbb{P}\{N \leq 2\} + 1 - \mathbb{P}\{N \leq 1\} = 2 - \Phi(2) - \Phi(1) = 2 - .9772 - .8413 = .1815 \end{aligned}$$

(d)

$$f_X(t) = \frac{\exp\left[-\frac{(t-1)^2}{8}\right]}{\sqrt{8\pi}} \quad t \in \mathbb{R}$$

(e)



(f) $\mathbb{P}\{Y \leq -2\} = \mathbb{P}(\emptyset) = 0$

(g)

$$\mathbb{P}\{Y \leq 3\} = \mathbb{P}\left\{X \leq \frac{3}{2}\right\} = \mathbb{P}\left\{2N + 1 \leq \frac{3}{2}\right\} = \mathbb{P}\left\{N \leq \frac{1}{4}\right\} = \Phi\left(\frac{1}{4}\right)$$

(h) $\mathbb{P}\{Y \leq 7\} = 1.$

(i)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < -1 \\ \Phi\left(-\frac{1}{2}\right) & \text{if } -1 \leq t < 0 \\ \Phi\left(\frac{t/2-1}{2}\right) & \text{if } 0 \leq t < 6 \\ 1 & \text{if } t \geq 6 \end{cases}$$

Note that $\Phi\left(-\frac{1}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right).$

(j) No; it has a jump at -1 and 6 .

Math 461, Section X13 and X14, Fall 2009
Final, December 15

Show all work; partial credit will be given
No calculators allowed

Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
200 points total.

1. 20 points This is question 7 on page 16 of the book.
 - (a) 5 points How many ways can we seat 3 boys and 3 girls in a row?
 - (b) 5 points How many ways can we seat 3 boys and 3 girls in a row if the boys and girls are each to sit together?
 - (c) 5 points How many ways can we seat 3 boys and 3 girls in a row if only the boys must sit together?
 - (d) 5 points How many ways can we seat 3 boys and 3 girls in a row if no two people of the same gender are allowed to sit together?
2. 15 points At a certain college, 54% of the students are female. We also know that 10% of the students are majoring in computer science, and we know that 7% of the students are women majoring in computer science.
 - (a) 10 points Randomly select a computer science student. What is the probability that it is a woman?
 - (b) 5 points In this data, is gender independent of the decision to study computer science?
3. 12 points Let X be a random variable with probability mass function

$$p(j) = \begin{cases} \frac{2}{9} & \text{if } j = -1 \\ \frac{1}{9} & \text{if } j = 0 \\ \frac{2}{9} & \text{if } j = 3 \\ \frac{4}{9} & \text{if } j = 7 \\ 0 & \text{else} \end{cases}$$

- (a) 6 points Compute $\mathbb{E}[X]$.
 - (b) 6 points Compute $\mathbb{E}[\min\{X, 1\}]$.
4. 40 points We will here develop some ideas relating to the *negative hypergeometric* distribution. This is similar in some ways to a negative binomial, and you might keep those calculations in mind.

Suppose that we have a box containing 20 black balls $\{B_1, B_2 \dots B_{20}\}$ and 30 red balls $\{R_1, R_2 \dots R_{30}\}$. Let's pick balls out of the box, one by one.

Let X_7 be the first time that we have 7 black balls.

- (a) 10 points Write a specific configuration where $X_7 = 13$
- (b) 10 points Compute the probability of this configuration.
- (c) 10 points Compute the probability that $\mathbb{P}\{X_7 = 13\}$.
- (d) 10 points What is the probability mass function of X_7 ?
5. 66 points Let U be a continuous random variable which is uniformly distributed on $(0, 1)$; i.e., it has density

$$f_U(t) = \begin{cases} 1 & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases}$$

Define $\varphi_1(u) \stackrel{\text{def}}{=} u^3$ for $u \in (0, 1)$, and define $X_1 \stackrel{\text{def}}{=} \varphi_1(U)$.

- (a) 5 points Graph φ_1
- (b) 7 points Compute $\mathbb{P}\{X_1 \leq \frac{2}{3}\}$.
- (c) 5 points Compute the cumulative of X_1 .
- (d) 4 points Does X_1 have a density? If so, compute it. If not, explain why.

Define $\varphi_2(u) \stackrel{\text{def}}{=} 1/u^3$ for $u \in (0, 1)$, and define $X_2 \stackrel{\text{def}}{=} \varphi_2(U)$.

- (e) 5 points Graph φ_2
- (f) 7 points Compute $\mathbb{P}\{X_2 \leq 5\}$.
- (g) 5 points Compute the cumulative of X_2 .
- (h) 4 points Does X_2 have a density? If so, compute it. If not, explain why.

Define $\varphi_3(u) \stackrel{\text{def}}{=} -\frac{1}{3} \ln u$ for $u \in (0, 1)$, and define $X_3 \stackrel{\text{def}}{=} \varphi_3(U)$.

- (i) 5 points Graph φ_3
- (j) 7 points Compute $\mathbb{P}\{X_3 \leq 7\}$.
- (k) 5 points Compute the cumulative of X_3 .
- (l) 4 points Does X_3 have a density? If so, compute it. If not, explain why.
- (m) 3 points X_3 is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poisson, or uniform. Which one?
6. 47 points Fix two parameters $p \in (0, 1)$ and $\lambda > 0$. Consider two discrete random variables X and Y with joint probability mass function

$$p_{X,Y}(i, j) = \begin{cases} \binom{j}{i} p^i (1-p)^{j-i} e^{-\lambda} \frac{\lambda^j}{j!} & \text{if } j \in \{0, 1, \dots\} \text{ and } i \in \{0, 1, \dots, j\} \\ 0 & \text{else} \end{cases}$$

(as usual, we define $\binom{0}{0} \stackrel{\text{def}}{=} 1$). In other words,

$$\mathbb{P}\{X = 2, Y = 7\} = \binom{7}{2} p^2 (1-p)^5 e^{-\lambda} \frac{\lambda^7}{7!}.$$

This is some combination of binomial and Poisson distributions, of course.

- (a) 10 points Compute $p_X(5) \stackrel{\text{def}}{=} \mathbb{P}\{X = 5\}$. Hint: recall that $\binom{j}{i} = \frac{j!}{i!(j-i)!}$. Recall also the Taylor formula for the exponential.
- (b) 5 points Compute the probability mass function of X .
- (c) 3 points X is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poisson, or uniform. Which one?
- (d) 10 points Compute $p_Y(7) \stackrel{\text{def}}{=} \mathbb{P}\{Y = 7\}$ Hint: recall the binomial theorem.
- (e) 5 points Compute the probability mass function of Y .
- (f) 3 points Y is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poisson, or uniform. Which one?
- (g) 4 points Are X and Y independent?
- (h) 7 points Compute $\mathbb{P}\{X = 5|Y = 7\}$.

ANSWERS

1. (a) $6!$
 (b) $2 \times 3! \times 3!$ (the 2 comes from which gender is left-most)
 (c) $4! \times 3!$ (group the boys as one unit, and then order within the unit)
 (d) $2 \times 3! \times 3!$ (the two comes from which gender is left-most).

2.

$$W = \{\text{woman}\} \quad \text{and} \quad C = \{\text{computer science}\}.$$

(a)

$$\mathbb{P}(W|C) = \frac{\mathbb{P}(W \cap C)}{\mathbb{P}(C)} = \frac{7}{10} = .7.$$

(b) No; $\mathbb{P}(W|C) = .7 \neq .54 = \mathbb{P}(W)$.

3. (a)

$$\mathbb{E}[X] = \frac{(-1) \times 2 + 0 \times 1 + 3 \times 2 + 7 \times 4}{9} = \frac{32}{9}.$$

(b)

$$\begin{aligned} \mathbb{E}[\min\{X, 1\}] &= \sum_j \min\{j, 1\}p(j) \\ &= \min\{-1, 1\}\frac{2}{9} + \min\{0, 1\}\frac{1}{9} + \min\{3, 1\}\frac{2}{9} + \min\{7, 1\}\frac{4}{9} \\ &= (-1)\frac{2}{9} + (0)\frac{1}{9} + (1)\frac{2}{9} + (1)\frac{4}{9} = \frac{-2 + 0 + 2 + 4}{9} = \frac{4}{9}. \end{aligned}$$

4. (a) $(B_4, B_2, R_9, R_{11}, R_3, B_3, R_7, B_{20}, B_{15}, R_{30}, B_8, R_1, B_1)$

(b)

$$\frac{1}{(50)_{13}}.$$

(c)

$$\binom{12}{6} \frac{(20)_7(30)_6}{(50)_{13}}.$$

(d)

$$p_{X_7}(j) = \begin{cases} \frac{(20)_7(30)_{j-7}}{(50)_j} \binom{j-1}{6} & \text{if } j \in \{7, 8 \dots 37\} \\ 0 & \text{else} \end{cases}$$

5. (a) increasing

(b)

$$\mathbb{P}\left\{X_1 \leq \frac{2}{3}\right\} = \mathbb{P}\left\{U \leq \left(\frac{2}{3}\right)^{1/3}\right\} = \left(\frac{2}{3}\right)^{1/3}.$$

(c)

$$F_{X_1}(t) = \mathbb{P}\{U^3 \leq t\} = \mathbb{P}\{U \leq t^{1/3}\} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

(d) X_1 is continuous and has density

$$f_{X_1}(t) = \begin{cases} \frac{1}{3}t^{-2/3} & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases}$$

(e) decreasing

(f)

$$\mathbb{P}\{X_2 \leq 5\} = \mathbb{P}\left\{\frac{1}{U^3} \leq 5\right\} = \mathbb{P}\left\{U \geq \frac{1}{5^{1/3}}\right\} = 1 - \frac{1}{5^{1/3}}.$$

(g) Since U takes values in $(0, 1)$, X_2 takes values in $(1, \infty)$. Thus $F_{X_2}(t) = 0$ for $t \leq 1$. For $t > 1$, we have that $F_{X_2}(t) = \mathbb{P}\left\{\frac{1}{U^3} \leq t\right\} = \mathbb{P}\{U \geq t^{-1/3}\} = 1 - t^{-1/3}$. Thus

$$F_{X_2}(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - t^{-1/3} & \text{if } t \geq 1 \end{cases}$$

(h) X_2 is continuous and has density

$$f_{X_2}(t) = \begin{cases} \frac{1}{3}t^{-4/3} & \text{if } t > 1 \\ 0 & \text{else} \end{cases}$$

(i) decreasing

(j)

$$\mathbb{P}\{X_3 \leq 7\} = \mathbb{P}\{\ln U \geq -35\} = \mathbb{P}\{U \geq e^{-35}\} = 1 - e^{-35}.$$

(k) Since U takes values in $(0, 1)$, X_3 takes values in $(0, \infty)$. Thus $F_{X_3}(t) = 0$ for $t \leq 0$. For $t > 0$, we have that $F_{X_3}(t) = \mathbb{P}\{\ln U \geq -5t\} = \mathbb{P}\{U \geq 1 - e^{-5t}\} = 1 - e^{-5t}$. Thus

$$F_{X_3}(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-5t} & \text{if } t \geq 0 \end{cases}$$

(l) X_3 is continuous and has density

$$f_{X_3}(t) = \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

6. (a)

$$\begin{aligned}\mathbb{P}\{X = 5\} &= \sum_j \mathbb{P}\{X = 5, Y = j\} = \sum_{j=5}^{\infty} \binom{j}{5} p^j (1-p)^{5-j} e^{-\lambda} \frac{\lambda^j}{j!} \\ &= \frac{e^{-\lambda} p^5}{5!} \sum_{j=5}^{\infty} \frac{(1-p)^{5-j} \lambda^j}{(5-j)!} = \frac{e^{-\lambda} (\lambda p)^5}{5!} \sum_{j=5}^{\infty} \frac{(\lambda(1-p))^{5-j}}{(5-j)!} \\ &= \frac{e^{-\lambda} (\lambda p)^5}{5!} \sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!} = \frac{\exp[-\lambda + \lambda(1-p)] (\lambda p)^5}{5!} \\ &= \frac{\exp[-\lambda p] (\lambda p)^5}{5!}.\end{aligned}$$

(b)

$$p_X(i) = \begin{cases} e^{-\lambda p} \frac{(\lambda p)^i}{i!} & \text{if } i \in \{0, 1, \dots\} \\ 0 & \text{else} \end{cases}$$

(c) X is Poisson.

(d)

$$\mathbb{P}\{Y = 7\} = \sum_i \mathbb{P}\{X = i, Y = 7\} = \sum_{i=0}^7 \binom{7}{i} p^i (1-p)^{7-i} e^{-\lambda} \frac{\lambda^7}{7!} = e^{-\lambda} \frac{\lambda^7}{7!}.$$

(e)

$$p_Y(j) = \begin{cases} e^{-\lambda} \frac{\lambda^j}{j!} & \text{if } j \in \{0, 1, \dots\} \\ 0 & \text{else} \end{cases}$$

(f) Y is Poisson

(g) They are not independent.

(h)

$$\begin{aligned}\mathbb{P}\{X = 5 | Y = 7\} &= \frac{\mathbb{P}\{X = 5, Y = 7\}}{\mathbb{P}\{Y = 7\}} = \frac{p_{X,Y}(5, 7)}{p_Y(7)} \\ &= \frac{\binom{7}{5} p^5 (1-p)^{7-5} e^{-\lambda} \frac{\lambda^7}{7!}}{e^{-\lambda} \frac{\lambda^7}{7!}} = \binom{7}{5} p^5 (1-p)^2.\end{aligned}$$