I. Syllabus

Math 461 Introduction to Probability Theory I
Summer 2005

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Textbook:
Sheldon Ross, A First Course in Probability, Sixth Edition, 2002, Prentice Hall.

We will cover most of chapters 1 to 7 and the topics include combinatorial analysis, axioms of probability, conditional probability and independence, random variables and expectation, as well as central limit theorems.

The grades will be allocated as follows:

Homework	20%
Quizzes	10%
Midterms	40%
Final	30%

Administration:

- There will be about 1 or 2 Homework per week.
- There will be about 5 quizzes at the end of some chapters (no chance to make up).
- The first midterm will cover chapter 1 to 3. The second midterm will cover chapter 4 and 5.
- The final will be comprehensive; though will concentrate on chapter 6 and 7.

HW #1 Page 15-18, Problems: 2-5, 7-9,11-12, 16-18, 24, 26-28, 31

Chap 1.

2. $6 \times 6 \times 6 \times 6 = 1296$
There are 4 steps to finish the process. There are 6 outcomes in each step.

3. $20! = 20 \times 19 \times \cdots \times 1$

4. If each of the boys can play any instrument, then there are $4! = 24$ possible arrangements. If Jay and Jack can only play piano and drums, then there are two ways to assign instruments to Jay and Jack, which leaves two possible assignments for John and Jim, so that there are $2 \times 2 = 4$ possibilities in this case.

5. Number of possible area codes: $8 \times 2 \times 9 = 144$ (Generalized counting principle: There are eight choices for the first digit, two for the second digit, and nine for the third digit.) Number of area codes starting with a 4: $1 \times 2 \times 9 = 18$

7. (a) There are six children altogether, so that there are $6! = 720$ ways to arrange them in a row.
(b) There are $3! = 6$ ways to arrange the girls (resp. boys) in a row, and there are two ways to arrange the two blocks of boys/girls (i.e., girls first or boys first), so that there are $3! \times 3! \times 2 = 72$ arrangements altogether.
(c) There are $4! = 24$ ways to arrange the girls and the block of boys in a row, and there are $3! = 6$ ways to arrange the boys within their block, so that there are $4! \times 3! = 144$ arrangements altogether.
(d) If no two people of the same sex are allowed to sit together, then either the girls are in the even seats, and the boys are in the odd seats, or vice versa. In total, there are $3! \times 3! \times 2 = 72$ possibilities.

8. (a) $5! = 120$ (five distinct letters)
(b) $\frac{7!}{2! \times 2!} = 1260$ (seven letters, including two Ps and two Os)
(c) $\frac{11!}{4! \times 4! \times 2!} = 34{,}650$ (eleven letters, including four Is, four Ss, and two Ps)
(d) $\frac{7!}{2! \times 2!} = 1260$ (seven letters, two As, two Rs)

9. There are $\frac{12!}{6! \times 4!} = 27{,}720$ possibilities.

11. (a) There are six books altogether, so that there are $6! = 720$ ways to arrange them.
(b) The number of possible arrangements is $3! \times 2! \times 3! = 72$.
(c) There are $3! \times 4! = 144$ possible arrangements.

12. (a) Each award can go to any one of 30 students, so that there are $30^5 = 24{,}300{,}000$ possibilities if a student can receive any number of awards.
(b) The first award can go to any one of 30 students, for the second award there are

29 choices left, etc. In total, there are $30 \times 29 \times 28 \times 27 \times 26 = 17{,}100{,}720$ possibilities.

16. (a) $\binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42$

(b) $6\times7 + 6\times4 + 4\times7 = 94$

17. $\binom{10}{7}\times 7! = 604{,}800$

18. The number of possibilities is $\binom{5}{2}\times\binom{6}{2}\times\binom{4}{3} = 600$

24. Binomial theorem:

$$(3x^2 + y)^5 = \sum_{k=0}^{k=5}\binom{5}{k}(3x^2)^k y^{5-k}$$
$$= y^5 + 15x^2y^4 + 90x^4y^3 + 270x^6y^2 + 405x^8y + 243x^{10}$$

26. Multinomial theorem:

$$(x_1 + 2x_2 + 3x_3)^4 = \sum_{\substack{n_1, n_2, n_3 = 0 \\ n_1 + n_2 + n_3 = 4}}^{4}\binom{4}{n_1, n_2, n_3}(x_1)^{n_1}(2x_2)^{n_2}(3x_3)^{n_3}$$
$$= 81x_3^4 + 216\,x_2x_3^3 + 216\,x_2^2x_3^2 + 96\,x_2^3x_3 + 16\,x_2^4 + 108\,x_1x_3^3 + 216\,x_1x_2x_3^2 + 144\,x_1x_2^2x_3 + 32\,x_1x_2^3 + 54x_1^2x_3^2 + 72x_1^2x_2x_3 + 24x_1^2x_2^2 + 12x_1^3x_3 + 8x_1^3x_2 + x_1^4$$

27. $\binom{12}{3,4,5} = \dfrac{12!}{3!4!5!} = 27{,}720$

28. Assuming teachers are distinct.

(a) $4^8 = 65{,}536$

(b) $\binom{8}{2,2,2,2} = \dfrac{8!}{2^4} = 2520$

31. (a) number of nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 8$. Hence, answer is $\binom{11}{3} = 165$

(b) here it is the number of positive solutions----hence answer is $\binom{7}{3}=35$

HW #2 Page 53-54, Problems: 1,3,5,6,7,8,9,10,11

HW #2 Solutions: (Chapter 2)

1. (a) $S = \{(r,r),(r,g),(r,b),(g,r),(g,g),(g,b),(b,r),(b,g),(b,b)\}$

   (b) $S = \{(r,g),(r,b),(g,r),(g,b),(b,r),(b,g)\}$

3. $EF = \{(1,2),(1,4),(1,6),(2,1),(4,1),(6,1)\}$.

   $E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1.

   $FG = \{(1,4),(4,1)\}$. EF^c is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

5. (a) $2^5 = 32$

   (b) W = {(1,1,1,1,1), (1,1,1,1,0), (1,1,1,0,1), (1,1,0,1,1), (1,1,1,0,0), (1,1,0,1,0)
   (1,1,0,0,1), (1,1,0,0,0), (1,0,1,1,1), (0,1,1,1,1), (1,0,1,1,0), (0,1,1,1,0)
   (0,0,1,1,1), (0,0,1,1,0), (1,0,1,0,1)}

   (c) 8

   (d) AW = {(1,1,1,0,0), (1,1,0,0,0)}

6. (a) S = {(1,g), (0,g), (1,f), (0,f), (1,s), (0,s)}

   (b) A = {(1,s), (0,s)}

   (c) B = {(0,g), (0,f), (0,s)}

   (d) {(1,s), (0,s), (1,g), (1,f)}

7. (a) 6^{15}

   (b) $6^{15}-3^{15}$

   (c) 4^{15}

8. (a) 0.8

   (b) 0.3

   (c) 0

9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

   $P(A \cup V) = P(A) + P(V) - P(AV) = 0.24 + 0.61 - 0.11 = 0.74$

   Therefore, 74 percent of the establishment's customers carry at least one of the two Types of credit cards that it accepts.

10. Let R and N denote the events, respectively, that the student wears a ring and wears a necklace.

    (a) $P(R \cup N) = 1 - 0.6 = 0.4$

    (b) $0.4 = P(R \cup N) = P(R) + P(N) - P(RN) = 0.2 + 0.3 - P(RN)$

    Thus, $P(RN) = 0.1$

11. Let A be the event that a randomly chosen person is a cigarette smoker and let B be The event that she or he is a cigar smoker.

    (a) $1 - P(A \cup B) = 1 - (0.07 + 0.28 - 0.05) = 0.7$. Hence, 70% smoke neither.

(b) $P(A^C B) = P(B) - P(AB) = 0.07 - 0.05 = 0.02$. Hence, 2% smoke cigars but not cigarettes.

HW #3 Page 55-58, Problems:21,27,28,29,32,36,37,43,50,53,54

HW #3 Solution:

21. (a) $p_1 = 4/20 = 1/5$, $p_2 = 8/20 = 2/5$, $p_3 = 5/20 = 1/4$, $p_4 = 2/20 = 1/10$, $p_5 = 1/20$
    (b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,
    $q_1 = 4/48 = 1/12$, $q_2 = 16/48 = 1/3$, $q_3 = 15/48 = 5/16$, $q_4 = 8/48 = 1/6$, $q_5 = 5/48$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3\cdot 9! + 7\cdot 6\cdot 3\cdot 7! + 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 5! + 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 3\cdot 3!}{10!} = 0.583$$

28. $$P\{same\} = \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}} = 0.089$$

$$P\{different\} = \frac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}} = 0.248$$

If sampling is with replacement

$$P\{same\} = \frac{5^3+6^3+8^3}{(19)^3} = 0.1243$$

$$P\{different\} = P(RBG) + P(BRG) + P(RGB) + \cdots + P(GBR) = \frac{6\cdot 5\cdot 6\cdot 8}{(19)^3} = 0.2099$$

29. (a) $\dfrac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$

(b) $\dfrac{n^2+m^2}{(n+m)^2}$

(c) Putting all terms over the common denominator $(n+m)^2(n+m-1)$ shows that
we must prove that
$$n^2(n+m-1)+m^2(n+m-1) \geq n(n-1)(n+m)+m(m-1)(n+m)$$
which is immediate upon multiplying through and simplifying.

32. $$\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$$

36. (a) $\dfrac{\binom{4}{2}}{\binom{52}{2}} \approx 0.0045$

(b) $\dfrac{13\binom{4}{2}}{\binom{52}{2}} = 1/17 \approx 0.0588$

37. (a) $\dfrac{\binom{7}{5}}{\binom{10}{5}} = 1/12 \approx 0.0833$

(b) $\dfrac{\binom{7}{4}\binom{3}{1}}{\binom{10}{5}} + 1/12 = ½$

43. $\dfrac{2(n-1)(n-2)}{n!} = \dfrac{2}{n}$ in a line

$\dfrac{2n(n-2)!}{n!} = \dfrac{2}{n-1}$ if in a circle, $n \geq 2$

50. $\dfrac{\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}}{\binom{52}{13}\binom{39}{13}} = 2.61\times10^{-6}$

53.Let A_i be the event that couple i sit next to each other. Then

$$P\left(\cup_{i=1}^{4} A_i\right) = 4\frac{2\cdot7!}{8!} - 6\frac{2^2\cdot6!}{8!} + 4\frac{2^3\cdot5!}{8!} - \frac{2^4\cdot4!}{8!}$$

and the desired probability is 1 minus the preceding. The result is 0.343.

54. $P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(C) - P(SH) - \cdots - P(SHDC)$

$$= \frac{4\binom{39}{13}}{\binom{52}{13}} - \frac{6\binom{26}{13}}{\binom{52}{13}} + \frac{4\binom{13}{13}}{\binom{52}{13}} = \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}$$

= 0.0511

HW #4 Page 104-115, Problems:1,5,6,9,10,18,20,40,43,46,47

1. P{6 | different} = P{6, different} / P{different}

$$= \frac{P\{1st = 6, 2nd \neq 6\} + P\{1st \neq 6, 2nd = 6\}}{5/6} = 1/3$$

5. $\frac{6}{15}\frac{5}{14}\frac{9}{13}\frac{8}{12} = \frac{6}{91}$ =0.066

6. In both cases the one black ball is equally likely to be in either of the 4 positions. Hence the answer is ½.

9.

$$P\{A = w \mid 2w\} = \frac{P\{A = w, 2w\}}{P\{2w\}} = \frac{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\}}{P\{2w\}}$$

$$= \frac{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4}}{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{2}{3}\frac{1}{4}} = 7/11$$

10. 11/50

18. (a) (.9)(.8)(.7) = .504

(b) Let F_i denote the event that she failed the ith exam.

$$P(F_2 \mid (F_1^c F_2^c F_3^c)^c) = \frac{P(F_1^c F_2)}{1 - 0.504} = \frac{(.9)(.2)}{.496} = .3629$$

20. (a) $\frac{5}{12}\frac{7}{14}\frac{7}{16}\frac{9}{18} = \frac{35}{768}$

(b) $\frac{35}{128}$

40. The jailer's reasoning is wrong. The probability of one prisoner being executed is 1/3 whether he knows any information or not. The two event is independent.

43.

$$P\{\text{All white}\} = \frac{1}{6}\left[\frac{5}{15} + \frac{5}{15}\frac{4}{14} + \frac{5}{15}\frac{4}{14}\frac{3}{13} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12} + \frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right] = 0.07576$$

$$P\{3 \mid \text{All white}\} = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P\{\text{All white}\}} = 0.048$$

46. Choose a person at random

$$P\{\text{they have accident}\} = P\{\text{acc.} \mid \text{good}\}P\{\text{g}\} + P\{\text{acc.} \mid \text{ave.}\}P\{\text{ave.}\} + P\{\text{acc.} \mid \text{bad}\}\ P\{\text{b}\}$$
$$= (.05)(.2) + (.15)(.5) + (.30)(.3) = 0.175$$
$$P\{A \text{ is good} \mid \text{no accident}\} = \frac{.95(.2)}{.825} = 0.23$$
$$P\{A \text{ is average} \mid \text{no accident}\} = \frac{(.85)(.5)}{.825} = 0.515$$

47. Let R be the event that she receives a job offer.

(a)
$$P(R) = P(R \mid \text{strong})P(\text{strong}) + P(R \mid \text{moderate})P(\text{moderate}\} + P(R \mid \text{weak}\}P(\text{weak})$$
$$= (.8)(.7) + (.4)(.2) + (.1)(.1) = 0.65$$

(b)
$$P(\text{strong} \mid R\} = \frac{P(R \mid \text{strong})P(\text{strong}\}}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}$$

Similarly,
$P(\text{moderate} \mid R\} = 8/65$, $P(\text{weak} \mid R) = 1/65$

(c)
$$P(\text{strong} \mid R^c) = \frac{P(R \mid \text{strong})P(\text{strong})}{P(R^c)} = \frac{(.2)(.7)}{.35} = \frac{14}{35}$$

Similarly,
$P(\text{moderate} \mid R^c) = 12/35$, $P(\text{weak} \mid R^c) = 9/35.$

HW #5 Page 104-115, Problems:53,55,58,59,62,63

53. (a) $2p(1-p)$

(b) $\binom{3}{2} p^2(1-p)$

(c) P\{up on first | up 1 after 3\}
= P\{up first, up 1 after 3\}/ $[3p^2(1-p)]$
= $p2p(1-p) / [3p^2(1-p)] = 2/3$

55. (a) 1/16

(b) 1/16

(c) The only way in which the pattern *H, H, H, H* can occur first is for the first 4 flips to all be heads, for once a tail appears it follows that a tail will precede the first run of 4 heads (and so *T, H, H, H* will appear first). Hence, the probability that *T, H, H, H* occurs first is 15/16.

58. (a) P\{both hit | at least one hit\} = $\dfrac{P\{bothhit\}}{P\{atleastonehit\}} = p_1p_2/(1-q_1q_2)$

(b) P\{Barb hit | at least one hit\} = $p_1/(1-q_1q_2)$. $Q_i = 1 - p_i$, and we have assumed that the outcomes of the shots are independent.

59. Consider the final round of the duel. Let $q_x = 1 - p_x$

(a) P\{A not hit\} = P\{A not hit | at least one is hit\}
= P\{A not hit, B hit\} / P\{at least one is hit\}
= $q_Bp_A / (1 - q_Aq_B)$

(b) P\{both hit\} = P\{both hit | at least one is hit\}
= P\{both hit\} /P\{at least one hit\}
= $p_Ap_B / (1 - q_Aq_B)$

(c) $(q_Aq_B)^{n-1}(1 - q_Aq_B)$

(d) P\{n rounds | A unhit\} = P\{n rounds, A unhit\}/P\{A unhit\}

$$= \frac{(q_Aq_B)^{n-1}p_Aq_B}{q_Bp_A/(1-q_Aq_B)} = (q_Aq_B)^{n-1}(1-q_Aq_B)$$

(e) P\{n rounds | both hit\} = P\{n rounds both hit\}/P\{both hit\}

$$= \frac{(q_Aq_B)^{n-1}p_Ap_B}{p_Bp_A/(1-q_Aq_B)} = (q_Aq_B)^{n-1}(1-q_Aq_B)$$

Note that (c), (d), and (e) all have the same answer.

62. (a) $[I - (1 - P_1P_2)(1 - P_3P_4)]P_5 = (P_1P_2 + P_3P_4 - P_1P_2P_3P_4)P_5$

(b) Let E_1 = \{1 and 4 close\}, E_2 = \{1, 3, 5 all close\}
E_3=\{2, 5 close\}, E_4=\{2, 3, 4 close\}.
The desired probability is

$P(E_1 \cup E_2 \cup E_3 \cup E_4)$

$= P(E_1) + P(E_2) + P(E_3) + P(E_4) - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) - P(E_2E_3)$
$- P(E_2E_4) - P(E_3E_4) + P(E_1E_3E_4) + P(E_2E_3E_4) + P(E_1E_2E_4) + P(E_1E_2E_3)$

$- \quad P(E_1E_2E_3E_4)$

$= P_1P_4 + P_1P_3P_5 + P_2P_5 + P_2P_3P_4 - P_1P_3P_4P_5 - P_1P_2P_4P_5 - P_1P_2P_3P_4$
$- P_1P_2P_3P_5 - P_2P_3P_4P_5 + 2P_1P_2P_3P_4P_5$

63. (a) $P_1P_2(1-P_3)(1-P_4) + P_1(1-P_2)P_3(1-P_4) + P_1(1-P_2)(1-P_3)P_4$
$+ (1-P_1)P_2P_3(1-P_4) + (1-P_1)P_2(1-P_3)P_4 + (1-P_1)(1-P_2)P_3P_4$
$P_1P_2P_3(1-P_4) + P_1P_2(1-P_3)P_4 + P_1(1-P_2)P_3P_4 + (1-P_1)P_2P_3P_4 + P_1P_2P_3P_4$

(c) $\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

HW #6 Page 171-180, Problems:1,4,5,13,14,21,22,23

1.

$$P\{X=4\}=\frac{\binom{4}{2}}{\binom{14}{2}}=\frac{6}{91} \qquad P\{X=0\}=\frac{\binom{2}{2}}{\binom{14}{2}}=\frac{1}{91} \qquad P\{X=2\}=\frac{\binom{4}{2}\binom{2}{1}}{\binom{14}{2}}=\frac{8}{91}$$

$$P\{X=-1\}=\frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}}=\frac{16}{91} \qquad P\{X=1\}=\frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}}=\frac{32}{91} \qquad P\{X=-2\}=\frac{\binom{8}{2}}{\binom{14}{2}}=\frac{4}{13}$$

4.

$$P\{X=1\}=1/2, \qquad P\{X=2\}=\frac{5}{10}\frac{5}{9}=\frac{5}{18}, \qquad P\{X=3\}=\frac{5}{10}\frac{4}{9}\frac{5}{8}=\frac{5}{36},$$

$$P\{X=4\}=\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{5}{7}=\frac{5}{84}, \quad P\{X=5\}=\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{5}{6}=\frac{5}{252}, \quad P\{X=6\}=\frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}=\frac{1}{252}$$

5. $n-2i, \quad i=0,1,\ldots,n$

13.

$$p(0)=P\{\textit{no sale on first}\text{ and }\textit{no sale on}\text{ second}\}=(0.7)(0.4)=0.28$$

$$p(500)=P\{1\,\textit{sale}\text{ and it is for standard}\}=P\{1\,\textit{sale}\}/2$$

$$=\left[\text{P}\{\text{sale, no sale}\}+P\{\textit{no sale, sale}\}\right]/2=\left[(0.3)(0.4)+(0.7)(0.6)\right]/2=0.27$$

$$p(1000)=P\{2\text{ standard sales}\}+P\{1\text{ sale for deluxe}\}=(0.3)(0.6)(1/4)+P\{1\text{ sale}\}/2$$

$$=0.045+0.27=0.315$$

$$p(1500)=P\{2\text{ sales, one deluxe and one standard}\}=(0.3)(0.6)(1/2)=0.09$$

$$p(2000)=P\{2\text{ sales, both deluxe}\}=(0.3)(0.6)(1/4)=0.045$$

14.

$$P\{X=0\}=P\{1\text{ loses to }2\}=1/2$$

$$P\{X=1\}=P\{\text{of 1,2,3: 3 has largest, then 1, then 2}\}=(1/3)(1/2)=1/6$$

$$P\{X=2\}=P\{\text{of 1,2,3,4: 4 has largest and 1 has next largest}\}=(1/4)(1/3)=1/12$$

$$P\{X=3\}=P\{\text{of 1,2,3,4,5: 5 has largest then 1}\}=(1/5)(1/4)=1/20$$

$$P\{X=4\}=P\{1\text{ has largest}\}=1/5$$

21. (a) E[X] since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

$P\{X = i\} = i/148,\ i = 40, 33, 25, 50$

(b) $E[X] = \left[(40)^2 + (33)^2 + (25)^2 + (50)^2\right]/148 \approx 39.28$

$E[Y] = (40 + 33 + 25 + 50)/4 = 37$

22. Let N denote the number of games played.

(a) $E[N] = 2\left[p^2 + (1-p)^2\right] + 3\left[2p(1-p)\right] = 2 + 2p(1-p)$

The final equality could also have been obtained by using that N=2+I, where I is 0 if two games are played and 1 if three are played. Differentiation yields that $\frac{d}{dp}E[N] = 2 - 4p$ and so the minimum occurs when $2 - 4p = 0$ or p=1/2.

(b)

$$E[N] = 3\left[p^3 + (1-p)^3\right] + 4\left[3p^2(1-p)p + 3p(1-p)^2(1-p)\right] + 5\left[6p^2(1-p)^2\right]$$

$$= 6p^4 - 12p^3 + 3p^2 + 3p + 3$$

Differentiation yields $\frac{d}{dp}E[N] = 24p^3 - 36p^2 + 6p + 3$

Its value at p=1/2 is easily seen to be 0.

23. (a) Use all your money to buy 500 ounces of the commodity and then sell after one week. The expected amount of money you will get is

$$E[money] = \frac{1}{2}500 + \frac{1}{2}2000 = 1250$$

(b) Do not immediately buy but use your money to buy after one week. Then

$$E[\text{ounces of commodity}] = \frac{1}{2}1000 + \frac{1}{2}250 = 625$$

HW #7 Page 174-180, Problems: 27,28,32,33,35,36,45,57,59,61

27. $C - Ap = \frac{A}{10} \rightarrow C = A\left(p + \frac{1}{10}\right)$

28. $3 \cdot \frac{4}{20} = 3/5$

32. If T is the number of tests needed for a group of 10 people, then

$$E[T] = (0.9)^{10} + 11\left[1 - (0.9)^{10}\right] = 11 - 10(0.9)^{10} = 7.513$$

33. Look at proposition 6.1 on page 145

K ≤ (n+1)*p*, K is an integer, K=Max(x)

35.If X is the amount that you win, then

$$P\{X = 1.10\} = 4/9 = 1 - P\{X = -1\}$$

$$E[X] = (1.1)4/9 - 5/9 = -0.6/9 \approx -0.067$$

$$Var(X) = (1.1)^2(4/9) + 5/9 - (0.6/9)^2 \approx 1.089$$

36. Using the representation N=2 + I

Where I is 0 if the first two games are won by the same team and 1 otherwise, we Have that $Var(N) = Var(I) = E[I]^2 - E^2[I]$

Now, $E[I]^2 = E[I] = P\{I = 1\} = 2p(1-p)$ and so

$$Var(N) = 2p(1-p)[1 - 2p(1-p)] = 8p^3 - 4p^4 - 6p^2 + 2p$$

Differentiation yields

$$\frac{d}{dp}Var(N) = 24p^2 - 16p^3 - 12p + 2$$

and it is easy to verify that this is equal to 0 when p=1/2

45. with 3:

$$P\{pass\} = \frac{1}{3}\left[\binom{3}{2}(0.8)^2(0.2) + (0.8)^3\right] + \frac{2}{3}\left[\binom{3}{2}(0.4)^2(0.6) + (0.4)^3\right] = 0.533$$

with 5: $P\{pass\} = \frac{1}{3}\sum_{i=3}^{5}\binom{5}{i}(0.8)^i(0.2)^{5-i} + \frac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(0.4)^i(0.6)^{5-i} = 0.3038$

57. (a) $1 - e^{-3} - 3e^{-3} - e^{-3}\frac{3^2}{2} = 1 - \frac{17}{2}e^{-3} = 0.5768$

(b) $P\{X \geq 3 \mid X \geq 1\} = \frac{P\{X \geq 3\}}{P\{X \geq 1\}} = \frac{1 - \frac{17}{2}e^{-3}}{1 - e^{-3}} = 0.607$

58. (a) $1-e^{-1/2}=0.393$

(b) $\frac{1}{2}e^{-1/2}=0.303$

(c) $a-b=0.0897$

61. $1-e^{-1.4}-1.4e^{-1.4}=0.4082$

HW #8 Page 228-231, Problems:1,4,6,7,10,15,19,21,28

1. (a) $c\int_{-1}^{1}\left(1-x^2\right)dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \frac{3}{4}\int_{-1}^{x}\left(1-x^2\right)dx = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right), \ -1 < x < 1$

4. (a) $\int_{20}^{\infty}\frac{10}{x^2}dx = \frac{-10}{x}\int_{20}^{\infty} = 1/2$

(b) $F(y) = \int_{10}^{y}\frac{10}{x^2}dx = 1 - \frac{10}{y}, \ y > 10. \qquad F(y) = 0 \ for \ y < 10.$

(c)

$\sum_{i=3}^{6}\binom{6}{i}\left(\frac{2}{3}\right)^{i}\left(\frac{1}{3}\right)^{6-i} \approx 0.9,$ since $P(x > 15) = \frac{2}{3}$. Assuming independence of the events that the devices exceed 15 hours.

6. (a) $E[X] = \frac{1}{4}\int_{0}^{\infty}x^2 e^{-x/2}dx = 2\int_{0}^{\infty}y^2 e^{-y}dy = 2G(3) = 4$

(b) By symmetry of $f(x)$ about $x = 0$, $E[X] = 0$

(c) $E[X] = \int_{5}^{\infty}\frac{5}{x}dx = \infty$

7.

$\int_{0}^{1}\left(a + bx^2\right)dx = 1$ or $a + \frac{b}{3} = 1$

$E[X] = \int_{0}^{1}x\left(a + bx^2\right)dx = \frac{3}{5}$ or $\frac{a}{2} + \frac{b}{4} = 3/5$. Hence, $a = \frac{3}{5}, \ b = \frac{6}{5}$

10. (a) $P\{\text{goes to A}\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$
= 2/3 since X is uniform (0,60).

(b) same answer as in (a).

15. (a) $\Phi(0.8333) = 0.7977$
(b) $2\Phi(1) - 1 = 0.6827$
(c) $1 - \Phi(0.3333) = 0.3695$
(d) $\Phi(1.6667) = 0.9522$
(e) $1 - \Phi(1) = 0.1587$

19. Letting Z = (X − 12)/2 then Z is a standard normal. Now,
$0.10 = P\{Z > (c - 12)/2\}$. But from Table 5.1, $P\{Z < 1.28\} = 0.90$ and so
$(c - 12)/2 = 1.28$ or $c = 14.56$

21. (a) 6 feet 2 inches equals 74 inches. Let x denotes the height of a 25-year-old.

$$P\{x > 74\} = P\{\frac{x-71}{\sqrt{6.25}} > \frac{74-71}{\sqrt{6.25}}\} = 1-\Phi(1.2) = 1-0.8849 = 0.1151 = 11.51\%$$

(b) 6 feet equals 72 inches. 6 feet 5 inches equals 77 inches.

$$P\{x > 77 \mid x > 72\} = \frac{P\{x>77\}}{P\{x>72\}} = \frac{1-\Phi\left(\frac{77-71}{\sqrt{6.25}}\right)}{1-\Phi\left(\frac{72-71}{\sqrt{6.25}}\right)} = \frac{1-\Phi(2.4)}{1-\Phi(0.4)} = \frac{0.0082}{0.3446} = 2.38\%$$

28. Let X equal the number of lefthanders. Assuming that *X* is approximately distributed as a binomial random variable with parameters n = 200, p = 0.12, then, With Z being a standard normal random variable,

$$P\{X > 19.5\} = P\left\{\frac{X-200(0.12)}{\sqrt{200(0.12)(0.88)}} > \frac{19.5-200(0.12)}{\sqrt{200(0.12)(0.88)}}\right\}$$

$$\approx P\{Z > -0.9792\}$$

$$\approx 0.8363$$

HW#9: page 290-293, Problems: 1,4,6,7,10,15,19,21,28

1. (a)

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{36} & 12 \ge y > x \ge 1 \ \ \& \ y = 2x \\ \frac{1}{18} & 12 > y > x \ge 1 \ \ \& \ \ y \ne 2x \\ 0 & \text{otherwise}\end{cases}$$

(b)

$$f_{X,Y}(x,y)=\begin{cases}\frac{x}{36} & 6 \ge y = x \ge 1 \\ \frac{1}{36} & 6 \ge y > x \ge 1 \\ 0 & \text{otherwise}\end{cases}$$

(c)

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{36} & 6 \ge y = x \ge 1 \\ \frac{1}{18} & 6 \ge y > x \ge 1 \\ 0 & \text{otherwise}\end{cases}$$

4.

a. $p(0,0)=(\frac{8}{13})^2$, $p(0,1)=p(1,0)=)(\frac{5}{13})(\frac{8}{13})$, $p(1,1)=(\frac{5}{13})^2$

b. $p(0,0,0)=(\frac{8}{13})^3$,

$p(1,0,0)=p(0,1,0)=p(0,0,1)=(\frac{8}{13})^2 \frac{5}{13}$

$p(1,1,0)=p(1,0,1)=p(0,1,1)=(\frac{5}{13})^2 \frac{8}{13}$

$p(1,1,1)=(\frac{5}{13})^3$

7. $P(x_1,x_2)=P(i,j)=(1-p)^i\, p(1-p)^j\, p=p^2(1-p)^{i+j}$

10. (a) $f_X(x)=e^{-x}$, $f_Y(y)=e^{-y}$, $0<x<\infty$, $0<y<\infty$ $P\{X<Y\}=1/2$

(b) $P\{X<a\}=1-e^{-a}$

15.

a. $\iint_R f(x,y)dxdy=1$

$\iint_R c\, dxdy=1 \rightarrow c\iint_R dxdy=1 \rightarrow \frac{1}{c}=\iint_R dxdy=$ area under R

b. $f(x,y)=c=\frac{1}{4}$, for $-1<x<1$, $-1<y<1$

$f_x(x)=\int_{-1}^{1} f(x,y)dy=\int_{-1}^{1}\frac{1}{4}dy=\frac{1}{2}$, for $-1<x<1$

$f_y(y)=\int_{-1}^{1} f(x,y)dx=\int_{-1}^{1}\frac{1}{4}dx=\frac{1}{2}$, for $-1<y<1$

$f(x,y)=\frac{1}{4}=f_x(x)\bullet f_y(y)$, for $-1<x<1$, $-1<y<1$

$\therefore$ X and Y are independent

c. $P(x^2+y^2\le 1)=\iint_{x^2+y^2\le 1}\frac{1}{4}\,dxdy=\frac{1}{4}\iint_{x^2+y^2\le 1}dxdy=\frac{1}{4}\,\pi\,1^2=\frac{\pi}{4}$

21. (a) We must show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dxdy = 1$. Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dxdy = \int_{0}^{1}\int_{0}^{1-y} 24xy\;dxdy = \int_{0}^{1} 12y(1-y)^2\,dy = 12\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = 1$$

(b) $E[X] = \int_{0}^{1} x f_X(x)\,dx = \int_{0}^{1} x \int_{0}^{1-x} 24xy\;dydx = \int_{0}^{1} 12x^2(1-x)^2\,dx = \frac{2}{5}$

(c) 2/5

28.

$f_{x_1}(x_1) = \lambda_1\ e^{-\lambda_1 x_1}$, for $x_1 > 0$

$f_{x_2}(x_2) = \lambda_2\ e^{-\lambda_2 x_2}$, for $x_2 > 0$

$f(x_1, x_2) = \lambda_1\ e^{-\lambda_1 x_1}\lambda_2\ e^{-\lambda_2 x_2}$, for $x_1 > 0\ x_2 > 0$

$Z = \frac{X_1}{X_2}$, for $z > 0$

a.

$$F_z(z) = P(Z \le z) = P(X_1 \le z\ X_2)$$

$$= \iint_{X_1 \le z\ X_2} f(x_1, x_2)\ dx_1\ dx_2$$

$$= \int_0^{\infty}\int_0^{z\ X_2} \lambda_1\ e^{-\lambda_1 x_1}\lambda_2\ e^{-\lambda_2 x_2}\,d x_1\ dx_2$$

$$= \frac{\lambda_1 Z}{\lambda_2 + \lambda_1 Z}, \qquad \text{for } z > 0$$

$$f_z(z) = \frac{dF_z(z)}{dz} = \frac{\lambda_1\lambda_2}{(\lambda_2 + \lambda_1 Z)^2}, \quad \text{for } z > 0$$

b.

$$P(X_1 < X_2) = P\left(\frac{X_1}{X_2} < 1\right) = P(Z \le 1) = F_z(1) = \frac{\lambda_1}{\lambda_2 + \lambda_1}$$

HW #9 Page 230-231, Problems: 22,24,32,33,37,39,40,41

22. (a) $P\{0.9000-0.005< X <0.9000+0.005\} = P\left\{-\frac{0.005}{0.003} < Z < \frac{0.005}{0.003}\right\}$

$= P\{-1.67< Z <1.67\} = 2\Phi(1.67)-1 = 0.9050$

Hence 9.5 percent will be defective (that is each will be defective with probability 1-0.9050=0.0950).

(b) $P\left\{-\frac{0.005}{s} < Z < \frac{0.005}{s}\right\} = 2\Phi\left(\frac{0.005}{s}\right)-1 = 0.99$ *when*

$\Phi\left(\frac{0.005}{s}\right) = 0.995 \Rightarrow \frac{0.005}{s} = 2.575 \Rightarrow s = 0.0019$

24. With C denoting the life of a chip, and ϕ the standard normal distribution function we have $P\{C < 1.8\times10^6\} = \Phi\left(\frac{1.8\times10^6 - 1.4\times10^6}{3\times10^5}\right) = \Phi(1.33) = 0.9082$

Thus, if N is the number of the chips whose life is less than 1.8×10^6 then N is a binomial random variable with parameters (100, 0.9082). Hence

$P\{N > 19.5\} \approx 1-\Phi\left(\frac{19.5-90.82}{90.82(0.0918)}\right) = 1-\Phi(-24.7) \approx 1$

32. (a) $e^{-1} = 0.3679$

(b) $e^{-1/2} = 0.6065$

33. $e^{-1} = 0.3679$

37. (a) $P\{|X| > 1/2\} = P\{X > 1/2\} + P\{X < -1/2\} = 1/2$

(b) $P\{|X| \le a\} = P\{-a \le X \le a\} = a, 0<a<1$. Therefore, $f_{|X|}(a) = 1, 0<a<1$

That is, $|X|$ is uniform on (0,1).

39. $F_Y(y) = P\{\log X \le y\} = P\{X \le e^y\} = F_X(e^y)$

$f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y}$

40. $F_Y(y) = P\{e^X \le y\} = F_X(\log y)$

$f_Y(y) = f_X(\log y)\frac{1}{y} = \frac{1}{y},\ 1< y < e$

41. $F_R(r) = P(R \le r) = P(A\sin q \le r) = P\left(q \le \sin^{-1}\left(\frac{r}{A}\right)\right) = F_q\left(\sin^{-1}\left(\frac{r}{A}\right)\right)$

Differentiate it:

$f_R(r) = f_q\left(\sin^{-1}\left(\frac{r}{A}\right)\right)\frac{1}{A\sqrt{1-\left(\frac{r}{A}\right)^2}} = \frac{1}{p\sqrt{A^2-r^2}}\ , \qquad r < A$

HW #10 Page 290-295, Problems:1, 2, 7-11, 13, 14, 18, 20-23

1. (a)

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{36} & 12\ge y>x\ge 1 \ \ \& \ y=2x\\ \frac{1}{18} & 12>y>x\ge 1 \ \& \ \ y\ne 2x\\ 0 & \text{otherwise}\end{cases}$$

(b)

$$f_{X,Y}(x,y)=\begin{cases}\frac{x}{36} & 6\ge y=x\ge 1\\ \frac{1}{36} & 6\ge y>x\ge 1\\ 0 & \text{otherwise}\end{cases}$$

(c)

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{36} & 6\ge y=x\ge 1\\ \frac{1}{18} & 6\ge y>x\ge 1\\ 0 & \text{otherwise}\end{cases}$$

2. (a) $p(0,0)=\frac{8\cdot 7}{13\cdot 12}=14/39$

$p(0,1)=p(1,0)=\frac{8\cdot 5}{13\cdot 12}=10/39$

$p(1,1)=\frac{5\cdot 4}{13\cdot 12}=5/39$

(b) $p(0,0,0)=\frac{8\cdot 7\cdot 6}{13\cdot 12\cdot 11}=28/143$

$p(0,0,1)=p(0,1,0)=p(1,0,0)=\frac{8\cdot 7\cdot 5}{13\cdot 12\cdot 11}=70/429$

$p(0,1,1)=p(1,0,1)=p(1,1,0)=\frac{8\cdot 5\cdot 4}{13\cdot 12\cdot 11}=40/429$

$p(1,1,1)=\frac{5\cdot 4\cdot 3}{13\cdot 12\cdot 11}=5/143$

7. $P(x_1, x_2) = P(i, j) = (1-p)^i \; p(1-p)^j \; p = p^2(1-p)^{i+j}$

8. (a) $f_Y(y) = c\int_{-y}^{y}\left(y^2 - x^2\right)e^{-y}dx = \frac{4}{3}cy^3e^{-y}, \quad 0 < y < \infty$

$\int_0^{\infty} f_Y(y)dy = 1 \Rightarrow c = 1/8$

(b) $f_Y(y) = \frac{y^3e^{-y}}{6}, \; 0 < y < \infty \qquad f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}\left(y^2 - x^2\right)e^{-y}dy = \frac{1}{4}e^{-|x|}\left(1 + |x|\right)$

upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

(c) $E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{4}e^{-|x|}\left(1 + |x|\right)dx = 0$ since the integrand is an odd function

9. (a) $\int_0^2\int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)dxdy = \frac{6}{7}\int_0^2 [(\frac{1}{3}x^3 + \frac{1}{4}x^2y)]_0^1 dy = \frac{6}{7}\left[\frac{1}{3}y + \frac{1}{8}y^2\right]_0^2 = 1$

This shows $\iint_{xy} f_{X,Y}(x, y) = 1$, so $f_{X,Y}(x, y)$ is a valid distribution function.

(b) $f_X(x) = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}\left(2x^2 + x\right)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d)

$$P\{Y > 1/2 \,|\, X < 1/2\} = \frac{P\{Y > 1/2, X < 1/2\}}{P\{X < 1/2\}} = \frac{\int_{1/2}^{2}\int_0^{1/2}\left(x^2 + \frac{xy}{2}\right)dxdy}{\int_0^{1/2}\left(2x^2 + x\right)dx} = \frac{0.1797}{\frac{5}{24}} = 0.86256$$

(e) $E[X] = \int_0^1 x \cdot \frac{6}{7}\left(2x^2 + x\right)dx = \left[\frac{6}{7}\left(\frac{1}{2}x^4 + \frac{1}{3}x^3\right)\right]_0^1 = \frac{5}{7}$

(f) $f_Y(y) = \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)dx = \frac{6}{7}\left(\frac{1}{3} + \frac{1}{4}y\right)$

$E[Y] = \int_0^2 y \cdot \frac{6}{7}\left(\frac{1}{3} + \frac{1}{4}y\right)dy = \left[\frac{6}{7}\left(\frac{1}{6}y^2 + \frac{1}{12}y^3\right)\right]_0^2 = \frac{8}{7}$

10. (a) $f_X(x) = e^{-x}, \; f_Y(y) = e^{-y}, \; 0 < x < \infty, \; 0 < y < \infty \qquad P\{X < Y\} = 1/2$

(b) $P\{X < a\} = 1 - e^{-a}$

11. $\frac{5!}{2!1!2!}(0.45)^2(0.15)(0.40)^2 = 0.1458$

13. (a) set the starting point at 12:00 pm and the ending point at 1:00 pm

$$f_X(x) = \frac{1}{30} \quad 15 \le x \le 45; \quad f_Y(y) = \frac{1}{60} \quad 0 \le y \le 60$$

$$P\{|x-y| \le 5\} = \int_{15}^{45}\int_{x-5}^{x+5} \frac{1}{30} \cdot \frac{1}{60} dydx = \frac{1}{6}$$

(b) $P\{x < y\} = \frac{1}{2}$ by symmetry

14.

Let X and Y denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{|Y-X| < a\} = P\{Y < X < Y + a\} + P\{X < Y < X + a\} = \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,L)} dxdy$$

$$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dxdy + \int_{L-a}^{L}\int_y^{L} dxdy\right] = 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a)$$

$$= \frac{a}{L}\left(2 - \frac{a}{L}\right), \quad 0 < a < L$$

18. $$P\{Y - X > L/3\} = \iint_{y-x>L/3} \frac{4}{L^2} dydx \qquad \frac{L}{2} < y < L,\ 0 < x < \frac{L}{2}$$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dydx + \int_{L/6}^{L/2}\int_{x+\frac{L}{3}}^{L} dydx\right] = \frac{4}{L^2}\left[\frac{L^2}{12} + \frac{5L^2}{24} - \frac{7L^2}{72}\right] = \frac{7}{9}$$

20. (a) yes: $f_X(x) = xe^{-x},\ f_Y(y) = e^{-y},\ 0 < x < \infty, 0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x,y)dy = 2(1-x),\ 0 < x < 1$

$$f_Y(y) = \int_0^y f(x,y)dx = 2y,\ 0 < y < 1$$

21. (a) We must show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$. Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^1\int_0^{1-y} 24xy\ dxdy = \int_0^1 12y(1-y)^2 dy = 12\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1$$

(b) $E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 x \int_0^{1-x} 24xy\; dydx = \int_0^1 12x^2(1-x)^2\; dx = \frac{2}{5}$

(c) 2/5

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)\,dy = x + \frac{1}{2},\;\; 0 < x < 1.$

(c) $P\{X+Y<1\} = \int_0^1 \int_0^{1-x} (x+y)\,dydx = \int_0^1 [x(1-x) + \frac{(1-x)^2}{2}]dx = \frac{1}{3}$

23. (a) yes $f_X(x) = 12x(1-x)\int_0^1 ydy = 6x(1-x),\; 0 < x < 1$

$f_Y(y) = 12y\int_0^1 x(1-x)\,dx = 2y,\; 0 < y < 1$

(b) $E[X] = \int_0^1 6x^2(1-x)\,dx = 1/2$

(c) $E[Y] = \int_0^1 2y^2 dy = 2/3$

(d) $Var(X) = \int_0^1 6x^3(1-x)\,dx - \frac{1}{4} = \frac{1}{20}$

(e) $Var(Y) = \int_0^1 2y^3 dy - \frac{4}{9} = \frac{1}{18}$

HW #11 Page 380-389, Problems: 5,6,7,11,18,30,33,37,38,39,49,50,51,75

5. The joint density of the point (*X*,*Y*) at which the accident occurs is

$$f(x,y) = \frac{1}{9} = f(x)\, f(y), \quad -\frac{3}{2} < x, y < \frac{3}{2}$$

where $f(a) = \frac{1}{3}, \quad -\frac{3}{2} < a < \frac{3}{2}$

Hence we may conclude that *X* and *Y* are independent and uniformly distributed on (-3/2, 3/2) Therefore,

$$E\big[|X| + |Y|\big] = 2\int_{-3/2}^{3/2} \frac{1}{3} x\,dx = \frac{4}{3}\int_{0}^{3/2} x\,dx = \frac{3}{2}$$

6. $E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 10(7/2) = 35$

7. Let X_i equal 1 if both choose item i and let it be 0 otherwise; let Y_i equal 1 if neither A nor B chooses item i and let it be 0 otherwise. Also, let W_i equal 1 if exactly one of A and B choose item i and let it be 0 otherwise. Let

$$X = \sum_{i=1}^{10} X_i, \quad Y = \sum_{i=1}^{10} Y_i, \quad W = \sum_{i=1}^{10} W_i$$

(a) $E[X] = \sum_{i=1}^{10} E[X_i] = 10(3/10)^2 = 0.9$

(b) $E[Y] = \sum_{i=1}^{10} E[Y_i] = 10(7/10)^2 = 4.9$

(c) Since $X + Y + W = 10$, we obtain from parts (a) and (b) that

$E[W] = 10 - 0.9 - 4.9 = 4.2$

Of course, we could have obtained $E[W]$ from

$$E[W] = \sum_{i=1}^{10} E[W_i] = 10(2)(3/10)(7/10) = 4.2$$

11. Let X_i equal 1 if a changeover occurs on the i^{th} flip and 0 otherwise. Then

$$E[X_i] = P\{i-1 \text{ is } H, i \text{ is } T\} + P\{i-1 \text{ is } T, i \text{ is } H\} = 2(1-p)\,p, \quad i \geq 2.$$

E[number of changeovers] = $E\left[\sum X_i\right] = \sum_{i=1}^{n-1} E[X_i] = 2(n-1)(1-p)p$

18. *E*[number of matches] = $E\left[\sum_{i=1}^{52} I_i\right] = 52 \cdot \frac{1}{13} = 4$ since $E[I_i] = 1/13$,

$$I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{---} \end{cases}$$

30. $E\left[(X - Y)\right]^2 = Var(X - Y) = Var(X) + Var(-Y) = 2s^2$

33. (a) $E\left[X^2+4X+4\right]=E\left[X^2\right]+4E[X]+4=Var(X)+E^2[X]+4E[X]+4=14$

(b) $Var(4+3X)=Var(3X)=9Var(X)=45$

37. Let $W_i,\ i=1,2,$ denote the i^{th} outcome.

$$\text{Cov}(X,Y)=\text{Cov}(W_1+W_2,\ W_1-W_2)=\text{Cov}(W_1,W_1)-\text{Cov}(W_2,W_2)$$
$$=\text{Var}(W_1)-\text{Var}(W_2)=0$$

38. $$E[XY]=\int_0^\infty\int_0^x y2e^{-2x}dydx=\int_0^\infty x^2e^{-2x}dx=\frac{1}{8}\int_0^\infty y^2e^{-y}dy=\frac{\Gamma(3)}{8}=\frac{1}{4}$$

$$E[X]=\int_0^\infty xf_X(x)dx=\frac{1}{2},\qquad f_X(x)=\int_0^x\frac{2e^{-2x}}{x}dy=2e^{-2x}$$

$$E[Y]=\int_0^\infty yf_Y(y)dy=\int_0^\infty\int_y^\infty y\frac{2e^{-2x}}{x}dxdy=\int_0^\infty\int_0^x y\frac{2e^{-2x}}{x}dydx=\int_0^\infty xe^{-2x}dx=\frac{1}{4}\int ye^{-2}dy$$
$$=\frac{\Gamma(2)}{4}=\frac{1}{4},\qquad f_Y(y)=\int_0^\infty\frac{2e^{-2x}}{x}dx$$

$$Cov(X,Y)=\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$$

39. $$Cov(Y_n,Y_n)=Var(Y_n)=3s^2$$
$$Cov(Y_n,Y_{n+1})=Cov(X_n+X_{n+1}+X_{n+2},X_{n+1}+X_{n+2}+X_{n+3})$$
$$=Cov(X_{n+1}+X_{n+2},X_{n+1}+X_{n+2})=Var(X_{n+1}+X_{n+2})=2s^2$$
$$Cov(Y_n,Y_{n+2})=Cov(X_{n+2},X_{n+2})=s^2$$
$Cov(Y_n,Y_{n+j})=0$ when $j\geq 3$

49. $$E\left[X\mid Y=i\right]=\begin{cases}3\cdot\frac{3}{5}=\frac{9}{5}, & i=1\\ 3\cdot\frac{2}{5}=\frac{6}{5}, & i=1\\ 3\cdot\frac{1}{5}=\frac{3}{5}, & i=1\\ 0\quad, & i=4\end{cases}$$

50. $$f_{X|Y}(x|y)=\frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\,dx}=\frac{1}{y}e^{-x/y},\qquad 0<x<\infty$$

Hence, given $Y=y$, X is exponential with mean y, and so $E\left[X^2\mid Y=y\right]=2y^2$

51. $$f_{X|Y}\left(x|y\right)=\frac{e^{-y}/y}{\int_0^y e^{-y}/y\,dx}=\frac{1}{y}, \qquad 0<x<y$$

$$E\left[X^3|Y=y\right]=\int_0^y x^3\frac{1}{y}dx=y^3/4$$

75. X is Poisson with mean $\lambda=2$ and Y is Binomial with parameters 10, ¾. Hence

(a) $P\{X+Y=2\}=P\{X=0\}P\{Y=2\}+P\{X=1\}P\{Y=1\}+P\{X=2\}P\{Y=0\}$

$$=e^{-2}\binom{10}{2}(3/4)^2(1/4)^8+2e^{-2}\binom{10}{1}(3/4)(1/4)^9+2e^{-2}(1/4)^{10}$$

(b) $P\{XY=0\}=P\{X=0\}+P\{Y=0\}-P\{X=Y=0\}$

$$=e^{-2}+(1/4)^{10}-e^{-2}(1/4)^{10}$$

(c) $E[XY]=E[X]E[Y]=2\cdot 10\cdot\frac{3}{4}=15$

III. Quiz

Quiz #1

Math461 Quiz #1
Thursday June 16, 2005
Chapter 1 Combinational Analysis

1. If 6 Americans, 5 Frenchmen, and 4 Englishmen are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other and Frenchmen and Americans could not sit together?

   Solution: = 2! 6! 5! 4! =4147200

2. A student is to answer 6 out of 10 questions in an exam.
   How many choices she has if she must answer at least 3 of the first 5 questions?

   Solution: $=\binom{5}{3}\binom{5}{3}+\binom{5}{4}\binom{5}{2}+\binom{5}{5}\binom{5}{1}$=155

3. There are 3 classes, each consisting of 10 students. From this group of 30 students, a group of 3 students is to be chosen. How many choices are possible if
   (a) no restrictions?
   (b) all 3 students are in the same class?
   (c) 2 of the 3 students are in the same class and the 3rd one from a different class?
   (d) all 3 students are from different classes?

Solution:

$a.\ \binom{30}{3}=4060$

$b.\ \binom{3}{1}\binom{10}{3}=360$

$c.\binom{3}{1}\binom{10}{2}\binom{20}{1}=2700$

$d.\binom{10}{1}\binom{10}{1}\binom{10}{1}=1000$

4. An art collection on auction consists of 5 Dallis, 6 Van Goghs, and 7 Picassos. At the art auction, there are 6 collectors. If a reporter noted only the number of Dallis, Van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all works were sold?

Solution: $=\binom{5+6-1}{6-1}\binom{5+6-1}{6-1}\binom{7+6-1}{6-1}=\binom{10}{5}\binom{11}{5}\binom{12}{5}$=92,207,808

Quiz #2

Quiz #2 for Chapter 2 Axioms of Probability
June 23, 2005

Name: ____

1. A customer visiting a suit department stork will purchase a suit with probability 0.25, a shirt 0.28, a tie 0.22. A suit and a shirt 0.11. A suit and a tie 0.12. A shirt and a tie 0.15. A suit, a shirt and a tie 0.05. What is the probability that he purchase:
   a) None of them?
   b) Exactly one of them?

2. A deck of 52 cards is shuffled completely. What is the probability that the top 3 cards have:
   a) Different denominations?
   b) Different suits?
   c) The same suits?

3. Urn A contains 4 red ball and 3 black balls. Urn B contains 5 red balls and 4 black balls. If a ball is randomly selected from each urn, what is the probability that the two balls will be the same color?

4. If a die is rolled 3 times, what is the probability that
   a. two of the three outcomes are the same?
   b. all three outcomes are the same?

5. If 3 couples to sit in a row, what is the probability that no couple sits next to each other?

Quiz #3

Quiz #3 Chapter 4 Random Variables
July 11, 2005

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1. Suppose Lakers and Spurs are in the NBA final of 2006, the chance of Lakers win a game is 0.6. The team that wins 4 games wins the championship. Let N denotes the total number of games played.
   (a). List the possible values of N and find the probability P(N=4).
   (b). Given the first 4 games are tied, N* is the total number of more games to be played. List the possible value of N*, and find the mean and variance of N*.

2. A coin is tossed once and the chance of observing a head is 0.6. A person will win $5 if it is a head and loss $7 if it is a tail. Denote X as money he wins or loses in the game. Find the expected value and variance of X. If the game is played 5 times and each toss is independent of another, what is the expected value of the total gain (or loss)?

3. Suppose the number of hurricanes is a Possion random variable and on average there are 4 hurricanes in a region within a year. What is the probability that in 2005:
   (a). There will be no hurricane?
   (b). There will be at most 3 hurricanes?
   (c). There will be at least 5 hurricanes?

4. Suppose X can take on one of the values 0, 2, 4.
   Given $P(X=4)=cP(X=2)=c^2 P(X=0)$, find c, E(X), and Var(X).

5. 7 different numbers are to be allocated to 7 players. Player 1 and player 2 compare their numbers and the winner then compares with player 3, and so on.

   Let X denote the number of times player 1 wins. Find

   (a). All the possible values of X
   (b). E(X)
   (c).Var(X)

Quiz #4

Quiz #4 Chapter 6 Jointly Distributed Random Variables

1. Two balls are chosen without replacement from an Urn containing 4 Red balls and 6 White balls. Let X be the number of Red balls chosen and Y be the number of White balls chosen. Find:
   a. P(x,y):the joint mass function of X and Y
   b. P(x) and P(y): the marginal mass function of X and Y
   c. Is X and Y independent of each other?
   d. E (XY): the expected value of XY.

2. Given the joint PDF of X and Y as:
   $f(x,y)=C_0(y^2-x^2)e^{-y}$ for $-y \le x \le y \quad 0 < y < \infty$.
   Find:
   a. C_0
   b. the marginal density of X and Y
   c. Is X and Y independent of each other?
   d. E(X), E(Y), and E (XY)

3. Two people A and B are to meet at a certain location around 12pm. If A arrives at a time uniformly distributed between 11:45am and 12:15pm, and B arrives at a time uniformly distributed between 11:40am and 12:20pm. Find:
   a. the joint probability density function f(x,y).
   b. the probability that the first to arrive will wait less than 5 minutes.
   c. the probability that A will wait at least 5 minutes for B.

4. Given X and Y are independent of each other. X follows a Binomial distribution B(10, 0.5), and Y follows a Binomial distribution B(20, 0.5). Find:
   a. P(X|X+Y=10)
   b. P(X>1|X+Y=10)

5. Given $f(x,y)=\begin{cases} 2, & 0<x<y, 0<y<1 \\ 0, & \text{Otherwise} \end{cases}$. Find:
   a. f(x|y).
   b. P(X>0.5|y=0.25).

Midterm #1 Chapter 1, 2 and 3 June 30, 2005

Name:

1. If 4 English books, 5 history books and 6 Engineering books are to be placed in a row. Assuming all the books are different, how many ways you can place them? if
   (a). there are no restraints.
   (b). books of the same subjects have to be placed next to each other.
   (c). only English books have to be together.

2. 5 cards are to be drawn from a well shuffled deck of 52 cards. What is the probability that:
   (a). 4 Aces are drawn?
   (b). a pair of Aces and a pair of Kings are drawn?
   (c). a royal flush (the same suit of 10, J, Q, K and A) are drawn?
   (d). a straight flush (consecutive cards from the same suit) are drawn?

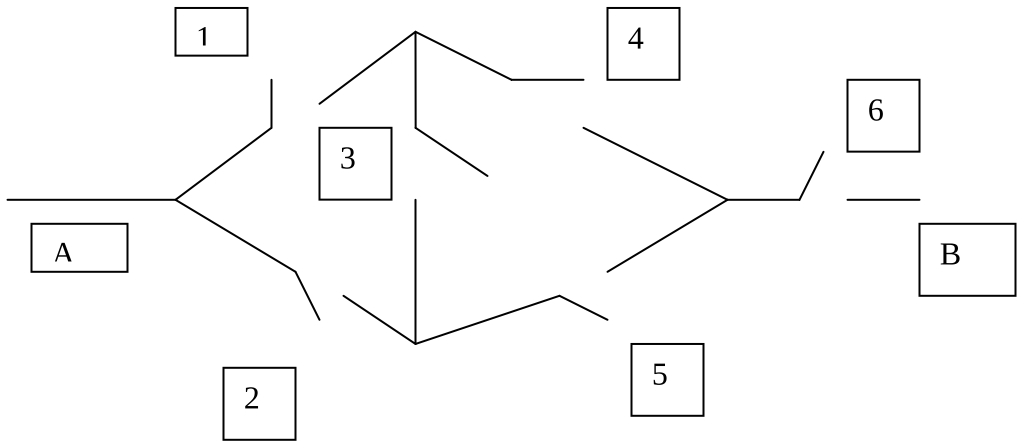
3. An investor has 15 units of wealth to allocate among 5 mutual funds. If only integer units can be allocated to each fund. How many ways he can allocate his wealth?, if
   (a). he has to allocate non-zero amount of wealth to each fund.
   (b). he can choose not to invest in some of the funds.

4. A student can take a English class with probability of 0.7, a French with probability of 0.6, and a German with probability of 0.4. The chance that he will take both English and French is 0.4, English and German 0.3, French and German 0.2, all the three 0.1. What is the probability that he will take:
   (a). none of them?
   (b). French only?
   (c). French and German only?

5. A student can choose to do all his homework with probability 0.7, just a part of the homework with probability 0.2, or not at all with probability 0.1. If he does all of his homework, his chance of getting A is 0.7; if he does only part, his chance is 0.2; if he does none, his chance is 0.05.
   (a). what is the chance that he will get A?
   (b). given he gets A, what the chance that he does at least part of the homework?

6. Urn A contains 5 red balls and 4 black balls. Urn B contains 6 red balls and 5 black balls. If two balls are drawn from each Urn, what is the probability that:
   (a). 2 red balls and 2 black balls are drawn?
   (b). given 2 red balls and 2 black balls are drawn, what is the probability that two red balls are drawn from Urn A?

7. Given all the relays functioning independently of each other, the probability of each working properly is p. What is the probability that:
   (a). a current can run from A to B?
   (b). given relay 2 and relay 4 not working, what is the probability that a current can run from A to B?



8. For independent flips of a coin, if the chance of observing a head is 0.6, find the probability that:

   (a). all the first three are heads?
   (b). two of the first three are heads?
   (c). HHH happens before THH?

9. Two gamblers Andy and Roger bet on the outcomes of successive independent flips of a coin. On each flip, if it is a Head, Andy collects $1 from Roger. if it is a Tail, Andy pays $1 to Roger. They continue until one of them runs out of money.

   Andy has $100 and Roger has $200. Find the probability that Andy runs out of his money when:

   (a). the probability of observing a Head is 0.5.
   (b). the probability of observing a Head is 0.6.

Midterm #2

Midterm #2 Chapter 4 and 5

1. A person takes out two balls (without replacement) from an urn containing 10 white, 6 black and 4 blue. He will win $3 dollar for each white ball, lose $5 dollar for each black ball. Let X denote his winning (or loss), what is the possible value of X, what is the mean and standard deviation of X?

2. 3 balls are to be randomly selected without replacement from an urn containing 10 balls numbered from 1 to 10. Let X be the smallest number selected, find the range of X and the probability mass function of X, the expected value and standard deviation of X.

3. If the number of floods in an area each year X follows a Possion distribution with parameter λ equal to 2. Find
   (a). the probability that there will be no flood in that area in 2005.
   (b). the probability that there will be no more than 3 floods from 2005 to 2006, assuming the floods in different years are independent of each other.

4. A person arrives at a train station at 8am every morning. Assume his train arrives at the station uniformly between 8 and 8:30am. What is the probability that he will have to wait more than 15 minutes? Given he has waited 5 minutes, what is the probability that he will wait for at least another 10 minutes?

5. Given 60% of people in the country support cutting tax. For a group of 100 people, Using Normal approximation, find the probability that
   (a). there are at least 50 people will support the cut?
   (b). between 40 and 80 inclusive will support the cut?

6. Given X follows an exponential distribution with parameter λ =0.5. If Y=10 X+5, What is the range of Y, f(y), Pr (30>Y>20), and the expected value and standard deviation of Y?

7. Given X follows a Normal distribution function with μ =0 and σ^2=4. Y=| X |, Find P(4>Y>1), the mean and standard deviation of Y.

Final Exam

Math 461 Final

Summer 2005

Name: ____

1. For a random variable X, given $F(x)=\begin{cases}0, & x<0\\ \frac{x}{4}, & 0\le x<1\\ \frac{x}{2}, & 1\le x<2\\ 1, & x\ge 2\end{cases}$.

   Find:

   $a.\ P(\frac{1}{2}\le x\le 1)$

   $b.\ P(\frac{3}{2}\le x\le 2)$

   $c.\ P(1<x<2)$

2. Given X is uniformly distributed between (-8, 8). Let $Y=X^3$, find the PDF of Y, and P(Y>1).

3. Given X and Y are independent and identical distributed random variables. If each is uniformly distributed between 0 and 1, and Z=|X-Y|. Find f(Z), E (Z), Var(Z), and P(Z<0.5).

4. Given X, Y and Z are independent random variables, X is Poisson with parameter $\lambda=2$, Y is Geometric with parameter p=0.5, and Z is a Standard Normal. Find Cov(X+Z, X+Y+Z).

5. Given the joint density function of X and Y as:

$$f(x,y)=\begin{cases} y^{-1}e^{-y}, 0<x<y,\ 0<y<\infty \\ 0, \quad \text{Otherwise} \end{cases}$$

Find f(x|y=1), E(X|y=1), E(X^2|y=1), Var(X|y=1).

6. Given X and Y independent of each other. X is Possion with parameter $\lambda_1 = 1$, and Y is Possion with parameter $\lambda_2 = 4$. Let Z=X+Y, Find:
   a. f(Z), E(Z) and Var(Z)
   b. f(X|Z=5), E(X|Z=5).

7. Given T=t, X is Poisson with parameter $\lambda_1 t$. If T is Exponential with parameter λ_2, find E(X|T=t), Var(X|T=t), E (X), and Var(X).

8. Given X and Y independent of each other, and the moment generating function of X as $M_X(t)=\exp(3e^t-3)$, and $M_Y(t)=(\frac{1}{2}e^t+\frac{1}{2})^{10}$, Find P(X+Y=1), P(XY=0) and E(XY).

9. Given X_i independent and identical distributed, for i=1 to 100, and E(X_i)=1, Var(X_i)=4. Let Y=X_1+X_2+…+X_{100}. Use the Central Limit Theorem to find P(80<Y<120).