BLOWUP FOR THE HEAT EQUATION WITH A NOISE TERM

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ABSTRACT

In this paper we study blowup of the equation $u_t = u_{xx} + u^\gamma \dot{W}_{tx}$, where $\dot{W}_{tx}$ is a two-dimensional white noise field and where Dirichlet boundary conditions are enforced. It is known that if $\gamma < 3/2$, then the solution exists for all time; in this paper we show that if γ is much larger than $3/2$, then the solution blows up in finite time with positive probability. We prove this by considering how peaks in the solution propagate. If a peak of high mass forms, we rescale the equation and divide the mass of the peak into a collection of peaks of smaller mass, and these peaks evolve almost independently. In this way we compare the evolution of u to a branching process. Large peaks are regarded as particles in this branching process. Offspring are peaks which are higher by some factor. We show that the expected number of offspring is greater than one when γ is much larger than $3/2$, and thus the branching process survives with positive probability, corresponding to blowup in finite time.

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0. Introduction.

In this paper we consider the behavior of the solution of the random heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u^\gamma \dot{W} \\ u(t,0) &= u(t,J) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J \\ u(0,\cdot) &= u_0. \end{aligned} \tag{0.1}$$

where $\gamma > 1$, $\dot{W}$ is a two-dimensional white noise term, and the initial condition u_0 is continuous and nonnegative but not identically zero. Mueller in [12] showed that if $\gamma < 3/2$, then the solution to (0.1) exists for all time and is finite. We here show that if γ is much larger than $3/2$, then the solution to (0.1) blows up in finite time with positive probability. Presumably, there is a γ_c with $\gamma_c \geq 3/2$ such that if $\gamma > \gamma_c$, the solution to (0.1) blows up with positive probability but if $\gamma < \gamma_c$, then blowup is impossible. At present, even the existence of such a γ_c is not known. Some computer simulations by Terry Lyons give some evidence that in fact $\gamma_c = 3/2$.

Note that blowup occurs in a similar deterministic problem. If v is the solution to

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + v^\gamma \\ v(t,0) &= v(t,J) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J \\ v(0,\cdot) &= cf \end{aligned}$$

where $\gamma > 1$ and where f is a nonnegative and nonzero continuous function on $[0, J]$, then if the constant c is large enough, v blows up in finite time (see, for example, [3], [4], [7]). Roughly, one can compare the solution v to the solution of the ODE $\dot{y} = y^\gamma$, which is easily seen to blow up in finite time. By analogy, we should then be able to study (0.1) via the Itô stochastic differential equation

$$dX_t = X_t^\gamma dB_t, \tag{0.2}$$

where B is a standard Wiener process. Note that here, however, X does not blow up for any γ. Indeed, X is a time-changed Brownian motion which is killed at zero, and so is bounded.

Our main line of argument in this paper is to compare the solution of (0.1) to a branching process. Large peaks are regarded as particles in this branching process. Offspring are peaks which are higher by some factor. We show that the expected number of offspring is greater than one when γ is much larger than $3/2$, and thus the branching process survives with positive probability, corresponding to blowup in finite time.

Nonlinear PDE's commonly have solutions which blow up in finite time, and such phenomena have been often studied. To our knowledge, there have been no results on blow-up for PDE's with random noise terms. In this paper we show blowup for the heat equation with a nonlinear noise term. The blow-up is due entirely to the effect of the noise, which can push the solution down as well as up. Thus, our work is not merely an extension of the deterministic case, and our techniques are substantially different.

1. A rigorous understanding of the problem.

We now recall the meaning of SPDE's such as (0.1), allowing us to rigorously state the main result of our paper.

First let us fix some notation. Let $\mathbb{R}_+ := [0,\infty)$. Let $\bar{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$ and $\bar{\mathbb{R}}_+ := \mathbb{R}_+\cup\{\infty\}$ with the usual topologies. For each $J > 0$, define $E(J)$ as the vector space consisting of those elements φ of $C([0,J])$ such that $\varphi(0) = \varphi(J) = 0$, and define $E_+(J)$ as the collection of those elements φ of $E(J)$ such that $\varphi(x) \geq 0$ for all $0 \leq x \leq J$. Also, let $E^\infty(J)$ be the cone of those elements φ of $C([0,J];\bar{\mathbb{R}})$ such that $\varphi(0) = \varphi(J) = 0$, and let $E^\infty_+(J)$ be the convex subset of $E^\infty(J)$ consisting of those elements $E^\infty(J)$ which are also in $C([0,J];\bar{\mathbb{R}}_+)$. We assume the standard topologies on these spaces such that they are Polish spaces. Also, for each $J > 0$, define $C_{E(J)}[0,\infty)$, $C_{E_+(J)}[0,\infty)$, $C_{E^\infty(J)}[0,\infty)$ and $C_{E^\infty_+(J)}[0,\infty)$ as the collection of continuous mappings from $\mathbb{R}_+$ into respectively $E(J)$, $E_+(J)$, $E^\infty(J)$ and $E^\infty_+(J)$; under appropriate metrics, these spaces are also Polish spaces (see [1], Chap 3.10). If φ is in $C_{E^\infty(J)}[0,\infty)$, then we denote by $\varphi(t,x)$ the value of φ at a specific time t and at a specific point x in $[0,J]$, and for each $t \geq 0$, we denote by $\varphi(t,\cdot)$ that element of $E^\infty(J)$ generated by the mapping $x \mapsto \varphi(t,x)$. Finally, we define $C^2_0([0,J])$ as the space of elements of $E(J)$ which possess a continuous second derivative, and we define for each φ in $E^\infty(J)$,

$$\|\varphi\| := \begin{cases} \sup_{x\in[0,J]} |\varphi(x)| & \text{if } \varphi \in E(J) \\ \infty & \text{otherwise.} \end{cases}$$

We can of course understand $\|\cdot\|$ as a function on $\cup_{J>0}E^\infty(J)$.

Now we quote the definition of a white noise on $\mathcal{B}(\mathbb{R}_+ \times [0,J])$. We assume an underlying probability triple $(\Omega,\mathcal{F},P)$. A white noise W on $\mathcal{B}(\mathbb{R}_+ \times [0,J])$ is a random set function on the Borel subsets of $\mathbb{R}_+ \times [0,J]$ of finite Lebesgue measure such that

i) for A a Borel subset of $\mathbb{R}_+ \times [0,J]$ with finite Lebesgue measure, $W(A)$ is a zero-mean Gaussian random variable *ii)* for A and B Borel subsets of $\mathbb{R}_+ \times [0, J]$ of finite Lebesgue measure,

$$E[W(A)W(B)] = \text{Leb}(A \cap B),$$

Leb being the Lebesgue measure on $(\mathbb{R}_+ \times [0, J], \mathcal{B}(\mathbb{R}_+ \times [0, J]))$.

The theory of integration against W is developed in [18]. For future reference, let us now define the filtration $\{\mathcal{F}_t^W : t \geq 0\}$ of $\mathcal{F}$ as

$$\mathcal{F}_t^W := \sigma\{W(A) : A \in \mathcal{B}([0, t] \times [0, J])\}. \qquad t \geq 0$$

Consider any random process v with sample paths in $C_{E^\infty(J)}[0, \infty)$. For each positive integer L, we define the random time

$$\sigma_L(v) := \inf\{t > 0 : \|v(t, \cdot)\| \geq L\}, \tag{1.1}$$

and we define the *explosion time* of v as

$$\sigma(v) := \lim_{L \to \infty} \sigma_L(v). \tag{1.2}$$

By considering all J's at once, the σ_L's and σ may be defined as functions from $\cup_{J>0} C_{E^\infty(J)}[0, \infty)$ to $\bar{\mathbb{R}}_+$. We say that a random process u with sample paths in $C_{E^\infty(J)}[0, \infty)$ is a *solution* of (0.1) if

(A.1) $u(t, \cdot)$ is $\mathcal{F}_t^W$-measurable for each $t \geq 0$ (i.e., the process $t \mapsto u(t, \cdot)$ is $\mathcal{F}_t^W$-adapted)

(A.2) for all $t \geq 0$ and all φ in $C_0^2([0, J])$,

$$\begin{aligned} \int_0^J u(t \wedge \sigma_L(u), x)\varphi(x)dx = &\int_0^J u_0(x)\varphi(x)dx \\ &+ \int_0^t \int_0^J u(s, x)\frac{\partial^2 \varphi}{\partial x^2}(x)\chi_{\{s \leq \sigma_L(u)\}} ds dx \\ &+ \int_0^t \int_0^J \varphi(x)u^\gamma(s, x)\chi_{\{s \leq \sigma_L(u)\}} W(ds, dx) \end{aligned} \tag{1.3}$$

P-a.s. for each $L = 1, 2, \ldots$.

Note that by **(A.1)**, $\sigma_L(u)$ is an $\mathcal{F}_t^W$-stopping time for each L, so that the stochastic integral in (1.3) is well-defined.

The existence and local uniqueness of the solution to (0.1) is given by the following localization procedure. Let $u^{(L)}$ be the solution of

$$\begin{aligned} &\frac{\partial u^{(L)}}{\partial t} = \frac{\partial^2 u^{(L)}}{\partial x^2} + |u^{(L)} \wedge L|^\gamma \dot{W} \\ &u^{(L)}(t, 0) = u^{(L)}(t, J) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J \\ &u^{(L)}(0, \cdot) = u_0; \end{aligned} \tag{1.4}$$

i.e., a random process with sample paths in $C_{E(J)}[0,\infty)$ which is $\mathcal{F}_t^W$-adapted and such that for all $t \geq 0$ and all φ in $C_0^2[0,J])$,

$$\int_0^J u^{(L)}(t,x)\varphi(x)dx = \int_0^J u_0(x)\varphi(x)dx + \int_0^t \int_0^J u^{(L)}(s,x)\frac{\partial^2 \varphi}{\partial x^2}(x)dsdx + \int_0^t \int_0^J \varphi(x)|u^{(L)}(s,x) \wedge L|^\gamma W(ds,dx). \tag{1.5}$$

Since $x \mapsto |x \wedge L|^\gamma$ is bounded and Lipschitz, (1.4) has a unique solution (see [18]). It is easily seen that for each L, $u^{(L)}$ satisfies (1.3) P-a.s. for all $t \geq 0$ and all φ in $C_0^2([0,J])$ and conversely that any solution u of (0.1) must agree with $u^{(L)}$ up to σ_L; more exactly

$$\sigma_L(u) = \sigma_L(u^{(L)}) \quad \text{and} \quad u(t,x) = u^{(L)}(t,x),$$

P-a.s. for all $0 \leq t \leq \sigma_L(u) = \sigma_L(u^{(L)})$ and all $0 \leq x \leq J$. Thus the solution of (0.1) is unique up to the explosion time $\sigma(u)$ and satisfies

$$u(t,x) = \lim_{L\to\infty} u^{(L)}(t,x). \qquad P-a.s., \quad 0 \leq t \leq \sigma(u), \quad 0 \leq x \leq J \tag{1.6}$$

Arguments of Mueller and Shiga (see [11] and [15]) show that for each L, $u^{(L)}$ is nonnegative when u_0 is nonnegative, so the solution u of (0.1) must also be nonnegative up to $\sigma(u)$; i.e. $u(t,x) \geq 0$ for all $0 \leq t \leq \sigma(u)$ and all $0 \leq x \leq J$.

We can now state our main result;

Theorem 1. *Let u be the solution of (0.1) where γ is much larger than $3/2$ and the initial condition u_0 in $E_+(J)$ is not identically zero. Then $P\{\sigma(u) < \infty\} > 0$.*

In our proofs we shall have reason to consider a class of SPDE's of a form slightly more general than (0.1). Unfortunately, the solutions of these auxiliary SPDE's are not known to be unique. We shall form a solution of (0.1) out of these auxiliary SPDE's, and to connect the resulting estimates to an estimate of (0.1), we need the following fact:

Proposition 1.1 *The solution of (0.1) is weakly unique up to explosion—i.e., if u_1 and u_2 are solutions respectively of equation*

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} + (u_1)^\gamma \dot{W}_1 \\ u_1(t,0) &= u_1(t,J) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J \\ u_1(0,\cdot) &= \varphi \end{aligned} \tag{1.7}$$

and

$$
\begin{aligned}
\frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} + (u_2)^\gamma \dot{W}_2 \\
u_2(t,0) &= u_2(t,J) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J \quad (1.8) \\
u_2(0,\cdot) &= \varphi
\end{aligned}
$$

where W_1 and W_2 are two white noises on $\mathcal{B}(\mathbb{R}_+ \times [0,J])$ and φ is some deterministic function in $E_+(J)$, then the law of $\{u_1(t,x) : 0 \leq t < \sigma(u_1), 0 \leq x \leq J\}$ is the same as the law of $\{u_2(t,x) : 0 \leq t < \sigma(u_2), 0 \leq x \leq J\}$.

Proof. Construct finite-difference approximations u_1^n and u_2^n of (1.7) and (1.8) as in [11], Section 3. By well-known results on the weak uniqueness of solutions of finite-dimensional SDE's, the laws of u_1^n and u_2^n will agree. By a calculation similar to that of [11], Theorem 3.2 and by the strong uniqueness up to explosion of (1.7) and (1.8), we then get that u_1 and u_2 have the same law up to explosion. We leave the details to the reader. ∎

2. Preliminary results.

In this section we gather some results and ideas, which form the bricks for building the ideas of Sections 3 and 4.

Our main line of argument will be to define a branching process in which the individuals in the population correspond to large peaks of the solution. We will wish to follow the evolution of these peaks by considering equations similar to (0.1) but on a finer and finer scale. To carry out our calculations, however, we need to consider a slightly more general form of (0.1). Let ξ be a $E_+(J)$-valued random process independent of the sigma field $\vee_{t\geq 0}\mathcal{F}_t^W$, and which is left-continuous with right-hand limits. Let $b : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be the mapping

$$
b(x,y) := \sqrt{(|x|+y)^{2\gamma} - y^{2\gamma}}. \qquad x \in \mathbb{R}, \quad y \in \mathbb{R}_+
$$

We can then consider the SPDE

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + b(u,\xi) \dot{W} \\
u(t,0) &= u(t,J) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J \quad (2.1) \\
u(0,\cdot) &= u_0.
\end{aligned}
$$

where the initial condition u_0 is continuous and nonnegative but not identically zero. If $\xi \equiv 0$, we have (0.1). Analogously to (0.1), for any $\mathcal{F}_t^W \vee \sigma\{\xi\}$-stopping time τ, we define a *solution* of (2.1) up to τ to be a random process with sample paths in $C_{E^\infty(J)}[0,\infty)$ such that

(B.1) $u(t,\cdot)$ is $\mathcal{F}_t^W \vee \sigma\{\xi\}$-measurable for each $t \geq 0$ (i.e., the process $t \mapsto u(t,\cdot)$ is $\mathcal{F}_t^W \vee \sigma\{\xi\}$-adapted)

(B.2) for all $t \geq 0$ and all φ in $C_0^2([0,J])$,

$$\begin{aligned}
&\int_0^J u(t \wedge \sigma_L(u) \wedge \sigma_L(\xi) \wedge \tau, x)\varphi(x)dx \\
&\qquad = \int_0^J u_0(x)\varphi(x)dx \\
&\qquad\quad + \int_0^t \int_0^J u(s,x)\frac{\partial^2 \varphi}{\partial x^2}(x)\chi_{\{s \leq \sigma_L(u)\wedge\sigma_L(\xi)\wedge\tau\}}dsdx \\
&\qquad\quad + \int_0^t \int_0^J \varphi(x)b(u(s,x),\xi(s,x))\chi_{\{s \leq \sigma_L(u)\wedge\sigma_L(\xi)\wedge\tau\}}W(ds,dx)
\end{aligned} \tag{2.2}$$

P-a.s. for each $L = 1, 2, \ldots$, where $\sigma_L(u)$ and $\sigma_L(\xi)$ are defined as in (1.1).

We say that u is a solution of (2.1) if it is a solution up to the stopping time $\tau \equiv \infty$. The following information about (2.2) is relevant to our study.

Lemma 2.1. *The SPDE (2.2) has a solution u up to $\sigma(u) \wedge \sigma(\xi)$ which is nonnegative up to $\sigma(u) \wedge \sigma(\xi)$.*

Proof. We give only the basic idea of the proof. We can define a nonnegative continuous-time, spatially discrete approximation u^n of (2.1) as in [11], Section 3. We may furthermore assume that u_n is stopped at the moment it reaches some height, which we denote by L. We may also assume that ξ is bounded by L. Under the assumption that u and ξ are bounded, we can show, by using some estimates on the regularity of the discrete and continuous time heat kernel, that the laws of the u_n's are tight and that the L^2 norm of the u_n's is bounded. An appropriate weak compactness argument as in [14], Chapter 3.1 gives us that we may find a subsequence u_n which weakly converges to some u which is $\mathcal{F}_t^W \vee \sigma\{\xi\}$-adapted. By the aforementioned tightness of the laws of the u_n's, u must be continuous, and furthermore we may pass to the limit in the integral equations corresponding to (2.2), showing that u will indeed solve (2.2). We then pass to the limit in L, which allows us to continue the solution to $\sigma(u) \wedge \sigma(\xi)$. Note that we have not implied that the solution of (2.2) is unique (see [15]). ∎

To develop the machinery needed for analyzing the structure of the peaks, let us define for each finite interval $I := [a,b]$ the fundamental solution of the heat equation on I with Dirichlet boundary conditions;

$$\frac{\partial G}{\partial t}(t,x,y;I) = \frac{\partial^2 G}{\partial x^2}(t,x,y;I)$$
$$G(t,a,y;I) = G(t,b,y;I) = 0 \qquad t \geq 0, \quad 0 \leq x \in I$$
$$G(0,\cdot,y;I) = \delta_y.$$

We shall need the following comparison results, which will give us some uniformity in our bounds;

Lemma 2.2. *For any $J > 0$ and any interval $I \subset [0,J]$,*

$$G(t,x,y;I) \leq G(t,x,y;[0,J]) \leq \frac{\exp\left[-\frac{(x-y)^2}{4t}\right] - \exp\left[-\frac{(x+y)^2}{4t}\right]}{\sqrt{4\pi t}} \leq \frac{\exp\left[-\frac{(x-y)^2}{4t}\right]}{\sqrt{4\pi t}}$$

for all $t > 0$ and all x and y in I.

Proof. We use the maximum principle and the nonnegativity of Green's functions. For any nonnegative function φ in $C(I)$,

$$\int_I G(t,x,z;I)\varphi(z)dz \leq \int_I G(t,x,z,[0,J])\varphi(z)dz$$

by the maximum principle and the positivity of $G(\cdot,\cdot,\cdot;[0,J])$. Now let φ tend in the distributional sense to δ_y for any y. This gives us the first inequality in the statement of the lemma. The second inequality follows from similar reasoning, by comparing the interval $[0,J]$ to $[0,\infty)$. The final inequality is obvious. ∎

For our later convenience, let us define

$$\bar{G}(t,x,z) := \frac{\exp\left[-\frac{(x-y)^2}{4t}\right]}{\sqrt{4\pi t}}$$

for all $t > 0$ and all x and y in $\mathbb{R}$.

We shall analyze the nonnegative solutions u of (2.1) by integrating it against a time-reversed heat kernel; define

$$\phi(t,x;y,J) := G(2-t,x,y;[0,J]) \qquad 0 \leq t \leq 2, \quad x \in [0,J]$$

for each $J > 0$ and each y in $[0,J]$. The following result will allow us to compare u to a solution of (0.2).

Lemma 2.3 *Take any y in $[0,J]$. Define*

$$U_t := \int_0^J \phi(t,x;y,J)u(t,x)dx. \qquad 0 \leq t < \sigma(u) \wedge \sigma(\xi) \wedge 1 \quad (2.3)$$

Then for each L, the process U^L given by

$$U_t^L := U_{t\wedge\sigma_L(u)\wedge\sigma_L(\xi)\wedge 1} \qquad t \geq 0, \qquad (2.4)$$

is a nonnegative, continuous $\mathcal{F}_t^W$-martingale with quadratic variation $\langle U^L\rangle$ satisfying

$$d\langle U^L\rangle_t \geq c(U_t^L)^{2\gamma}dt \qquad (2.5)$$

for all $0 \leq t \leq \sigma_L(u)\wedge\sigma_L(\xi)\wedge 1$, where c is a constant which does not depend upon y, γ, J, or L.

Proof. By using (2.2), we can easily argue that

$$U_t^L = \int_0^t\int_0^J \phi(s,x;y,J)b(u(s,x),\xi(s,x))\chi_{\{s\leq\sigma_L(u)\wedge\sigma_L(\xi)\}}W(ds,dx) + \int_0^J \phi(0,x;y,J)u_0(x)dx \qquad t\geq 0$$

for all $t \geq 0$. Thus U^L is an $\mathcal{F}_t^W$-martingale with quadratic variation

$$\langle U^L\rangle_t = \int_0^t\int_0^J \phi^2(s,x;y,J)b^2(u(s,x),\xi(s,x))\chi_{\{s\leq\sigma_L(u)\wedge\sigma_L(\xi)\}}dsdx. \qquad t\geq 0$$

Now since u and ξ are assumed to be nonnegative, clearly for all x and all $0 \leq t < \sigma(u)\wedge\sigma(\xi)$,

$$(u(t,x)+\xi(t,x))^{2\gamma} \geq (u(t,x))^{2\gamma} + (\xi(t,x))^{2\gamma}$$

and hence by the definition of b,

$$b^2(u(t,x),\xi(t,x)) \geq (u(t,x))^{2\gamma}.$$

By an application of Jensen's inequality,

$$\left(\int_0^J \phi(s,x;y,J)u(s,x)dx\right)^{2\gamma} \leq \left(\int_0^J \phi(s,x;y,J)^{1-(2\gamma-1)^{-1}}dx\right)^{2\gamma-1}\int_0^J \phi^2(s,x;y,J)u^{2\gamma}(s,x)dx$$

for all $0 \leq s \leq \sigma_L(u)\wedge 1$. But now by Lemma 2.2, we have that for all $0 \leq s \leq 1$

$$\begin{aligned}&\left(\int_0^J \phi(s,x;y,J)^{1-(2\gamma-1)^{-1}}dx\right)^{2\gamma-1}\\ &\leq \left\{\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{4\pi(2-s)}}\exp\left[-\frac{(x-y)^2}{4\pi(2-s)}\right]\right)^{1-(2\gamma-1)^{-1}}dx\right\}^{2\gamma-1}\\ &= \frac{\sqrt{4\pi(2-s)}}{(1-1/(2\gamma-1))^{(2\gamma-1)/2}}.\end{aligned}$$

But since $\eta := \inf_{0<x\leq 1/2}(1-x)^{1/x}$ is positive, as is easily checked, we have that

$$\left(\int_0^J \phi(s,x;y,J)^{1-(2\gamma-1)^{-1}}\,dx\right)^{2\gamma-1} \leq \sqrt{\frac{8\pi}{\eta}}$$

for all $0 \leq s \leq 1$. Thus

$$\begin{aligned} d\langle U^L\rangle_t &\geq \left(\int_0^J \phi^2(t,x;y,J)u^{2\gamma}(t,x)dx\right)^{2\gamma} \chi_{\{t\leq\sigma_L(u)\wedge\sigma_L(\xi)\}}dxdt \\ &\geq \sqrt{\frac{\eta}{8\pi}}\left(\int_0^J \phi(t,x;y,J)u(t,x)dx\right)^{2\gamma} \chi_{\{t\leq\sigma_L(u)\wedge\sigma_L(\xi)\}}dt \\ &\geq \sqrt{\frac{\eta}{8\pi}}(U_t^L)^{2\gamma}\chi_{\{t\leq\sigma_L(u)\wedge\sigma_L(\xi)\}}dt. \end{aligned}$$

This completes the proof if we set

$$c := \sqrt{\frac{\inf_{0<x\leq 1/2}(1-x)^{1/x}}{8\pi}}.$$

■

If a high peak exists, then we shall describe the next stage of our process by focusing in on the region of the high peak. The following rescaling lemma will allow us to do this;

Lemma 2.4 *Suppose that u solves (2.1) up to some $\mathcal{F}_t^W \vee \sigma\{\xi\}$ stopping time τ. Let $L > 0$. Then let*

$$\tilde{v}(t,x) := L^{-1}u\left(tL^{4(1-\gamma)}, xL^{2(1-\gamma)}\right) \qquad t\geq 0, \quad 0\leq x\leq JL^{2(\gamma-1)}$$

solves

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} &= \frac{\partial^2 \tilde{v}}{\partial x^2} + b(\tilde{v},\tilde{\xi})\dot{\tilde{W}} \\ \tilde{v}(t,0) &= \tilde{v}\left(t, JL^{2(\gamma-1)}\right) = 0 \qquad t\geq 0, \quad 0\leq x\leq JL^{2(\gamma-1)} \\ \tilde{v}(0,\cdot) &= \tilde{v}_0 \end{aligned}$$

up to the $\mathcal{F}_t^{\tilde{W}} \vee \sigma\{\xi\}$-stopping time $\tau L^{2(1-\gamma)}$, where $\tilde{v}_0(x) := L^{-1}u_0\left(xL^{2(1-\gamma)}\right)$ for all $0 \leq x \leq JL^{2(\gamma-1)}$, $\tilde{\xi}(t,x) = L^{-1}\xi(tL^{4(1-\gamma)}, xL^{2(1-\gamma)})$ for all $t \geq 0$ and $0 \leq x \leq JL^{2(\gamma-1)}$, and $\tilde{W}$ is the white noise on $\mathcal{B}(\mathbb{R}_+ \times [0, JL^{2(\gamma-1)}])$ defined by

$$\tilde{W}(A) := L^{3(\gamma-1)}\int_{\mathbb{R}_+\times[0,J]} \chi_A(tL^{4(\gamma-1)}, xL^{2(\gamma-1)})W(dt,dx)$$

for all A in $\mathcal{B}(\mathbb{R}_+ \times [0, JL^{2(\gamma-1)}])$ with finite Lebesgue measure.

Proof. Write out (2.2) and carry out the obvious rescaling arguments. ∎

The next stage of the branching process will come from splitting up the mass of a scaled large peak, in such a way that each piece will evolve independently. The following idea will be at the heart of this.

Lemma 2.5. *Consider the N recursively-defined equations*

$$\begin{aligned} \frac{\partial u^i}{\partial t} &= \frac{\partial^2 u^i}{\partial x^2} + b\left(u^i, \sum_{j=1}^{i-1} u^j\right) \dot{W}^i \\ u^i(t,0) &= u^i(t,J) = 0 \\ u^i(0,\cdot) &= u_0^i. \end{aligned} \qquad t \geq 0, \quad 0 \leq x \leq J, \quad i = 1, 2, \ldots, N \tag{2.6}$$

where $u^0 \equiv 0$ by definition. Here the W^i's are independent white noises on $\mathcal{B}(\mathbb{R}_+ \times [0, J])$ and the u_0^i's are some collection of nonnegative initial conditions. We consider the solutions of the SPDE's (2.6) to be those given by Lemma 2.1. Let us then define the $E_+^\infty(J)$-valued process

$$\tilde{u}(t,\cdot) := \begin{cases} \sum_{i=1}^N u^i(t,\cdot) & \text{for } 0 \leq t < \min\{\sigma(u^i) : i = 1, 2, \ldots, N\} \\ \infty & \text{otherwise} \end{cases}$$

all $t \geq 0$. Then $\tilde{u}$ is a solution of

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial^2 \tilde{u}}{\partial x^2} + (\tilde{u})^\gamma \dot{\tilde{W}} \\ \tilde{u}(t,0) &= \tilde{u}(t,J) = 0 \\ \tilde{u}(0,\cdot) &= \sum_{i=1}^N u_0^i \end{aligned} \qquad t \geq 0, \quad 0 \leq x \leq J$$

for some white noise $\tilde{W}$ which is a linear combination of the W^i's.

In some sense one might think of the u^i's as 'subsolutions' of the SPDE (0.1).

Proof. By the recursive nature of the Lemma, it sufficies to prove the result for $N = 2$. Then we can set

$\tilde{W}(A)$

$$:= \int_{(t,x)\in A} \chi_{\{0<u^1(t,x)+u^2(t,x),\, t<\sigma(u^1)\wedge\sigma(u^2)\}} \sqrt{\frac{\left(u^1(t,x)+u^2(t,x)\right)^{2\gamma} - \left(u^1(t,x)\right)^{2\gamma}}{\left(u^1(t,x)+u^2(t,x)\right)^{2\gamma}}} W^1(dt,dx)$$

$$+\int_{(t,x)\in A} \chi_{\{0<u^1(t,x)+u^2(t,x),\, t<\sigma(u^1)\wedge\sigma(u^2)\}} \left(\frac{u^1(t,x)}{u^1(t,x)+u^2(t,x)}\right)^\gamma W^2(dt,dx)$$

$$+\int_{(t,x)\in A} \chi_{\{u^1(t,x)+u^2(t,x)=0 \ or \ t\geq\sigma(u^1)\wedge\sigma(u^2)\}} W^1(dt,dx)$$

for all sets A in $\mathcal{B}(\mathbb{R}_+\times[0,J])$ with finite Lebesgue measure. It is not difficult to see that $\tilde{W}$ is a white noise. One should consider the covariance structure of $\tilde{W}$, and one can use Levy's martingale characterization of Brownian motion (see [6], Section 3.3.3.B) to see that the rectangular increments of $\tilde{W}$ are Gaussian. This is sufficient. One can then prove the result in the expected way. ∎

We finish this section by deriving several bounds on the heat kernel with Dirichlet conditions. These estimates will be necessary for the transitions between steps of the branching process.

Lemma 2.6. *Define*

$$\alpha_1 := \min\left\{\inf_{-1/2\leq x\leq 1/2} G(2,x,0;[-1,1]),\ \inf_{-1\leq x\leq 1} G(2,x,0;[-2,2])\right\}$$

and

$$\alpha_2 := \inf_{0<x\leq 1} \frac{G(2,x,1;[0,2])}{x}.$$

Then the lower bounds α_1 and α_2 are positive. We also have corresponding finite upper bounds

$$\alpha_1' := \sup_{\substack{0\leq x\leq J\\ 0\leq y\leq J\\ J\geq 2\\ t\geq 1}} G(t,x,y;[0,J])$$

and

$$\alpha_2' := \sup_{\substack{0<x\leq 1\\ 1\leq y\leq J\\ J\geq 2\\ t\geq 1}} \frac{G(t,x,y;[0,J])}{x}.$$

Proof. The positivity of α_1 and α_2 are consequences of strong maximum principles for parabolic PDE's. They can be deduced from Theorems 2 and 3 of Chapter 3 of [13]. The assertions concerning α_1' and α_2' can be deduced from the comparison results of Lemma 2.2 and some obvious calculations. We leave the details of these arguments to the reader. ∎

3. Characterizing peaks in the solution and studying their propagation.

In this section we use the results of Section 2 to characterize peaks of high mass in the solution of (0.1). We then rescale the SPDE to obtain a finer resolution. We finally show how a peak of large mass in the original SPDE can be broken up into peaks of smaller mass which allow us to restart our arguments. In Section 4 we then rigorously define a branching process in which peaks correspond to individuals in a population. If γ is much larger than $3/2$, then this branching process will survive with positive probability, and this will correspond to blowup of the solution u of (0.1).

First let us define several constants whose use will be apparent later. Define

$$M := \max\left\{\frac{4\alpha_2'\sqrt{4\pi}}{\alpha_2}, \frac{4\alpha_1'\sqrt{4\pi}}{\alpha_1}\right\} \tag{3.1}$$

where α_1, α_2, α_1' and α_2' are as in Lemma 2.7, and

$$K := M\left(4 + \frac{4}{\sqrt{4\pi}}\right) \tag{3.2}$$

With these constants in hand, we now begin our arguments.

As a preliminary remark, we may use the scaling of Lemma 2.4, to assume that the interval $[0, J]$ satisfies $J \geq 4$. We shall also assume that

$$\int_0^J \phi(0, x; z, J)u_0(x)dx \geq 2 \tag{3.3}$$

for some z in $[1, J-1]$. Later we will show that if u_0 is not identically zero, then with positive probability we can restart the process, by virtue of the Markov property, assuming (3.3) to be true.

Now we describe how a large peak in u the solution of (2.1) gives rise to even higher peaks. Letting u be a solution of (2.1), we define

$$U_t := \int_0^J \phi(t, x; z, J)u(t, x)dx. \qquad 0 \leq t < 1 \wedge \sigma(u) \wedge \sigma(\xi) \tag{3.4}$$

Let us consider the random time

$$\tau := \begin{cases} \inf\{t > 0 : U_t > K\} & \text{if } \sup_{0 \leq t < 1 \wedge \sigma(u) \wedge \sigma(\xi)} U_t > K \\ \infty & \text{otherwise,} \end{cases} \tag{3.5}$$

i.e., the first time at which U_t reaches the high level K. This corresponds to a large peak in u, since

$$\int_0^J \phi(t, x; z, J)dx \leq 1,$$

as one can easily show using Lemma 2.2. We are interested in showing that $P\{\tau\wedge\sigma(u)\wedge\sigma(\xi)\leq 1\}$ is not too small; i.e., we want to bound from below the probability that before time 1 either U reaches a high level or u or ξ blows up. By Lemma 2.3, we can compare U to a time-changed Brownian motion $\tilde{B}$ with the estimate (2.5) giving us a bound on the time change. With this, we can use some level-crossing probabilities for a standard Brownian motion to bound $P\{\tau\wedge\sigma(u)\wedge\sigma(\xi)\leq 1\}$ from below. To emphasize some uniformity of the lower bound, we shall let B be a standard Brownian motion starting at 0.

We begin by defining $\{U_t^L : t\geq 0\}$ as in (2.3)–(2.4) where u is now a nonnegative solution of the more general equation (2.1). For convenience, set

$$\zeta := U_0 \tag{3.6}$$

and $c' := c(1\wedge(\zeta/2)^{2\gamma})$ and $x' := 1\wedge(\zeta/2)$, where c is as in Lemma 2.3. We simply note in passing that our definitions of c' and x' are made to handle several usages (see Proposition 3.1 and the comments at the end of Section 4 for relaxing the requirement (3.3)). The important relationship between c' and x' is that $c'\geq c$ and

$$c(x')^{2\gamma}\geq c'. \tag{3.7}$$

By Lemma 2.3, U^L is a continuous martingale and hence a time-changed Brownian motion. Thus there is a Brownian motion $\tilde{B}^L$ and a random nondecreasing mapping $\rho^L:\mathbb{R}_+\to\mathbb{R}_+$ with

$$\dot{\rho}^L(t)\geq c(U_t^L)^{2\gamma}\chi_{\{t\leq\sigma_L(u)\wedge\sigma_L(\xi)\wedge 1\}} \tag{3.8}$$

such that $U_t^L=\tilde{B}^L_{\rho^L(t)}$ for all $0\leq t\leq\sigma_L(u)\wedge\sigma_L(\xi)\wedge 1$ (see [9]). Consider now the set $\{\sup_{0\leq t\leq c'}\tilde{B}^L>K, \inf_{0\leq t\leq c'}\tilde{B}^L>x', \sigma_L(\xi)>1\}$. On this set, the time change should be bounded from below, since $\tilde{B}_t^L>x'$ for all $0\leq t\leq c'$, and thus $\tilde{B}^L$ reaching the level K before time c' should imply that U^L reaches K in a similar time interval, so τ cannot be too large. The exact result we shall use is:

Lemma 3.1 *Setting*

$$\tau_L := \begin{cases}\inf\{t>0: U_t^L>K\} & \textit{if } \sup_{t\geq 0}U_t^L>K\\ \infty & \textit{otherwise}\end{cases}$$

for each $L=1,2,\ldots$, we have that

$$\left\{\sup_{0\leq t\leq c'}\tilde{B}_t^L>K,\ \inf_{0\leq t\leq c'}\tilde{B}_t^L>x'\right\}\subset\{\tau_L\wedge\sigma_L(u)\wedge\sigma_L(\xi)\leq 1\}$$

for each $L = 1, 2, \ldots$, *and thus*

$$P\{\tau \wedge \sigma(u) \wedge \sigma(\xi) \leq 1\} \geq \lim_{L\to\infty} P\left\{ \sup_{0\leq t\leq c'} \tilde{B}_t^L > K,\ \inf_{0\leq t\leq c'} \tilde{B}_t^L > x' \right\}.$$

Proof. It is easy to see that for each $L = 1, 2, \ldots$,

$$\tau_L \wedge \sigma_L(u) \wedge \sigma_L(\xi) \leq \tau_{L+1} \wedge \sigma_{L+1}(u) \wedge \sigma_{L+1}(\xi) \leq \tau \wedge (u) \wedge \sigma(\xi).$$

Hence

$$\{\tau_{L+1} \wedge \sigma_{L+1}(u) \wedge \sigma_{L+1}(\xi) \leq 1\} \subset \{\tau_L \wedge \sigma_L(u) \wedge \sigma_L(\xi) \leq 1\} \subset \{\tau \wedge \sigma(u) \wedge \sigma(\xi) \leq 1\}$$

and thus

$$P\{\tau \wedge \sigma(u) \wedge \sigma(\xi) \leq 1\} \geq \lim_{L\to\infty} P\{\tau_L \wedge \sigma_L(u) \wedge \sigma_L(\xi) \leq 1\}.$$

Now for each $L = 1, 2, \ldots$, we have that

$$\left\{ \sup_{0\leq t\leq c'} \tilde{B}_t^L > K,\ \inf_{0\leq t\leq c'} \tilde{B}_t^L > x' \right\} \subset \{\sigma_L(u) \wedge \sigma_L(\xi) \leq 1\}$$
$$\cup \left\{ \sup_{0\leq t\leq c'} \tilde{B}_t^L > K,\ \inf_{0\leq t\leq c'} \tilde{B}_t^L > x',\ \sigma_L(u) \wedge \sigma_L(\xi) > 1 \right\}.$$

Consider now the case where $\inf_{0\leq t\leq c'} \tilde{B}_t^L > x'$ and $\sigma_L(u) \wedge \sigma_L(\xi) > 1$. We argue that then $\rho_L(1) \geq c'$. Assuming the contrary, that $\rho_L(1) < c'$, then by the mean value theorem,

$$\dot{\rho}_L(s) = \rho_L(1) - \rho_L(0) < c' \tag{3.9}$$

for some s in $(0, 1)$. But under the assumption that $\inf_{0\leq t\leq c'} \tilde{B}_t^L > x'$ and $\sigma_L(u) \wedge \sigma_L(\xi) > 1$, we would then have by (3.7) and (3.8) that

$$\dot{\rho}^L(s) \geq c(\tilde{B}^L_{\rho_L(s)})^{2\gamma} \chi_{\{s\leq\sigma_L(u)\wedge\sigma_L(\xi)\}} \geq c\left(\inf_{0\leq t\leq c'} \tilde{B}_t^L\right)^{2\gamma} \geq c',$$

which contradicts (3.9). Thus indeed $\rho_L(1) \geq c'$ if $\inf_{0\leq t\leq c'} \tilde{B}_t^L > x'$ and $\sigma_L(u) \wedge \sigma_L(\xi) > 1$. Hence if also $\sup_{0\leq t\leq c'} \tilde{B}_t^L > K$, then

$$\sup_{0\leq t\leq 1} U_t^L = \sup_{0\leq t\leq \rho_L(1)} \tilde{B}_t^L \geq \sup_{0\leq t\leq c'} \tilde{B}_t^L > K.$$

Note that if $\sigma_L(u) \wedge \sigma_L(\xi) > 1$ and $\sup_{0\le t\le 1} U_t^L > K$, then clearly $\tau_L \le 1$. Hence the inclusion

$$\left\{ \sup_{0\le t\le c'} \tilde{B}_t^L > K, \inf_{0\le t\le c'} \tilde{B}_t^L > x', \sigma_L(u) \wedge \sigma_L(\xi) > 1 \right\} \subset \{\tau_L \le 1\},$$

so indeed we can see that

$$\begin{aligned} \left\{ \sup_{0\le t\le c'} \tilde{B}_t^L > K, \inf_{0\le t\le c'} \tilde{B}_t^L > x' \right\} &\subset \{\sigma_L(u) \wedge \sigma_L(\xi) \le 1\} \cup \{\tau_L \le 1\} \\ &\subset \{\sigma_L(u) \wedge \sigma_L(\xi) \wedge \tau_L \le 1\}. \end{aligned}$$

■

From this lemma we get

Proposition 3.1. *Defining U as in (3.4) and ζ as in (3.6), we have that*

$$P\{\tau \wedge \sigma(u) \wedge \sigma(\xi) \le 1\} \ge \beta(\zeta)$$

where for each $x > 0$, $\beta(x) > 0$ is defined as

$$\beta(x) := P\left\{ \sup_{0\le t\le c(1\wedge (x/2)^{2\gamma})} (B_t + x) > K, \inf_{0\le t\le c(1\wedge (x/2)^{2\gamma})} (B_t + x) > 1 \wedge (x/2) \right\}.$$

This is our desired bound on $P\{\tau \wedge \sigma(u) \wedge \sigma_L(\xi) \le 1\}$. Note for future reference that

$$\inf_{2\le x\le K} \beta(x) \ge \bar{\beta} \tag{3.10}$$

where

$$\bar{\beta} := P\left\{ \sup_{0\le t\le c} B_t > K-2, \inf_{0\le t\le c} B_t > -1 \right\} \tag{3.11}$$

since

$$\left\{ \sup_{0\le t\le c} B_t > K-2, \inf_{0\le t\le c} B_t > -1 \right\} \subset \left\{ \sup_{0\le t\le c} (B_t + x) > K, \inf_{0\le t\le c} (B_t + x) > 1 \wedge (x/2) \right\} \tag{3.12}$$

for all $2 \le x \le K$. Standard results give us the following exact formulae for $\bar{\beta}$ and $\beta(x)$ with $0 < x < 2$;

Lemma 3.2. *We have that*

$$\bar{\beta} = \int_0^c \int_0^{K-1} \frac{K-2}{\pi(c-s)s^3} \exp\left[-\frac{w^2}{2(c-s)} - \frac{(K-2)^2}{2s}\right] dwds$$

and that for $0 < x < 2$,

$$\beta(x) = \int_0^{c(x/2)^{2\gamma}} \int_0^{K-(x/2)} \frac{K-x}{\pi(c(x/2)^{2\gamma}-s)s^3} \exp\left[-\left(\frac{w^2}{2(c(x/2)^{2\gamma}-s)} + \frac{(K-x)^2}{2s}\right)\right] dwds.$$

Proof. For any $a < 0 < b$ and any $t' > 0$, we have that

$$P\left\{\sup_{0\le t\le t'} B_t > b,\ \inf_{0\le t\le t'} B_t > a\right\} = P\{\tau_a > t',\ \tau_b < t'\}$$

where

$$\tau_a := \begin{cases} \inf\{t > 0 : B_t < a\} & \text{if } \inf_{t\ge 0} B_t < a \\ \infty & \text{otherwise} \end{cases}$$

and

$$\tau_b := \begin{cases} \inf\{t > 0 : B_t > b\} & \text{if } \sup_{t\ge 0} B_t > b \\ \infty & \text{otherwise.} \end{cases}$$

Now, using $\{\mathcal{G}_t : t \ge 0\}$ to denote the filtration generated by B,

$$\begin{aligned} P\{\tau_a > t',\ \tau_b < t'\} &= P\{\tau_a - \tau_b > t' - \tau_b,\ \tau_b < t'\} \\ &= E\left[P\{\tau_a - \tau_b > t' - \tau_b | \mathcal{G}_{\tau_b}\}\chi_{\{\tau_b < t'\}}\right] \\ &= E\left[\int_0^{b-a} \frac{2}{\sqrt{2\pi(t'-\tau_b)}} \exp\left[-\frac{w^2}{2(t'-\tau_b)}\right] dw \chi_{\{\tau_b<t'\}}\right] \\ &= \int_0^{t'} \int_0^{b-a} \frac{2}{\sqrt{2\pi(t'-s)}} \exp\left[-\frac{w^2}{2(t'-s)}\right] dw \frac{b}{\sqrt{2\pi s^3}} \exp\left[-\frac{b^2}{2s}\right] ds. \end{aligned}$$

We have used here Problem 8.2, Theorem 6.16, and Proposition 8.5 of Chapter 2 of [6]. Inserting $a = -1$, $b = K - 2$ and $t' = c$ into this, we get the formula for $\bar{\beta}$. Inserting $a = -(x/2)$, $b = K - x$, and $t' = c(x/2)^{2\gamma}$ into this, we get the formula for $\beta(x)$ with $0 < x < 2$. ∎

Now let us consider how we can start the whole process again if $\tau \wedge \sigma(u) \le 1$ and $\sigma(\xi) > 1$. If $\tau \wedge \sigma(u) = \sigma(u) \le 1$, then the solution u of (2.1) has already blown up. If $\tau \wedge \sigma(u) = \tau \le 1$, then we use the strong Markov property of the solution of (2.1) along with the scaling result of Lemma 2.4

to allow us to repeat our arguments at a finer scale of resolution. More clearly, if $\tau \wedge \sigma(u) = \tau \leq 1$, set

$$v_0(x) := \frac{1}{2} u\left(\tau, x 2^{2(1-\gamma)}\right). \qquad 0 \leq x \leq J 2^{2(\gamma-1)} \quad (3.13)$$

Then by the Markov property of the solution to (0.1) and Lemma 2.4, we have that the conditional law of the process $\{(1/2)u(t2^{4(1-\gamma)} + \tau, x2^{2(1-\gamma)}) : t \geq 0, 0 \leq x \leq J2^{2(\gamma-1)}\}$, given $\mathcal{F}_\tau^W$, is the same up to explosion as the law of a solution of

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + b(v, \tilde{\xi})\dot{\tilde{W}} \\ v(t,0) &= v(t, J2^{2(\gamma-1)}) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J2^{2(\gamma-1)} \\ v(0,\cdot) &= v_0 \end{aligned}$$

where $\tilde{\xi}(t,x) := 2^{-1}\xi(t2^{4(1-\gamma)}, x2^{2(1-\gamma)})$ for all $t \geq 0$ and $0 \leq x \leq J2^{2(\gamma-1)}$, and with $\tilde{W}$ a white noise on $\mathbb{R}_+ \times [0, J2^{2(\gamma-1)}]$ independent of $\tilde{\xi}$. By the transformation (3.13) we have made the peaks smaller but wider. By the definition of τ and an obvious change of variables, we have that

$$\int_0^{J2^{2(\gamma-1)}} \phi\left(\tau, x2^{2(1-\gamma)}; z, J\right) v_0(x) dx > K 2^{2\gamma-3}. \quad (3.14)$$

Now an easy argument based upon Lemma 2.2 gives us that

$$\int_0^{J2^{2(\gamma-1)}} \phi\left(t, x2^{2(1-\gamma)}; z, J\right) dx \leq 2^{2(\gamma-1)} \quad (3.15)$$

for all $0 \leq t \leq 1$, so by comparing (3.14) and (3.15) we see that the mass of v_0 cannot be too small. We wish to split the mass of v_0 into a collection of smaller masses, each of which satisfies (3.3) with J replaced by $J2^{2(\gamma-1)}$ and z replaced by some appropriate point in $[1, \bar{J} - 1]$. Doing this will enable us to repeat the arguments starting with (3.3), allowing us to study the propagation of peaks.

The lemma we need to break up the mass of v_0 is the following one, which shows us that if a function v_0 satisfies (3.14), then it can be broken up into no less than

$$N := \lfloor 2^{2\gamma-3} \rfloor \quad (3.16)$$

smaller pieces, each of which satisfies the appropriate modification of (3.3). Here $\lfloor\cdot\rfloor$ denotes the greatest integer function. We note for future reference that the number N is of critical importance to the construction and analysis of the branching process in Section 4.

Proposition 3.2. *Let $J > 4$ be fixed. Set $\bar{J} := J2^{2(\gamma-1)}$. If f_0 in $E_+^\infty(\bar{J})$ satisfies*

$$\int_0^{\bar{J}} \phi\left(t, x2^{2(1-\gamma)}; z_0, J\right) f_0(x)dx > K2^{2\gamma-3},$$

for some z_0 in $[1, J-1]$ and some $0 \leq t \leq 1$, then there are $\{f_i : i = 1, 2, \ldots, N\}$ in $E_+^\infty(\bar{J})$ such that

$$f_0 = \sum_{i=1}^{N} f_i$$

and for each $i = 1, 2, \ldots, N$,

$$\int_0^{\bar{J}} \phi(0, x; z_i, \bar{J}) f_i(x)dx \geq 2. \tag{3.17}$$

for some z_i in $[1, \bar{J}-1]$.

Proof. The f_i's will come from multiplying f by the elements of a partition of unity. In this proof, we define "cutoff functions" as those elements η of $C^\infty([0, \bar{J}])$ such that $0 \leq \eta(x) \leq 1$ for all x in $[0, \bar{J}]$. Also, for convenience we shall define

$$\bar{\phi}(x) := \phi\left(t, x2^{2(1-\gamma)}; z_0, J\right)$$

for all $0 \leq x \leq \bar{J}$ for any fixed but arbitrary t in $[0, 1]$. Finally, we set $\bar{z}_0 := z_0 2^{2(\gamma-1)}$; note that thus (recall that $\gamma \gg 3/2$)

$$\bar{z}_0 \in [2, \bar{J} - 2]. \tag{3.18}$$

We begin by breaking $[0, \bar{J}]$ into a collection of small intervals on which f_0 has mass at most M and width at most 1. Set $p_0 := \bar{z}_0$, $p_1 := \bar{z}_0 + 1$, $p_{-1} := \bar{z}_0 - 1$,

$$p_i := \begin{cases} \inf\{z > p_{i-1} : \int_{p_{i-1}}^{z} f_0(x)dx \geq M\} \wedge (p_{i-1} + 1) & \text{if } \int_{p_{i-1}}^{\bar{J}-1} f_0(x)dx \geq M \\ \bar{J} - 1 & \text{otherwise} \end{cases}$$

for all $i > 1$, and

$$p_i := \begin{cases} \sup\{z < p_{i+1} : \int_{z}^{p_{i+1}} f_0(x)dx \geq M\} \vee (p_{i+1} - 1) & \text{if } \int_{1}^{p_{i+1}} f_0(x)dx \geq M \\ 1 & \text{otherwise} \end{cases}$$

for all $i < -1$. Note that for all i,

$$1 \leq p_i \leq \bar{J} - 1. \tag{3.19}$$

We would like to count up respectively those intervals $[p_i, p_{i+1}]$ and $[p_{i-1}, p_i]$ for $i \geq 1$ and $i \leq -1$ respectively on which the mass of f_0 is M. We hope that there will be on the order of N such intervals. To do this, define $\mathcal{I}^+$ and $\mathcal{I}^-$ as those sets of respectively positive and negative integers i for which f_0 has mass M on respectively $[p_i, p_{i+1}]$ and $[p_{i-1}, p_i]$; i.e.,

$$\mathcal{I}^+ := \{i \geq 1 : \int_{p_i}^{p_{i+1}} f_0(x)dx \geq M\}$$

and

$$\mathcal{I}^- := \{i \leq -1 : \int_{p_{i-1}}^{p_i} f_0(x)dx \geq M\}.$$

The following lemma shows us that if i is in $\mathcal{I}^+$ or $\mathcal{I}^-$, we can define an f_i as in the statement of the Proposition by multiplying f by a cutoff function.

Lemma 3.3. *If i is in $\mathcal{I}^+$ or $\mathcal{I}^-$, then there exists a z_i in respectively $[p_i, p_{i+1}]$ or $[p_{i-1}, p_i]$ and a cutoff function η_i with support in respectively $[p_i, p_{i+1}]$ or $[p_{i-1}, p_i]$ such that*

$$\int_0^{\bar{J}} \phi(0, x; z_i, \bar{J}) f_0(x) \eta_i(x) dx \geq 2.$$

Proof. Set $z_i := (p_i + p_{i+1})/2$ if i is in $\mathcal{I}^+$ and $z_i := (p_{i-1} + p_i)/2$ if i is in $\mathcal{I}^-$. By (3.19), $1 \leq z_i \leq \bar{J} - 1$; note that thus $[z_i - 1, z_i + 1] \subset [0, \bar{J}]$ for all i in $\mathcal{I}^+$ or $\mathcal{I}^-$. Let η_i be a cutoff function with support in $(z_i - 1/2, z_i + 1/2)$ such that

$$\int_{z_i-1/2}^{z_i+1/2} \bar{\phi}(x) f_0(x) \eta_i(x) dx \geq \frac{1}{2} \int_{z_i-1/2}^{z_i+1/2} \bar{\phi}(x) f_0(x) dx. \tag{3.20}$$

By the comparison results of Lemma 2.2 and Lemma 2.6,

$$\begin{aligned}\int_{z_i-1/2}^{z_i+1/2} \phi(0, x; z_i, \bar{J}) f_0(x) \eta_i(x) dx &\geq \int_{z_i-1/2}^{z_i+1/2} G(2, x, z_i; [z_i - 1, z_i + 1]) f_0(x) \eta_i(x) dx \\ &\geq \alpha_1 \int_{z_i-1/2}^{z_i+1/2} f_0(x) \eta_i(x) dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{z_i-1/2}^{z_i+1/2} \bar{\phi}(x) f_0(x) \eta_i(x) dx &= \int_{z_i-1/2}^{z_i+1/2} G(2 - t, x2^{2(1-\gamma)}, z_0; [0, J]) f_0(x) \eta_i(x) dx \\ &\leq \alpha_1' \int_{z_i-1/2}^{z_i+1/2} f_0(x) \eta_i(x) dx.\end{aligned}$$

Combining these two inequalities and (3.20) and the definition of M, we get that

$$\begin{aligned}\int_{z_i-1/2}^{z_i+1/2} \phi(0,x;z_i,\bar{J}) f_0(x)\eta_i(x)dx &\geq \frac{\alpha_1}{\alpha_1'} \int_{z_i-1/2}^{z_i+1/2} \bar{\phi}(x) f_0(x)\eta_i(x)dx \\ &\geq \frac{\alpha_1}{2\alpha_1'} \int_{z_i-1/2}^{z_i+1/2} \bar{\phi}(x) f_0(x)dx \\ &\geq \frac{M\alpha_1}{2\alpha_1'} \\ &\geq 2.\end{aligned}$$

■

Now let us show that the cardinality $|\mathcal{I}^+|$ and $|\mathcal{I}^-|$ of respectively $\mathcal{I}^+$ and $\mathcal{I}^-$ must be on the order of $2^{2\gamma-3}$. We have

$$\begin{aligned}K2^{2\gamma-3} &\leq \int_0^{\bar{J}} \bar{\phi}(x) f_0(x)dx \\ &= \int_0^1 \bar{\phi}(x) f_0(x)dx + \int_{p_0-1}^{p_0+1} \bar{\phi}(x) f_0(x)dx + \int_{\bar{J}-1}^{\bar{J}} \bar{\phi}(x) f_0(x)dx \\ &\quad + \sum_{i\geq 1} \int_{p_i}^{p_{i+1}} \bar{\phi}(x) f_0(x)dx + \sum_{i\leq -1} \int_{p_{i-1}}^{p_i} \bar{\phi}(x) f_0(x)dx.\end{aligned} \tag{3.21}$$

Note from Lemma 2.6 that $\bar{\phi}(x) \leq (4\pi)^{-1/2}$ and, from Lemma 2.2, that

$$\begin{aligned}\bar{\phi}(x) &\leq \bar{G}\left(2-t, x2^{2(1-\gamma)}, z_0\right) \\ &= 2^{2(\gamma-1)}\bar{G}\left((2-t)2^{4(1-\gamma)}, x, \bar{z}_0\right).\end{aligned} \qquad 0 \leq x \leq J$$

Also, from the explicit formula for $\bar{G}$, if $x \leq x' \leq \bar{z}_0$,

$$\bar{G}\left((2-t)2^{4(1-\gamma)}, x, \bar{z}_0\right) \leq \bar{G}\left((2-t)2^{4(1-\gamma)}, x', \bar{z}_0\right)$$

whereas if $\bar{z}_0 \leq x \leq x'$,

$$\bar{G}\left((2-t)2^{4(1-\gamma)}, x, \bar{z}_0\right) \geq \bar{G}\left((2-t)2^{4(1-\gamma)}, x', \bar{z}_0\right).$$

These bounds give us that

$$\sum_{i\geq 1}\int_{p_i}^{p_{i+1}}\bar{\phi}(x)f_0(x)dx = \sum_{i\geq 1, i\in\mathcal{I}^+}\int_{p_i}^{p_{i+1}}\bar{\phi}(x)f_0(x)dx + \sum_{i\geq 1, i\notin\mathcal{I}^+}\int_{p_i}^{p_{i+1}}\bar{\phi}(x)f_0(x)dx$$
$$\leq \frac{M|\mathcal{I}^+|}{\sqrt{4\pi}} + M2^{2(\gamma-1)}\sum_{i\geq 1, i\notin\mathcal{I}^+}\bar{G}\left((2-t)2^{4(\gamma-1)}, p_i, \bar{z}_0\right)$$
$$\leq \frac{M|\mathcal{I}^+|}{\sqrt{4\pi}} + M2^{2(\gamma-1)}\sum_{i\geq 1, i\notin\mathcal{I}^+}\int_{p_i-1}^{p_i}\bar{G}\left((2-t)2^{4(\gamma-1)}, x, \bar{z}_0\right)dx.$$

Note also that if $1\leq i<j$ where neither i nor j is in $\mathcal{I}^+$, then

$$p_i + 1 = p_{i+1} \leq p_j$$

so $[p_i-1, p_i]$ and $[p_j-1, p_j]$ are disjoint. Thus since the heat kernel integrates to 1,

$$\sum_{i\geq 1}\int_{p_i}^{p_{i+1}}\bar{\phi}(x)f_0(x)dx \leq \frac{M|\mathcal{I}^+|}{\sqrt{4\pi}} + M2^{2(\gamma-1)}.$$

Similarly

$$\sum_{i\leq -1}\int_{p_{i-1}}^{p_i}\bar{\phi}(x)f_0(x)dx \leq \frac{M|\mathcal{I}^-|}{\sqrt{4\pi}} + M2^{2(\gamma-1)}.$$

Inserting these bounds into (3.21),

$$K2^{2\gamma-3} \leq \frac{M}{\sqrt{4\pi}}(|\mathcal{I}^+|+|\mathcal{I}^-|) + 2M2^{2(\gamma-1)}$$
$$+\int_0^1\bar{\phi}(x)f_0(x)dx + \int_{p_0-1}^{p_0+1}\bar{\phi}(x)f_0(x)dx + \int_{\bar{J}-1}^{\bar{J}}\bar{\phi}(x)f_0(x)dx$$

and hence

$$\frac{M}{\sqrt{4\pi}}(|\mathcal{I}^+|+|\mathcal{I}^-|) + \int_0^1\bar{\phi}(x)f_0(x)dx + \int_{p_0-1}^{p_0+1}\bar{\phi}(x)f_0(x)dx + \int_{\bar{J}-1}^{\bar{J}}\bar{\phi}(x)f_0(x)dx$$
$$\geq (K-4M)2^{2\gamma-3}.$$

Thus, modulo the integrals near p_0 and the left and right endpoints, $|\mathcal{I}^+|$ and $|\mathcal{I}^-|$ have the correct order of magnitude. We now need to consider the endpoints and the neighborhood of p_0. Define

$$I_l := \left\lfloor \frac{\sqrt{4\pi}}{M}\int_0^1\bar{\phi}(x)f_0(x)dx\right\rfloor$$

$$I_r := \left\lfloor \frac{\sqrt{4\pi}}{M} \int_{\bar{J}-1}^{\bar{J}} \bar{\phi}(x) f_0(x) dx \right\rfloor$$

and

$$I_c := \left\lfloor \frac{\sqrt{4\pi}}{M} \int_{p_0-1}^{p_0+1} \bar{\phi}(x) f_0(x) dx \right\rfloor$$

corresponding to the left, right, and center integrals. Then by a simple calculation,

$$\begin{aligned} \frac{M}{\sqrt{4\pi}}(|\mathcal{I}^+| + |\mathcal{I}^-| + I_l + I_r + I_c) &> (K - 4M)2^{2\gamma-3} - \frac{3M}{\sqrt{4\pi}} \\ &> \left(K - 4M - \frac{3M}{\sqrt{4\pi}}\right) 2^{2\gamma-3}. \end{aligned}$$

In the last line here, we have used the fact that $2\gamma - 3 > 0$. Thus

$$|\mathcal{I}^+| + |\mathcal{I}^-| + I_l + I_r + I_c \geq \sqrt{4\pi}\left(K/M - 4 - \frac{3}{\sqrt{4\pi}}\right) 2^{2\gamma-3}.$$

At this point our reason for defining K as in (3.2) is clear. We have that

$$|\mathcal{I}^+| + |\mathcal{I}^-| + I_l + I_r + I_c \geq 2^{2\gamma-3} \geq \lfloor 2^{2\gamma-3} \rfloor. \tag{3.22}$$

In Lemma 3.3 we showed that there must be $|\mathcal{I}^+| + |\mathcal{I}^-|$ functions f_i in $E_+^\infty(\bar{J})$ and points z_i in $[1, \bar{J} - 1]$ satisfying (3.17). To finish the proof of the proposition, we need to do something similar at the endpoints and near p_0. The following two lemmas will suffice.

Lemma 3.4. *There exist I_l and I_r positive cutoff functions $\{\eta_i^l\}$ functions and $\{\eta_i^r\}$ respectively, with support in $(0,1)$ and $(\bar{J} - 1, \bar{J})$ respectively, such that*

$$\sum_{1 \leq i \leq I_l} \eta_i^l(x) \leq 1 \tag{3.23}$$

for all $0 < x < 1$ and

$$\sum_{1 \leq i \leq I_r} \eta_i^r(x) \leq 1$$

for all $\bar{J} - 1 < x < \bar{J}$, and such that respectively

$$\int_0^1 \phi(0, x; 1, \bar{J}) f_0(x) \eta_i^l(x) dx \geq 2 \tag{3.24}$$

and

$$\int_{\bar{J}-1}^{\bar{J}} \phi(0,x;\bar{J}-1,\bar{J}) f_0(x)\eta_i^r(x)dx \geq 2$$

for all $1 \leq i \leq I_l$ *and* $1 \leq i \leq I_r$ *respectively.*

Proof. Consider I_l. If $I_l = 0$, there is nothing to do, so we assume that $I_l \geq 1$. Let η be a cutoff function with support in $(0,1)$ such that

$$\int_0^1 \bar{\phi}(x) f_0(x)\eta(x)dx \geq \frac{1}{2}\int_0^1 \bar{\phi}(x) f_0(x)dx.$$

Define

$$\eta_i^l(x) := \frac{1}{I_l}\eta(x). \qquad 0 \leq x \leq \bar{J}$$

Clearly (3.23) is satisfied, so we need only to check (3.24). By the comparison results of Lemma 2.2 and Lemma 2.6 and an argument similar to that of Lemma 3.3, we have that

$$\begin{aligned}\int_0^1 \phi(0,x;1,\bar{J}) f_0(x)\eta_i^l(x)dx &\geq \frac{\alpha_2}{\alpha_2'}\int_0^1 \bar{\phi}(x) f_0(x)\eta_i^l(x)dx \\ &\geq \frac{\alpha_2}{2I_l\alpha_2}\int_0^1 \bar{\phi}(x) f_0(x)dx \\ &\geq \frac{M\alpha_2}{2\alpha_2'\sqrt{4\pi}}\frac{I_l}{I_l} \\ &\geq 2.\end{aligned} \tag{3.25}$$

The calculations for I_r are analogous. ∎

Lemma 3.5 *There exist* I_c *positive cutoff functions* $\{\eta_i^c\}$ *with support in* $(\bar{z}_0-1,\bar{z}_0+1)$ *such that*

$$\sum_{1\leq i\leq I_c} \eta_i^c(x) \leq 1$$

for all $\bar{z}_0 - 1 < x < \bar{z}_0 + 1$ *and such that*

$$\int_{\bar{z}_0-1}^{\bar{z}_0+1} \phi(0,x;\bar{z}_0,\bar{J}) f_0(x)\eta_i^l(x)dx \geq 2$$

and for all $1 \leq i \leq I_c$.

Proof. If $I_c = 0$, there is nothing to do, so we assume that $I_c \geq 1$. Let η be a cutoff function with support in $(\bar{z}_0 - 1, \bar{z}_0 + 1)$ such that

$$\int_{\bar{z}_0-1}^{\bar{z}_0+1} \bar{\phi}(x) f_0(x)\eta(x)dx \geq \frac{1}{2}\int_{\bar{z}_0-1}^{\bar{z}_0+1} \bar{\phi}(x) f_0(x)dx.$$

Define

$$\eta_i^c(x) := \frac{1}{I_c}\eta(x). \qquad 0 \leq x \leq \bar{J}$$

for all $1 \leq i \leq I_c$. Recall (3.18). The analogue of calculation (3.25) is

$$\begin{aligned}\int_{\bar{z}_0-1}^{\bar{z}_0+1} \phi(0, x; \bar{z}_0, \bar{J}) f_0(x)\eta_i^c(x)dx &\geq \frac{\alpha_1}{\alpha_1'}\int_{\bar{z}_0-1}^{\bar{z}_0+1} \bar{\phi}(x) f_0(x)\eta_i^c(x)dx \\ &\geq \frac{\alpha_1}{2I_c\alpha_1'}\int_{\bar{z}_0-1}^{\bar{z}_0+1} \bar{\phi}(x) f_0(x)dx \\ &\geq \frac{M\alpha_1}{2\alpha_1'\sqrt{4\pi}}\frac{I_c}{I_c} \\ &\geq 2.\end{aligned}$$

■

The exact result of Proposition 3.2 is given by considering the set

$$A := \{\eta_i : i \in \mathcal{I}^+ \cup \mathcal{I}^-\} \cup \{\eta_i^l : i = 1, 2, \ldots, I_l\} \cup \{\eta_i^r : i = 1, 2, \ldots, I_r\} \cup \{\eta_i^c : i = 1, 2, \ldots, I_c\}$$

of elements of $E_+^\infty(J)$. By (3.22) we know that $|A| \geq N$. After ordering these functions in any way, we then take

$$f_i(x) := f_0(x)\eta_i(x)$$

for $i = 1, 2, \ldots, N-1$, and

$$f_N(x) := f_0(x)\left(1 - \sum_{1\leq i\leq N-1} \eta_i(x)\right)$$

for each $0 \leq x \leq \bar{J}$. ■

4. A branching process.

In this section we show how to connect together all the ideas of Section 3 to form a branching process which describes the propagation of peaks of u a solution of (0.1). When γ is much larger than 3/2, then this branching process survives, which probabilistically implies that u blows up. Our technique is to study a collection of auxiliary 'subsolutions' of the form (2.1) on finer and finer scales, using the process (3.4) and Proposition 3.1 to get uniform lower bounds on the growth rate of the auxiliary of SPDE's. We then use Lemma 2.5 to construct a solution of our original SPDE (0.1). We keep track of the evolution of the peaks in our collection of subsolutions by means of a branching process. When a branch survives, the scaling of our subsolutions will give us blowup in finite time.

A few notational conventions will simplify our presentation. Let $\mathbb{N} := \{1, 2, \dots\}$ be the collection of natural numbers and let $\mathcal{N}_m := \{1, 2, \dots, N\}^m$ be the m-fold product of $\{1, 2, \dots, N\}$ for each $m = 1, 2, \dots,$ where $N := \lfloor 2^{2\gamma-3} \rfloor$ is as in (3.16). The set $\mathcal{N}_m$ is to be interpreted as the collection of all family histories of the m-th generation. Note that we are assuming that individuals can have at most N offspring. We shall denote elements of the $\mathcal{N}_m$'s by boldface letters. Let us also define $\mathcal{N} := \cup_{m \in \mathbb{N}} \mathcal{N}_m$ as the collection of all family histories of all generations. If $\mathbf{i} = (i_1, i_2, \dots, i_m)$ is in $\mathcal{N}_m$ for some m in $\mathbb{N}$, then for any $j = 1, 2, \dots, N$, we let $\mathbf{i}, j$ denote the obvious element $(i_1, i_2, \dots, i_m, j)$ of $\mathcal{N}_{m+1}$. Additionally, for any $\mathbf{i} = (i_1, i_2, \dots, i_m)$ in $\mathcal{N}$, we define $g(\mathbf{i})$ to be m—the function g tells us to which generation $\mathbf{i}$ belongs. For any two branches $\mathbf{i} = (i_1, i_2, \dots, i_m)$ and $\mathbf{j} = (j_1, j_2, \dots, j_n)$ in $\mathcal{N}$, we say that $\mathbf{i} < \mathbf{j}$ if $i_l \neq j_l$ for some $l = 1, 2, \dots, \min\{n, m\}$ and if $i_l < j_l$ for the least such l. We only note by way of explanation that we shall use Lemma 2.5 to construct a solution of (0.1) by considering a collection of subsolutions of (0.1) indexed by our tree, so the recursive nature of Lemma 2.5 requires that we have some notion of order on our tree. As final notational conveniences, we let $l_m := 2^{m-1}$ and $J_m := J l_m^{2(\gamma-1)}$ for all m in $\mathbb{N}$. For each $\mathbf{i}$ in $\mathcal{N}$, the $\mathbf{i}$-th subsolution of (0.1) will be an SPDE on the interval $[0, J_{g(\mathbf{i})}]$, and $l_{g(\mathbf{i})}$ will be the scaling between this subsolution and the original SPDE (0.1), with Lemma 2.4 allowing this rescaling.

We make some general remarks. In order to allow us to use Lemma 2.5 and the time change techniques of Section 3 to consider independent evolution of peaks, we expand our probability space as necessary to support collections $\{W^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}\}$ and $\{B^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}\}$ of independent white noises and Brownian motions. For any $\mathbf{i}$ in $\mathcal{N}$, $W^{\mathbf{i}}$ is a white noise on $\mathcal{B}(\mathbb{R}_+ \times [0, J_{g(\mathbf{i})}])$. Let us also fix two large numbers L and L' which will represent some truncations we will need to make to construct our branching process. The number L will correspond to a truncation of the tree, and L' will correspond to truncating the subsolutions of (2.1) at some high level. We will need to truncate the tree in order to have somewhere to begin our recursion in Lemma 2.5, and we will need to truncate the solutions of (2.1) so that we can get around the possible blowup of ξ in Lemma 2.1.

Let us now explain the structure of our branching process by studying the first several generations. For the moment, we will be more interested in explanation than rigor. Considering the original SPDE (0.1), which is the same as the SPDE (2.1) with $\xi \equiv 0$, we assume that $J > 4$ and that there is a z in $[1, J-1]$ such that (3.3) holds. For consistency, we label the solution of (0.1) as u^1, and we similarly note that $J = J_1$. We construct the process U^1 from u^1 as in (3.4) and define τ^1 as in (3.5) as the first time at which U^1 reaches the level K, with K defined by (3.2). Then $P\{\tau^1 \wedge \sigma(u^1) \leq 1\}$ is bounded from below by $\bar{\beta}$ as in (3.11)—see Proposition 3.1. Consider the the case when $\tau^1 \wedge \sigma(u^1) \leq 1$. If $\sigma(u^1) \leq 1$, then u^1 has already blown up by time 1. Consider the other case, when $\tau^1 \leq 1 < \sigma(u^1)$. In this case we use Proposition 3.2 with

$$f_0(x) := \frac{1}{2} u^1(\tau^1, x 2^{2(1-\gamma)}) \qquad 0 \leq x \leq J_2$$

to find functions $\{u_0^{1,i} : i = 1, 2, \ldots, N\}$ in $E_\infty^+(J_2)$ which add up to f_0 and points $\{z^{1,i} : i = 1, 2, \ldots, N\}$ such that

$$\int_0^{J_2} \phi(0, x; z^{1,i}, J_2) u_0^{1,i}(x) dx \geq 2$$

for each $i = 1, 2, \ldots, N$. We can then use Lemma 2.1 to find nonnegative solutions of the recursive equations

$$\begin{aligned} \frac{\partial u^{1,i}}{\partial t} &= \frac{\partial^2 u^{1,i}}{\partial x^2} + b(u^{1,i}, \tilde{u}^{1,i}) \dot{W}^{1,i} \\ u^{1,i}(t, 0) &= u^{1,i}(t, J_2) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J_2, \quad i = 1, 2, \ldots, N \\ u^{1,i}(0, \cdot) &= u_0^{1,i} \end{aligned} \tag{4.1}$$

with

$$\tilde{u}^{1,i} := \sum_{j=1}^{i-1} u^{1,j} \tag{4.2}$$

for $i = 2, 3, \ldots, N$ and

$$\tilde{u}^{1,1} \equiv 0 \tag{4.3}$$

by definition (recall that solutions of (4.1) are not known to be unique; we can take any solutions). By combining the results of Lemmas 2.4 and 2.5, we can see that

$$\tilde{u}(t, x) := \sum_{i=1}^{N} 2 u^{1,i} \left(t 2^{4(\gamma-1)}, x 2^{2(\gamma-1)} \right) \qquad t \geq 0, \quad 0 \leq x \leq J_1 \tag{4.4}$$

solves

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} + (\tilde{u})^\gamma \dot{\tilde{W}}$$
$$\tilde{u}(t,0) = \tilde{u}(t,J_1) = 0 \qquad t \geq 0, \quad 0 \leq x \leq J_1 \quad (4.5)$$
$$\tilde{u}(0,\cdot) = u^1(\tau,\cdot)$$

for some white noise $\tilde{W}$ on $\mathcal{B}(\mathbb{R}_+ \times [0,J_1])$. As a helpful practice, for consistency we shall use tildes to represent sums of subsolutions. This explains the intentional similarity of notation between $\tilde{u}^{1,i}$ and $\tilde{u}$. In view of the uniqueness of Proposition 1.1, we thus can study the law of $u^1 = u$ after τ^1 by studying the law of the processes $\{u^{1,i} : i = 1,2,\ldots,N\}$. We can construct the processes $\{U^{1,i} : i = 1,2,\ldots,N\}$ from the $u^{1,i}$'s as in (3.4) and define the $\tau^{1,i}$'s from the $U^{1,i}$'s as in (3.5). In particular, note that if $\tau^{1,i} \wedge \sigma(u^{1,i}) \leq 1$, then either $\sigma(u^{1,i}) \leq 1$ so $u^{1,i}$ blows up by time 1 so

$$\sup_{0 \leq t < 1 \wedge \sigma(u^{1,i})} \|u^{1,i}(t,\cdot)\| = \infty,$$

or $\tau^{1,i} \leq 1 < \sigma(u^{1,i})$, in which case

$$K \leq \int_0^{J_2} \phi(\tau^{1,i},x;z^{1,i},J_2) u^{1,i}(\tau^{1,i},x)dx \leq \sup_{0 \leq t < 1 \wedge \sigma(u^{1,i})} \|u^{1,i}(t,\cdot)\|$$

by using the fact that the heat kernel $\phi(\tau^{1,i},\,\cdot\,;z^{1,i},J_2)$ is integrable with integral not more than 1. In either case, $\tau^{1,i} \wedge \sigma(u^{1,i}) \leq 1$ implies that

$$\sup_{0 \leq t < 1 \wedge \sigma(u^{1,i})} \|u^{1,i}(t,\cdot)\| \geq K.$$

Now since the $u^{1,j}$'s are nonnegative, we have that $2u^{1,i}$ is less than $\tilde{u}$ as given in (4.4)—thus if $\tau^{1,i} \wedge \sigma(u^{1,i}) \leq 1$, then $\tilde{u}$ exceeds the level $2K$ by the time (in our original scale) $2^{4(1-\gamma)}$. By Lemma 2.5, $\tilde{u}$ has the same distribution as u (for $t \geq \tau^1$), and thus $\tau^{1,i} \wedge \sigma(u^{1,i}) \leq 1$ probabilistically implies that actually u exceeds the level $2K$ by the time $1 + 2^{4(1-\gamma)}$. Heuristically, if the 'peaks' survive to the second level, the solution of (0.1) should reach the level $2K$ by time $1 + 2^{4(1-\gamma)}$. We shall make this idea more rigorous later. We now would like to proceed to the next stage and to analyze the peaks of the $u^{1,i}$'s independently, repeating our arguments. Thanks to the nonlinearity b in the equations of (4.1), however, the $u^{1,i}$'s are not independent, so this is not directly possible. But note that for each $i = 1,2,\ldots,N$, $U^{1,i}$ is an $\mathcal{F}_t^{W^{1,i}}$-martingale. Thus by Knight's theorem (see [6], Theorem 3.4.13), we may time change then into *independent* Brownian motions, analogously to our comments preceeding Lemma 3.1. Thus although the $u^{1,i}$'s are not independent, the evolutions of their peaks is in some sense independent. This will be crucial to our analysis.

Now let us look a bit more closely at the evolution of the $u^{1,i}$'s as defined by (4.1). Here is where we begin to see some complications. Quite apart from any probabilistic argument, we have a problem that the process $u^{1,2}$, for example, is defined only up to $\sigma(u^{1,1})$. To get around this problem, we stop $u^{1,1}$ at $\sigma_{L'}(u^{1,1})$ in the definition of $u^{1,2}$. Instead of (4.1), we are led to study the recursion

$$\begin{aligned} \frac{\partial u^{1,i}}{\partial t} &= \frac{\partial^2 u^{1,i}}{\partial x^2} + b(u^{1,i}, \tilde{u}^{1,i}_{L'})\dot{W}^{1,i} \\ u^{1,i}(t,0) &= u^{1,i}(t,J_2) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J_2, \quad i = 1,2,\dots,N \\ u^{1,i}(0,\cdot) &= u^{1,i}_0 \end{aligned} \tag{4.6}$$

where

$$\tilde{u}^{1,i}_{L'}(t,x) := \tilde{u}^{1,i}(t \wedge \sigma_{L'}(\tilde{u}^{1,i}), x) \tag{4.7}$$

for all $t \geq 0$ and all $0 \leq x \leq J_2$ and all $i = 1,2,\dots,N$ with the $\tilde{u}^{1,i}$'s as in (4.2)–(4.3).

The next problem is what to do when, for example, $u^{1,2}$ reaches a high level but $u^{1,1}$ is still finite; i.e., $\tau^{1,2} \leq 1 < \sigma_{L'}(u^{1,1})$. We should use Proposition 3.2 to break up

$$f_0(x) := \frac{1}{2}u^{1,2}(\tau^{1,2}, x2^{2(1-\gamma)}) \qquad\qquad 0 \leq x \leq J_3$$

into N components and repeat our arguments, defining now $\{u^{1,2,i} : i = 1,2,\dots,N\}$. But since $u^{1,3}$ depends upon $u^{1,2}$, this will affect the evolution of $u^{1,3}$. This is particularly true since uniqueness for (2.1) is not known—we don't know that the law of $u^{1,2}$, which we initially use in defining $u^{1,3}$ is the same as the law of the appropriately scaled sum of the $u^{1,2,i}$'s, which we should use after the moment $\tau^{1,2}$ when $u^{1,2}$ should be split. Due to this nonuniqueness, we should at that moment define $\tilde{u}^{1,2}$ as the appropriately scaled sum of the $u^{1,2,i}$'s. A little thought shows that we should in fact at that moment define $u^{1,3}$ using *all* the (appropriately scaled) $u^{\mathbf{i}}$'s such that $\mathbf{i} < 1,3$; this occurs when the solution blows up along one branch of the tree. The difficulty of this is that then the recursion in (4.1) has no finite starting place. We overcome this problem by truncating the tree, and defining the $\mathbf{i}$-th branch of the tree using only those relevant branches from the first L

generations. Thus we replace the recursion (4.6) now by

$$\begin{aligned}
&\frac{\partial u^{1,i}}{\partial t} = \frac{\partial^2 u^{1,i}}{\partial x^2} + b\left(u^{1,i}, \tilde{u}_{L'}^{*|1,i}\right) \dot{W}^{1,i} \\
&u^{1,i}(t,0) = u^{1,i}(t,J_2) = 0 \qquad\qquad t \geq 0, \quad 0 \leq x \leq J_2, \quad i = 1,2,\ldots,N \\
&u^{1,i}(0,\cdot) = u_0^{1,i}
\end{aligned} \tag{4.8}$$

where now the $\tilde{u}^{*|1,i}$'s are analogous to (4.7) except that we now sum over all (appropriately scaled) $u^{\mathbf{j}}$'s with $\mathbf{j}$ in $\mathcal{N}_L$ such that $\mathbf{j} < 1, i$. The scaling is done so that the $u^{\mathbf{j}}$'s will be on the same scale as $u^{1,i}$.

Having rather loosely defined our branching process, let us list several quantities we will need to keep track of, as a preliminary to a rigorous definition of our branching process. For each fixed nonnegative integer L, we shall construct a branching process with L generations. The individuals in our branching process, i.e., the 'peaks' of (0.1) will be denoted by a collection $\{X^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$ of $\{0,1\}$-valued random variables. The 'subsolutions' of (0.1) will be denoted as $\{u^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$. To keep track of the domains of definition of these functions, shall use a collection $\{r^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$ of nonnegative times. For each $\mathbf{i}$ in $\mathcal{N}_L$, $u^{\mathbf{i}}$ will be a function defined on $[0, r^{\mathbf{i}}) \times [0, J_{g(\mathbf{i})}]$. We will need also to keep track of the 'location' of the peaks in our subsolutions and the shape of these peaks. The locations will be denoted by a collection $\{z^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$ and the shapes by a collection $\{u_0^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$. For any $\mathbf{i}$ in $\mathcal{N}_L$, $z^{\mathbf{i}}$ is some element of $[1, J_{g(\mathbf{i})} - 1]$ and $u_0^{\mathbf{i}}$ is some element of $E_+^\infty(J_{g(\mathbf{i})})$. We will also want to convert between the time scales of the $u^{\mathbf{i}}$'s and our original time scale. We will do this with a collection $\{s^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$ of times. Essentially, the interval $[0,1]$ on the time scale of $u^{\mathbf{i}}$ for some $\mathbf{i}$ in $\mathcal{N}_L$ will correspond to the interval $[s^{\mathbf{i}}, s^{\mathbf{i}} + l_{g(\mathbf{i})}^{4(1-\gamma)}]$. The form of Lemma 2.5 also indicates that we will want to keep a running sum of all of our subsolutions. We will do this with a collection $\{\tilde{u}^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$. By summing up *all* of the subsolutions, we have by Lemma 2.5 a solution of (0.1) for some white noise. Finally, we need to keep track of the possibility that the subsolutions actually blow up in finite time. We do this via a collection $\{\delta^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$ of $\{0,1\}$-valued random variables.

Let us now begin to define the recursion of our branching process. Fixing an $\mathbf{i}$ in $\mathcal{N}_L$, assume that we know $X^{\mathbf{i}}$, $z^{\mathbf{i}}$, $u_0^{\mathbf{i}}$, $\delta^{\mathbf{i}}$, and $s^{\mathbf{i}}$. Assume also that we know $r^{\mathbf{j}}$, $u^{\mathbf{j}}$, $\delta^{\mathbf{j}}$, and $s^{\mathbf{j}}$ for all $\mathbf{j}$ in $\mathcal{N}_L$ such that $\mathbf{j} < \mathbf{i}$ (note that such $\mathbf{j}$ need not exist—consider the edges of the tree). We can then define

$$\tilde{u}^{\mathbf{i}}(t,x) := \sum_{\substack{\mathbf{j} \in \cup_{m \leq L} \mathcal{N}_m \\ \mathbf{j} < \mathbf{i}}} l_{g(\mathbf{j})} u^{\mathbf{j}} \left((t - s^{\mathbf{j}}) l_{g(\mathbf{j})}^{-4(1-\gamma)}, x l_{g(\mathbf{j})}^{-2(1-\gamma)} \right) \chi_{\{0 \leq (t-s^{\mathbf{j}}) l_{g(\mathbf{j})}^{-4(1-\gamma)} < r^{\mathbf{j}}\}} \tag{4.9}$$

for each $0 \le x \le J$ and $t \ge 0$, and with

$$\sum_{\emptyset} := 0$$

by definition. The random field $\tilde{u}^{\mathbf{i}}$ is simply the scaled sum of all the subsolutions which are 'alive' at time t, with the scaling to be that of our original SPDE (0.1).

Consider first the case when $\delta^{\mathbf{i}} = 1$. In our rules, this corresponds to blowup at some time in the past. We then set $X^{\mathbf{i},j} = 1$ and $\delta^{\mathbf{i},j} = 1$ for all $j = 1, 2, \ldots, N$ to record that blowup has already occurred at some previous time. As default values, we set $r^{\mathbf{i}} = 1$ and $u^{\mathbf{i}} = 0$ on $[0, r^{\mathbf{i}}) \times [0, J_{g(\mathbf{i})}]$. Similarly, we set $z^{\mathbf{i},j} = 1$, $u_0^{\mathbf{i}} \equiv 0$, and $s^{\mathbf{i},j} = s^{\mathbf{i}} + l_{g(\mathbf{j})}^{4(1-\gamma)}$ for all $j = 1, 2, \ldots, N$.

The second case is that $X^{\mathbf{i}} = 1$ but $\delta^{\mathbf{i}} = 0$. This corresponds to the case that the solution has not yet blown up, but large peaks have propagated along the geneological tree. In this case we study the propagation of the peak. We use Lemma 2.1 to find a solution of

$$\begin{aligned} \frac{\partial u^{\mathbf{i}}}{\partial t} &= \frac{\partial^2 u^{\mathbf{i}}}{\partial x^2} + b(u^{\mathbf{i}}, \tilde{u}_{L'}^{*|\mathbf{i}})\dot{W}^{\mathbf{i}} \\ u^{\mathbf{i}}(t,0) &= u^{\mathbf{i}}(t, J_{g(\mathbf{i})}) = 0 \qquad\qquad t \ge 0, \quad 0 \le x \le J_{g(\mathbf{i})} \\ u^{\mathbf{i}}(0,\cdot) &= u_0^{\mathbf{i}}. \end{aligned} \tag{4.10}$$

with explosion time being denoted as $\sigma(u^{\mathbf{i}})$ and where $\tilde{u}_{L'}^{*|\mathbf{i}}$ is defined as

$$\tilde{u}_{L'}^{*|\mathbf{i}}(t,x) := l_{g(\mathbf{i})}^{-1}\tilde{u}^{\mathbf{i}}\left(\left(tl_{g(\mathbf{i})}^{4(1-\gamma)} + s^{\mathbf{i}}\right) \wedge \sigma_{L'}(\tilde{u}^{\mathbf{i}}), xl_{g(\mathbf{i})}^{2(1-\gamma)}\right). \qquad t \ge 0, \quad 0 \le x \le J_{g(\mathbf{i})} \tag{4.11}$$

Note that $\tilde{u}_{L'}^{*|\mathbf{i}}$ is measurable with respect to $\sigma\{W^{\mathbf{j}} : \mathbf{j} < \mathbf{i}\}$ for each $\mathbf{i}$. We then define

$$U_t^{\mathbf{i}} := \int_0^{J_{g(\mathbf{i})}} \phi(t, x; z_{\mathbf{i}}, J_{g(\mathbf{i})}) u^{\mathbf{i}}(t,x)dx \qquad 0 \le t < 1 \wedge \sigma(u^{\mathbf{i}}) \wedge \sigma(\tilde{u}_{L'}^{*|\mathbf{i}}) \tag{4.12}$$

and

$$\tau^{\mathbf{i}} := \begin{cases} \inf\{t > 0 : U_t^{\mathbf{i}} > K\} & \text{if } \sup_{0 \le t < \sigma(u^{\mathbf{i}}) \wedge \sigma(\tilde{u}_{L'}^{*|\mathbf{i}})} U_t^{\mathbf{i}} > K \\ \infty & \text{otherwise} \end{cases} \tag{4.13}$$

in analogy to (3.4) and (3.5). We set $r^{\mathbf{i}} := \tau^{\mathbf{i}} \wedge \sigma(u^{\mathbf{i}}) \wedge 1$, and $s^{\mathbf{i},j} := s^{\mathbf{i}} + l_{g(\mathbf{i})}^{4(1-\gamma)} r^{\mathbf{i}}$ for all $j = 1, 2, \ldots, N$. Consider first the case where $u^{\mathbf{i}}$ blows up before $U^{\mathbf{i}}$ reaches level K and before time 1; in this case $\tau^{\mathbf{i}} \wedge \sigma(u^{\mathbf{i}}) = \sigma(u^{\mathbf{i}}) < 1$. We record that a large (in fact infinite) peak has been reached by setting $X^{\mathbf{i},i} = 1$ for all $i = 1, 2, \ldots, N$. We record the fact that blowup has already occurred by setting $\delta^{\mathbf{i},i} = 1$ for all $j = 1, 2, \ldots, N$. We set $z^{\mathbf{i},i} = 1$ for all $i = 1, 2, \ldots, N$ as a default. Consider next the case where $U^{\mathbf{i}}$ reaches level K before time 1 and before the solution blows up, i.e., $\tau^{\mathbf{i}} \wedge \sigma(u^{\mathbf{i}}) = \tau^{\mathbf{i}} < 1$. We record the formation of a large but finite peak by setting $X^{\mathbf{i},j} = 1$ and $\delta^{\mathbf{i},j} = 0$ for all $j = 1, 2, \ldots, N$. We record information about the shape of the peak by using Proposition 3.2 with $\bar{J} = J_{g(\mathbf{i})+1}$ and $f_0(x) = (1/2)u^{\mathbf{i}}(\tau^{\mathbf{i}}, x2^{2(1-\gamma)})$ for all $0 \le x \le J_{g(\mathbf{i})+1}$ to define $\{u_0^{\mathbf{i},j} : j = 1, 2, \ldots, N\}$ and $\{z^{\mathbf{i},j} : j = 1, 2, \ldots, N\}$. Finally, consider the case where $U^{\mathbf{i}}$ does not reach K before time 1 nor does the solution blow up before time 1; i.e., the peak dies and $\tau^{\mathbf{i}} \wedge \sigma(u^{\mathbf{i}}) \ge 1$. We record this by setting $X^{\mathbf{i},j} = 0$ and $\delta^{\mathbf{i},j} = 0$ for all $j = 1, 2, \ldots, N$. In this case we set $z^{\mathbf{i},i} = 1$ for all $i = 1, 2, \ldots, N$ as a default. We record the shape of the solution by setting $u_0^{\mathbf{i},j}(x) := N^{-1}u^{\mathbf{i}}(1, x2^{2(1-\gamma)})$ for all $0 \le x \le J_{g(\mathbf{i})+1}$, for all $j = 1, 2, \ldots, N$. (Thus if we sum $u_0^{\mathbf{i},j}$ over all the j's, we get back $u^{\mathbf{i}}(1, \cdot)$, appropriately scaled.)

The final possible situation we need to address is when $X^{\mathbf{i}} = 0$. This corresponds to the "death" of a peak at some point in the past of the geneological tree. We disallow the spontaneous generation of peaks and propagate the death of the branch by setting $X^{\mathbf{i},i} = 0$ and $\delta^{\mathbf{i},j} = 0$ for all $j = 1, 2, \ldots, N$. To retain the evolution of the shapes of our subsolutions, we again solve (4.9), define explosion time $\sigma(u^{\mathbf{i}})$, and define $u_{L'}^{*|\mathbf{i}}$, $U^{\mathbf{i}}$, and $\tau^{\mathbf{i}}$ as in (4.11)—(4.13). We set $r^{\mathbf{i},j} = \tau^{\mathbf{i}} \wedge \sigma(u^{\mathbf{i}}) \wedge 1$, $z^{\mathbf{i},j} = 1$ and $s^{\mathbf{i},j} = s^{\mathbf{i}} + l_{g(\mathbf{i})}^{4(1-\gamma)} r^{\mathbf{i}}$ for all $j = 1, 2, \ldots, N$, and set $u_0^{\mathbf{i},j}(x) := N^{-1}u^{\mathbf{i}}(1, x2^{2(1-\gamma)})$ for all $0 \le x \le J_{g(\mathbf{i})+1}$, for all $j = 1, 2, \ldots, N$.

Note that we have clumped all the information about the branches $\mathbf{j}$ with $\mathbf{j} < \mathbf{i}$ into the definition of $\tilde{u}^{\mathbf{i}}$. Thus, for each the $\mathbf{i}$, we need only to know $X^{\mathbf{i}}$, $z^{\mathbf{i}}$, $u_0^{\mathbf{i}}$, $\delta^{\mathbf{i}}$, $s^{\mathbf{i}}$, and $\tilde{u}^{\mathbf{i}}$ in order to define the children of the $\mathbf{i}$-th branch.

It remains now only to specify the initial conditions for our branching process. In light of the above remarks, it suffices to set $X^1 = 1$ and $\delta^1 = 0$ to indicate that the presence of an initial peak. As noted before, we assume that $J > 4$ and that (3.3) is satisfied for some z. We set $u_0^1 = u_0$ and set z^1 to be any z satisfying (3.3). We set $s^1 = 0$, so that the scale of the first branch coincides with the scale of our original SPDE. To correspond to the fact that we have only one large initial peak, we set $X^j = \delta^j = 0$ for all $j = 2, 3, \ldots, N$, and as default values we set $u_0^j \equiv 0$, $z^j = 1$, and $s^j = 0$ for all $j = 2, 3, \ldots, N$.

Given the above rules, we now define our branching process as

$$Y_m^{L,L'} := \sum_{\mathbf{i} \in \mathcal{N}_m} X_{\mathbf{i}}, \qquad m = 1, 2, \ldots, L \quad (4.14)$$

i.e., $Y_m^{L,L'}$ is the population size of $\{X\}$ at the m-th stage. We here have reintroduced the dependence of our branching process upon the truncation constants L and L'. We do this in anticipation of letting L and L' tend to infinity. To complete the proof of Theorem 1, we need to do several things. We need to show that the survival of $Y^{L,L'}$ implies the growth of u. We also need to show that $Y^{L,L'}$ is a branching process with positive probability of survival if γ is large enough. In fact, our calculations will be somewhat involved, since we can do neither of these things directly. We must remove the truncations of L and L'. And we must overcome the problem that $Y^{L,L'}$ is *not* a classical branching process, due to the above complicated rules—we will *couple* $Y^{L,L'}$ to an auxiliary classical branching process and apply known branching theory results to the auxiliary process.

Let us first see how the survival of $Y^{L,L'}$ implies the growth of u. We define

$$\tilde{u}^{L,L'}(t,x) := \sum_{\mathbf{j}\in\cup_{m\le L}\mathcal{N}_m} l_{g(\mathbf{j})} u^{\mathbf{j}}\left(\left(t-s^{\mathbf{j}}\right) l_{g(\mathbf{j})}^{-4(1-\gamma)}, x l_{g(\mathbf{j})}^{-2(1-\gamma)}\right) \chi_{\{0\le(t-s^{\mathbf{j}})l_{g(\mathbf{j})}^{-4(1-\gamma)}<r^{\mathbf{j}}\}} \tag{4.15}$$

for each $0 \le x \le J$ and $t \ge 0$; then, modulo some technical difficulties which we shall clean up, $\tilde{u}^{L,L'}$ has the same evolution as the solution u of (0.1). Let us also define the random time

$$\rho_L := \min\{\{s^{\mathbf{i}} + l_{g(\mathbf{i})}^{4(1-\gamma)} r^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}; \tag{4.16}$$

then the interval $[0, \rho_L]$ is the largest interval where we are tracing $u^{\mathbf{i}}$ on all generations of our L-generation of our tree. One should think of ρ_L as the first time at which we should consider a larger tree of subsolutions. Fix now an $m = 2, 3, \ldots, L$. If $Y_m^{L,L'} \ge 1$, then $X^{\mathbf{i}} = 1$ for some $\mathbf{i}$ in $\mathcal{N}_m$. If $\delta^{\mathbf{i}} = 0$, then the $u^{\mathbf{j}}$'s have blown up at some past moment in the $\mathbf{i}$-th branch of the tree, so $\sup_{t\ge0}\|\tilde{u}^{L,L'}(t,\cdot)\| = \infty$. If $\delta^{\mathbf{i}} = 0$, then by the rules defining our branching process,

$$\int_0^{J_m} \phi(0, x; z^{\mathbf{i}}, J_m) u_0^{\mathbf{i}}(x)dx \ge 2.$$

Since $x \mapsto \phi(0, x; z^{\mathbf{i}}, J_m)$ has integral not more than 1 (see Lemma 2.2), we thus have that $\|u_0^{\mathbf{i}}\| \ge 2$, so by (4.15) and the nonnegativity of the $u^{\mathbf{j}}$'s,

$$\|\tilde{u}^{L,L'}(s^{\mathbf{i}},\cdot)\| \ge l_m \|u_0^{\mathbf{i}}\| \ge 2^m.$$

In either case, we see that if $Y_m^{L,L'} \ge 1$, then $\sup_{t\ge0}\|\tilde{u}^{L,L'}(t,\cdot)\| \ge 2^m$. Thus

$$P\{Y_m^{L,L'} \ge 1\} \le P\left\{\sup_{0\le t<\rho_L} \|\tilde{u}^{L,L'}(t,\cdot)\| \ge 2^m\right\}. \tag{4.17}$$

This connects $Y^{L,L'}$ with $\tilde{u}^{L,L'}$; we next need to connect $\tilde{u}^{L,L'}$ to u the solution of (0.1). The key to this clearly is to use Lemmas 2.4 and 2.5 to see that the evolution of $\tilde{u}^{L,L'}$ is similar to the evolution of the solution u of (0.1). The following result will suffice:

Lemma 4.1. *If* $2^m < L'$,

$$P\left\{\sup_{0\le t\le \sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L} \|\tilde{u}^{L,L'}(t,\cdot)\| \ge 2^m\right\} \le P\left\{\sup_{0\le t<\bar{t}_\gamma\wedge\sigma(u)} \|u(t,\cdot)\| \ge 2^m\right\}$$

where

$$\bar{t}_\gamma := \left(1 - 2^{4(1-\gamma)}\right)^{-1} \tag{4.18}$$

and where u is the solution of (0.1).

Proof. We must connect the evolution of $\tilde{u}^{L,L'}$ to the evolution of the solution u of (0.1). The primary difficulty is that we are stopping the $u^{\mathbf{i}}$'s in (4.11) when they reach the level L'. That this does not seriously affect our calculations is a consequence of the following observations. First, if $u^{\mathbf{i}}$ reaches the level L' for some $\mathbf{i}$ in $\mathcal{N}_L$, then $\tilde{u}^{L,L'}$ must also have reached the level L' (compare (4.9) and (4.15) and use the fact that all the $u^{\mathbf{i}}$'s are nonnegative). Thus if we consider the evolution of $\tilde{u}^{L,L'}$ on $[0,\sigma_{L'}(\tilde{u}^{L,L'}))$, or alternately the evolution of $\tilde{u}^{L,L'}$ stopped at $\sigma_{L'}(\tilde{u}^{L,L'})$, the stopping times in (4.11) will not yet have affected the $\tilde{u}_{L'}^{*|\mathbf{i}}$'s and we may use Lemmas 2.4 and 2.5. Second, whether or not we stop $\tilde{u}^{L,L'}$ at $\sigma_{L'}(\tilde{u}^{L,L'})$ is irrelevant if we are only interested, as we are in (4.17), with the probability that $\tilde{u}^{L,L'}$ reaches the level 2^m—if $2^m < L'$. As a final observation, we always have that

$$\rho_L \le \sum_{j=0}^{L} l_j^{4(1-\gamma)} \le \sum_{j=0}^{\infty} l_j^{4(1-\gamma)} \le \bar{t}_\gamma \tag{4.19}$$

where $\bar{t}_\gamma$ is as in (4.18).

The following calculations make the above ideas rigorous. One can show that for all $t\ge 0$ and all φ in $C_0^2([0,J])$,

$$\begin{aligned}\int_0^J \tilde{u}^{L,L'}(t\wedge\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L,x)\varphi(x)dx &= \int_0^J u_0(x)\varphi(x)dx \\ &+\int_0^t\int_0^J \tilde{u}^{L,L'}(s,x)\frac{\partial^2\varphi}{\partial x^2}(x)\chi_{\{0\le s\le\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L\}}dsdx \\ &+\int_0^t\int_0^J \varphi(x)(\tilde{u}^{L,L'}(s,x))^\gamma\chi_{\{0\le s\le\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L\}}\tilde{W}(ds,dx)\end{aligned} \tag{4.20}$$

P-a.s., where $\tilde{W}$ is some white noise on $\mathcal{B}(\mathbb{R}_+ \times [0,J])$ given by a random linear combination of the $W^{\mathbf{i}}$'s, appropriately translated and scaled. The details of this, which we shall for the most part leave to the reader, require arranging the set $\{s^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}_L\}$ into a random set $\{\rho_j : 1 \le j \le N^L\}$ of nondecreasing times and then using Lemmas 2.4 and 2.5 on each interval $[\rho_j, \rho_{j+1}]$ such that $[\rho_j, \rho_{j+1}] \subset [0, \sigma_{L'}(\tilde{u}^{L,L'}) \wedge \rho_L]$. One should think of these ρ_j's as exactly those instants when we restart the evolution of the $u^{\mathbf{i}}$'s and if necessary break up the large peaks. The equality (4.20) formalizes the first observation above. The second and third observations can be formalized by expanding our probability space as necessary to include a white noise $\hat{W}$ on $\mathcal{B}(\mathbb{R}_+ \times [0,J])$ which is independent of the $W^{\mathbf{i}}$'s, then solving

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} &= \frac{\partial^2 \hat{u}}{\partial x^2} + \hat{u}^\gamma \dot{\hat{W}} \\ \hat{u}(t,0) &= \hat{u}(t,J) = 0 \qquad\qquad t \ge 0, \quad 0 \le x \le J \\ \hat{u}(0,\cdot) &= \tilde{u}^{L,L'}(\sigma_{L'}(\tilde{u}^{L,L'}) \wedge \rho_L, \cdot). \end{aligned}$$

and finally defining

$$u(t,\cdot) := \tilde{u}^{L,L'}(t,\cdot)\chi_{\{t<\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L\}} + \hat{u}(t - \sigma_{L'}(\tilde{u}^{L,L'}) \wedge \rho_L, \cdot)\chi_{\{t\ge\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L\}}$$

for all $t \ge 0$. One can see that u is a solution of (0.1) for some white noise W. But as a result of this and in view of (4.19),

$$P\left\{ \sup_{0\le t\le\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L} \|\tilde{u}^{L,L'}(t,\cdot)\| \ge 2^m \right\} \le P\left\{ \sup_{0\le t<\sigma(u)\wedge\bar{t}_\gamma} \|u(t,\cdot)\| \ge 2^m \right\}$$

which is exactly the result we seek. ∎

Combining finally (4.17) and Lemma 4.1, we get

Proposition 4.1. *We have*

$$\limsup_m \limsup_L \limsup_{L'} P\{Y_m^{L,L'} \ge 1\} \le P\{\sigma(u) \le \bar{t}_\gamma\}.$$

Proof. We have that for $2^m < L'$,

$$\begin{aligned} P\{Y_m^{L,L'} \ge 1\} &\le P\left\{ \sup_{0\le t<\rho_L} \|\tilde{u}^{L,L}(t,\cdot)\| \ge 2^m \right\} \\ &= P\left\{ \sup_{0\le t<\sigma_{L'}(\tilde{u}^{L,L'})\wedge\rho_L} \|\tilde{u}^{L,L}(t,\cdot)\| \ge 2^m \right\} \\ &\le P\left\{\sigma_{2^m}(u) \le \bar{t}_\gamma\right\}. \end{aligned}$$

First take L', then L, and then m to infinity. ∎

Now let us study the $Y^{L,L'}$'s to show that $P\{Y_m^{L,L'} \geq 1\}$ is bounded from below uniformly in L, L', and m, for γ large enough. We claim that $Y^{L,L'}$ behaves similarly to a Galton-Watson branching process (see [2] and [5] for a discussion of Galton-Watson proceses), and that if γ is large enough, then the branching process survives with positive probability.

The classical theory of branching processes gives us that if the mean number of offspring of a parent in a classical Galton-Watson branching process is greater than 1, then the branching process survives with positive probability. To start then, let us study the expected number of offspring of the system we have defined (we are considering a fixed L and L'). By the rules defining our processes $X^{\mathbf{i}}$'s, we have that if $X^{\mathbf{i}} = 1$ for some $\mathbf{i}$, then $X_{\mathbf{i}}$ can have either N offspring or no offspring. Thus

$$E\left[\sum_{j=1}^{N} X^{\mathbf{i},j} \middle| X^{\mathbf{i}}, \delta^{\mathbf{i}}\right] \chi_{\{X^{\mathbf{i}}=1\}} = NP\left\{\sum_{j=1}^{N} X^{\mathbf{i},j} = N \middle| X^{\mathbf{i}}, \delta^{\mathbf{i}}\right\} \chi_{\{X^{\mathbf{i}}=1\}}$$

where χ_A is the indicator function of the set A. Now if $\delta^{\mathbf{i}} = 1$, then $\sum_{j=1}^{N} X^{\mathbf{i},j} = N$, so

$$P\left\{\sum_{j=1}^{N} X^{\mathbf{i},j} = N \middle| X^{\mathbf{i}}, \delta^{\mathbf{i}}\right\} \chi_{\{X^{\mathbf{i}}=1,\delta^{\mathbf{i}}=1\}} = (1)\chi_{\{X^{\mathbf{i}}=1,\delta^{\mathbf{i}}=1\}}, \tag{4.21}$$

and if $\delta^{\mathbf{i}} = 0$, then we use Proposition 3.1 and (3.10) to see that

$$\begin{aligned} &P\left\{\sum_{j=1}^{N} X^{\mathbf{i},j} = N \middle| X^{\mathbf{i}}, \delta^{\mathbf{i}}\right\} \chi_{\{X^{\mathbf{i}}=1,\delta^{\mathbf{i}}=0\}} \\ &\quad \geq E\left[\beta\left(\int_0^{J_{g(\mathbf{i})}} \phi(0,x;z^{\mathbf{i}}, J_{g(\mathbf{i})}) u_0^{\mathbf{i}}(x)dx\right) \middle| X^{\mathbf{i}}, \delta^{\mathbf{i}}\right] \chi_{\{X^{\mathbf{i}}=1,\delta^{\mathbf{i}}=0\}} \\ &\quad \geq \bar{\beta}\chi_{\{X^{\mathbf{i}}=1,\delta^{\mathbf{i}}=0\}}, \end{aligned} \tag{4.22}$$

where β is as given in Proposition 3.1 and $\bar{\beta}$ is as in Lemma 3.2. Combining (4.18) and (4.19), we have that

$$P\left\{\sum_{j=1}^{N} X^{\mathbf{i},j} = N \middle| X^{\mathbf{i}}\right\} \chi_{\{X^{\mathbf{i}}=1\}} \geq \bar{\beta}\chi_{\{X^{\mathbf{i}}=1\}}$$

and so

$$E\left[\sum_{j=1}^{N} X^{\mathbf{i},j} \middle| X^{\mathbf{i}}\right] \chi_{\{X^{\mathbf{i}}=1\}} \geq \bar{\beta}N\chi_{\{X^{\mathbf{i}}=1\}}.$$

Our aim is to couple $Y^{L,L'}$ with a classical Galton-Watson branching $\tilde{Y}$ such that $\tilde{Y}_m \leq Y_m^{L,L'}$ for all $m = 1, 2, \ldots, L$, and where $\tilde{Y}$ has either N offspring or none with mean offspring $\bar{\beta}N$ (see [8] for a discussion of coupling). We then find conditions under which the auxiliary branching process survives with positive probability—namely, conditions under which $\bar{\beta}N > 1$ (see [2] and [5]). To be more precise, let $\{\xi_i : i \in \mathbb{N}\}$ be a collection of $\{0, N\}$ random variables with

$$P\{\xi_j = N\} = \bar{\beta} \quad \text{and} \quad P\{\xi_j = 0\} = 1 - \bar{\beta},$$

for all $j = 1, 2, \ldots$, and define recursively define a process $\tilde{Y}$ as

$$\begin{aligned} \tilde{Y}_1 &= 1 \\ \tilde{Y}_{m+1} &= \sum_{i=1}^{\tilde{Y}_m} \xi_i. \end{aligned} \qquad m \in \mathbb{N} \quad (4.23)$$

We have the following coupling result:

Lemma 4.2. *By expanding the probability space, we can define a process Y^* having the same law as $\tilde{Y}$ of (4.23) and such that $Y_m^* \leq Y_m^{L,L'}$ for all $m = 1, 2, \ldots, L$.*

Proof. By reviewing the construction of the $X^{\mathbf{i}}$'s, we see that in fact for each $m = 1, 2, \ldots, L-1$,

$$X^{\mathbf{i},j} = \chi_{\{r^{\mathbf{i}}<1, X^{\mathbf{i}}=1, \delta^{\mathbf{i}}=0\}} + \chi_{\{\delta^{\mathbf{i}}=1\}}$$

for all $\mathbf{i}$ in $\mathcal{N}_m$ and all $j = 1, 2, \ldots, N$. Let us expand our probability space as necessary to support a collection $\{\tilde{B}^{\mathbf{i}} : \mathbf{i} \in \mathcal{N}\}$ of Brownian motions with the following properties. We require that for each $\mathbf{i}$ in $\cup_{m \leq L} \mathcal{N}_m$, if $X^{\mathbf{i}} = 1$ and $\delta^{\mathbf{i}} = 0$, then

$$U_t^{\mathbf{i}} = \tilde{B}^{\mathbf{i}}_{\rho^{\mathbf{i}}(t)} \qquad 0 \leq t < r^{\mathbf{i}}$$

where $U^{\mathbf{i}}$ is as in (4.12) and where $\rho^{\mathbf{i}}$ is some random monotone nondecreasing mapping of $\mathbb{R}_+$ into itself. We also require that the $\{\tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} : t \geq 0\}$'s be mutually independent as $\mathbf{i}$ ranges over $\cup_{m \leq L} \mathcal{N}_m$. We leave the details of this construction to the reader. We only note that Knight's Theorem ([6], Theorem 3.4.13) is fundamental to the independence of the $\tilde{B}^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}}$'s, since the $u^{\mathbf{i}}$'s depend upon one another. By using Lemma 3.1 and a calculation analogous to (3.12), we see that

$$\begin{aligned} &\left\{ \sup_{0 \leq t \leq c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > K - 2,\ \inf_{0 \leq t \leq c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > -1,\ X^{\mathbf{i}} = 1,\ \delta^{\mathbf{i}} = 0 \right\} \\ &\qquad \subset \left\{ \sup_{0 \leq t \leq c} \tilde{B}_t^{\mathbf{i}} > K,\ \inf_{0 \leq t \leq c} \tilde{B}_t^{\mathbf{i}} > -1\ X^{\mathbf{i}} = 1,\ \delta^{\mathbf{i}} = 0 \right\} \\ &\qquad \subset \{\tau^{\mathbf{i}} \wedge \sigma(u^{\mathbf{i}}) < 1\ X^{\mathbf{i}} = 1,\ \delta^{\mathbf{i}} = 0\} \\ &\qquad \subset \{r^{\mathbf{i}} < 1,\ X^{\mathbf{i}} = 1,\ \delta^{\mathbf{i}} = 0\}. \end{aligned} \qquad (4.24)$$

From this we are led to set $X_1^* = 1$ and $X_i^* = 0$ for $i = 2, 3, \ldots, N$, and recursively define

$$X_*^{\mathbf{i},j} := \chi_{\{\sup_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > K,\, \inf_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > -1\} \cap \{X_*^{\mathbf{i}} = 1\}} \tag{4.25}$$

for all $\mathbf{i}, j$ in $\mathcal{N}$. Constructing X_* in this way, we have that $X_*^{\mathbf{i}} \le X^{\mathbf{i}}$ for all i in $\mathcal{N}_1$, and furthermore, if $X_*^{\mathbf{i}} \le X^{\mathbf{i}}$ for each i in $\cup_{m\le L}\mathcal{N}_m$, then $X_*^{\mathbf{i},j} \le X^{\mathbf{i},j}$ for all $j = 1, 2, \ldots, L$ since, via (4.24),

$$\begin{aligned}
\{X_*^{\mathbf{i}} \le X^{\mathbf{i}}\} &\cap \left\{ \sup_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > K - 2,\ \inf_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > -1,\ X_*^{\mathbf{i}} = 1 \right\} \\
&\subset \left\{ \sup_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} > K,\ \inf_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} > 1,\ X^{\mathbf{i}} = 1 \right\} \\
&\subset \left\{ \sup_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} > K,\ \inf_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} > 1,\ X^{\mathbf{i}} = 1,\ \delta^{\mathbf{i}} = 0 \right\} \cup \{\delta^{\mathbf{i}} = 1\} \\
&\subset \{r^{\mathbf{i}} < 1,\ X^{\mathbf{i}} = 1,\ \delta^{\mathbf{i}} = 0\} \cup \{\delta^{\mathbf{i}} = 1\}.
\end{aligned}$$

Hence, defining

$$Y_m^* := \sum_{\mathbf{i}\in\mathcal{N}_m} X_*^{\mathbf{i}}, \qquad m \in \mathbb{N}$$

we have that $Y_m^* \le Y_m$ for all $m = 1, 2, \ldots, L$. To finish the proof of the lemma, we shall show that Y^* is a classical branching process whose law coincides with the law of $\tilde{Y}$ of (4.22).

To show that Y^* and $\tilde{Y}$ have the same law, it is enough to show that for any m and k in $\mathbb{N}$,

$$P\{Y_{m+1}^* = kN | Y_1^*, Y_2^*, \ldots, Y_m^*\} = \binom{Y_m^*}{k} \bar{\beta}^k (1-\bar{\beta})^{Y_m^* - k} \chi_{\{k \le Y_m^*\}}. \tag{4.26}$$

Defining

$$E_{\mathbf{i}} := \left\{ \sup_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > K,\ \inf_{0\le t\le c} \tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} > -1 \right\}$$

for all $\mathbf{i}$ in $\cup_{m\le L}\mathcal{N}_m$, we can calculate from (4.24) that

$$\begin{aligned}
P\{Y_{m+1}^* = kN | Y_1^*, Y_2^*, \ldots, Y_m^*\} &= P\left\{ \sum_{\mathbf{i}\in\mathcal{N}_m} \chi_{E_{\mathbf{i}} \cap \{X_*^{\mathbf{i}} = 1\}} = k \,\middle|\, Y_1^*, Y_2^*, \ldots, Y_m^* \right\} \\
&= P\left\{ \sum_{\substack{\mathbf{i}\in\mathcal{N}_m \\ X_*^{\mathbf{i}} = 1}} \chi_{E_{\mathbf{i}}} = k \,\middle|\, Y_1^*, Y_2^*, \ldots, Y_m^* \right\} \\
&= E\left[P\left\{ \sum_{\substack{\mathbf{i}\in\mathcal{N}_m \\ X_*^{\mathbf{i}} = 1}} \chi_{E_{\mathbf{i}}} = k \,\middle|\, X_*^{\mathbf{j}} : \mathbf{j} \in \mathcal{N}_m \right\} \,\middle|\, Y_1^*, Y_2^*, \ldots, Y_m^* \right].
\end{aligned}$$

In the last line here we have used an obvious iterated conditioning argument. One can show from (4.25) that for each $\mathbf{j}$ in $\mathcal{N}_m$, $X_*^{\mathbf{j}}$ is measurable with respect to $\bigvee_{k\in\mathcal{N}_{g(\mathbf{j})-1}} \sigma\{\tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} : t \geq 0\}$; since the $\{\tilde{B}_t^{\mathbf{i}} - \tilde{B}_0^{\mathbf{i}} : t \geq 0\}$'s are mutually independent, $E_{\mathbf{i}}$ is consequently independent of $\sigma\{X_*^{\mathbf{j}} : \mathbf{j} \in \mathcal{N}_m\}$ for each $\mathbf{i}$ in $\mathcal{N}_m$; since $P(E_{\mathbf{i}}) = \bar{\beta}$, we can conclude (4.26). ∎

Thus we can combine our results thus far to prove the basic version of Theorem 1. Note that all of our reasoning thus far in this section has assumed that (3.3) holds for some z in $[1, J-1]$.

Proposition 4.2 *If (3.3) is true for some* $1 \leq z \leq J-1$, *and if*

$$\bar{\beta}2^{2\gamma-3} > 1, \tag{4.27}$$

where $\bar{\beta}$ *is as in (3.11), then* $P\{\sigma(u) \leq \bar{t}_\gamma\} > 0$. *In this case, in fact* $P\{\sigma(u) \leq \bar{t}_\gamma\} \geq 1 - s^*$, *where* s^* *is the unique solution in* $[0,1)$ *of*

$$s^* = (1-\bar{\beta}) + \bar{\beta}(s^*)^{\lfloor 2^{2\gamma-3} \rfloor}. \tag{4.28}$$

Proof. We simply write down the chain of inequalities

$$\begin{aligned} P\{\tilde{Y} \text{ survives }\} &= \lim_m P\{\tilde{Y}_m \geq 1\} \\ &\leq \limsup_m \limsup_L \limsup_{L'} P\{Y_m^{L,L'} \geq 1\} \\ &\leq P\{\sigma(u) \leq \bar{t}_\gamma\}. \end{aligned}$$

We have used Proposition 4.1 in the last inequality. From classical branching process theory, $\tilde{Y}$ survives if $\bar{\beta}N > 1$, which is (4.27). More exactly, $P\{\tilde{Y} \text{ survives }\} = 1 - s^*$, where s^* solves (4.28) (see any standard text, such as [2] or [5]). ∎

Note that the bound on $P\{\sigma(u) \leq \bar{t}_\gamma\}$ does not depend upon the z chosen in (3.3) nor upon J nor upon the specific shape of u_0.

The final difficulty we must overcome is the requirement that (3.3) be true for some $1 \leq z \leq J-1$. This is not too difficult. If $u_0 \not\equiv 0$, then

$$\int_0^J \phi(0, x; z, J) u_0(x) dx > 0$$

for all $1 \leq z \leq J-1$ by the positivity of Green's functions. Fix any such z and define U and τ as in (3.4) and (3.5). Now by Proposition 3.1, $P\{\tau \wedge \sigma(u) \leq 1\} \geq \beta(\zeta) > 0$, where ζ is as in (3.6). If

$\tau \wedge \sigma(u) = \sigma(u) \leq 1$, then u blows up before time 1; in particular,

$$P\{\sigma(u) \leq 1 | \mathcal{F}_\tau^W\} \chi_{\{\tau\wedge\sigma(u)=\sigma(u)<1\}} = (1)\chi_{\{\tau\wedge\sigma(u)=\sigma(u)<1\}}. \tag{4.29}$$

Thus trivially $\sigma(u) \leq 1 + 2^{4(\gamma-1)}\bar{t}_\gamma$ in this case. Alternately, if $\tau \wedge \sigma(u) = \tau \leq 1$, we look at the process

$$v(t,x) := \frac{1}{2}u\left(t2^{4(1-\gamma)} + \tau, x2^{2(1-\gamma)}\right) \qquad t \geq 0,\ 0 \leq x \leq J2^{2(\gamma-1)}$$

(where $(1/2) \cdot \infty = \infty$ by convention). Now by the semigroup property of solutions to (0.1) and the continuity of (0.1) with respect to initial conditions, we can make an easy argument involving Proposition 4.2 to see that

$$P\{\sigma(v) \leq \bar{t}_\gamma | \mathcal{F}_\tau^W\} \chi_{\{\tau\wedge\sigma(u)=\tau<1\}} \geq (1-s^*)\chi_{\{\tau\wedge\sigma(u)=\tau<1\}} \tag{4.30}$$

and by the strong Markov property of the solutions to (0.1) along with Lemma 3.1, we can see that

$$\begin{aligned} P\{\sigma(v) < \bar{t}_\gamma | \mathcal{F}_\tau^W\}\chi_{\{\tau\wedge\sigma(u)=\tau<1\}} &= P\left\{\tau \leq \sigma(u) < \tau + 2^{4(\gamma-1)}\bar{t}_\gamma \middle| \mathcal{F}_\tau^W\right\}\chi_{\{\tau\wedge\sigma(u)=\tau<1\}} \\ &\leq P\left\{\sigma(u) \leq \tau + 2^{4(\gamma-1)}\bar{t}_\gamma \middle| \mathcal{F}_\tau^W\right\}\chi_{\{\tau\wedge\sigma(u)=\tau<1\}}. \end{aligned} \tag{4.31}$$

Combining the possibilities that $\tau \wedge \sigma(u) = \sigma(u) < 1$ and $\tau \wedge \sigma(u) = \tau < 1$, we have from (4.29)–(4.31) that

$$\begin{aligned} P\{\sigma(u) < \tau + 2^{4(\gamma-1)}\bar{t}_\gamma\} &\geq P\left\{\sigma(u) < \tau + 2^{4(\gamma-1)}\bar{t}_\gamma,\ \tau\wedge\sigma(u) < 1\right\} \\ &= E\left[P\left\{\sigma(u) < \tau + 2^{4(\gamma-1)}\bar{t}_\gamma \middle| \mathcal{F}_\tau^W\right\}\chi_{\{\tau\wedge\sigma(u)=\tau<1\}}\right] \\ &\quad + E\left[P\left\{\sigma(u) < \tau + 2^{4(\gamma-1)}\bar{t}_\gamma \middle| \mathcal{F}_\tau^W\right\}\chi_{\{\tau\wedge\sigma(u)=\sigma(u)<1\}}\right] \\ &\geq (1-s^*)P\{\tau\wedge\sigma(u) = \tau < 1\} + P\{\tau\wedge\sigma(u) = \sigma(u) < 1\} \\ &\geq (1-s^*)\beta(\zeta). \end{aligned}$$

Thus we now have the full result.

Theorem 2. *Assume that u_0 in (0.1) is not equal to zero. If*

$$\gamma > \frac{3}{2} - \frac{1}{2}\log_2\left(\bar{\beta}\right) \tag{4.32}$$

where $\bar{\beta}$ is as in Lemma 3.2, then $P\{\sigma(u) < \infty\} > 0$. If (4.32) holds and

$$\int_0^J \phi(0,x;z,J)u_0(x)dx > 2 \tag{4.33}$$

is true for some $1 \le z \le J-1$, then $P\{\sigma(u) \le \bar{t}_\gamma\} \ge 1 - s^*$, where s^* is as in Theorem 1, whereas if (4.32) holds but (4.33) is not true for any $1 \le z \le J-1$, then

$$P\{\sigma(u) \le 1 + 2^{4(\gamma-1)}\bar{t}_\gamma\} \ge (1-s^*)\beta\left(\sup_{1\le z\le J-1}\int_0^J \phi(0,x;z,J)u_0(x)dx\right)$$

where the function β is as in Proposition 3.1.

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