# CENTRAL LIMIT THEOREM RESULTS FOR A REACTION-DIFFUSION EQUATION WITH FAST-OSCILLATING BOUNDARY PERTURBATIONS

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## 0. Introduction.

In this paper we consider some Central Limit theorems for stochastic reaction-diffusion equations (RDE's) of the form

$$\begin{aligned} \frac{\partial u^{\epsilon}}{\partial t} &= \Delta u^{\epsilon} + f(x, u^{\epsilon}) \\ u^{\epsilon}(0, \cdot) &= u_0 \\ \left.\frac{\partial u^{\epsilon}}{\partial \nu}\right|_{[0,T]\times S^1} &= \zeta^{\epsilon} \end{aligned} \tag{1}$$

where the space variable $x$ takes values on the unit disc $D^2 := \{x \in I\!R^2 : \|x\| \leq 1\}$ and where the Neumann data $\zeta^{\epsilon}$ is a properly normalized fast-oscillating random field on the boundary $[0, T] \times S^1$.

Infinite-dimensional evolution equations may be perturbed in many more natural ways than are possible with ordinary differential equations. Much work has been done on equations of the form

$$\begin{aligned} \frac{\partial u^{\epsilon}}{\partial t} &= \Delta u^{\epsilon} + f(u^{\epsilon}) + \zeta^{\epsilon} \\ u^{\epsilon}(0, \cdot) &= u_0 \end{aligned} \tag{2}$$

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for $x$ in a one-dimensional manifold (see [2], [10] and [11]), but only recently have boundary perturbations been studied (see [3], [9] and [10]). Indeed, in a wide number of cases, such as for example several questions connected with nerve impulse propogation, the correct model for a physical phenomenon is (1) and not (2).

In this short note we shall investigate the behavior of $u^\epsilon$ of (1) where $\zeta^\epsilon$ is a fast-oscillating (in some sense) random field. We of course expect that the limiting behavior will be given by the solution of the corresponding RDE

$$\begin{aligned} \frac{\partial u^0}{\partial t} &= \Delta u^0 + f(x, u^0) \\ u^0(0,\cdot) &= u_0 \\ \left.\frac{\partial u^0}{\partial \nu}\right|_{[0,T]\times S^1} &= \zeta^0 \end{aligned} \tag{3}$$

where $\zeta^0$ is a (perhaps generalized) Gaussian field on $[0,T] \times S^1$ such that $\zeta^\epsilon$ tends to $\zeta^0$ in law. What is of interest here is the formulation of these results in the proper functional spaces; central limit theorems and other asymptotic results for randomly-perturbed processes must be understood as statements about probability measures on the appropriate function spaces. A more complete discussion of the motivation and goals of these types of questions may be found in [3]. We shall in this paper identify the functional spaces associated with the way in which the solution of (1) converges in law to the solution of (3).

The organization of this paper is as follows. In Section 1 we introduce two classes of rapidly-oscillating boundary perturbations which are natural models either of a white noise field on the boundary or of a field which has independent increments in the time direction but is smooth in the space direction. For both types of random fields, we define a Banach space in which to study convergence in law of the random boundary perturbation. In Section 2, we consider the RDE (1) with $f \equiv \mathbf{0}$; for this linear problem we write the solution as a linear transformation of the boundary perturbation; using the Banach space formalism introduced in Section 1, we then define Banach spaces which contain the solutions of (1) and in which the mapping of $\zeta^\epsilon$ to $u^\epsilon$ is continuous. This allows us to conclude, using the convergence results of Section 1, the convergence in law of the solutions of (1) when $f \equiv \mathbf{0}$. Finally, in Section 3, we consider the fully nonlinear version of (1); it is easy to represent the solution of (1) as a continuous transformation of the solution of (1) with $f \equiv \mathbf{0}$; this allows us to immediately achieve the final goal of this paper; to understand the central limit theorem results for (1).

## 1. Two Classes of Boundary Perturbations.

In this section, we introduce two fairly general types of rapidly-oscillating boundary perturbations, and define the appropriate Banach spaces in which to study the asymptotics of the laws of these random fields.

To introduce these perturbations, we shall assume an underlying probability triple $(\Omega, \mathcal{F}, P)$ and an i.i.d. collection $\{\xi_i\}$ of real-valued second order stationary Markov processes indexed by $\mathbb{R}_+ := [0, \infty)$, and an i.i.d. collection $\{W_i\}$ of Wiener processes on $[0,T]$. Our general boundary perturbation shall be of the form

$$\zeta^\epsilon(t,x) := \sum_{i=0}^{\infty} c_{i,\epsilon} \phi_i(x) \epsilon^{-1} \xi_i(t\epsilon^{-2}) \qquad (t,x) \in [0,T] \times S^1 \tag{4}$$

for each $\epsilon > 0$, where $\{c_{i,\epsilon}\}$ are nonrandom constants such that $\sum_{i=1}^{\infty} |c_{i,\epsilon}| < \infty$, and where $\{\phi_i\}$ are the eigenfunctions of the Laplacian operator on $S^1$; i.e.,

$$\phi_i((\cos\theta, \sin\theta)) = (\cos i\theta, \sin i\theta)$$

for all $\theta$ in $[0, 2\pi)$ and all $i$. We assume that there is a $\{c_{i,0}\}$ such that $\lim_\epsilon c_{i,\epsilon} = c_{i,0}$. Under some natural statistical assumptions to be specified in the next paragraph, we then expect that $\zeta^\epsilon$ will tend in law to the Gaussian field $\zeta^0$, where formally

$$\zeta^0(t,x) = \sum_{i=0}^{\infty} c_{i,0}\phi_i(x)\dot{W}_k(t). \tag{5}$$

If $c_{i,0} = 1$ for all $i$, then $\zeta^0$ will be a white noise field on $[0,T] \times S^1$. If on the other hand $\lim_i c_{i,0} = 0$, this decay being fast enough, $\zeta^0$ will be smooth in the $x$ variable, but will have independent increments in the time variable. Our two classes of boundary perturbations, $\mathcal{C}_1$, and $\mathcal{C}_2$ will correspond to these two possibilities.

The exact statistical assumptions which we shall require of $\{\xi_0\}$ (and thus of $\{\xi_i\}$ for all $i = 0, 1, \ldots$, since the $\{\xi_i\}$ are i.i.d.) are the following. We assume that

**(A.1)** The process $\xi$ is second-order stationary with $E\xi_0(0) = 0$ and $\int_0^\infty |E[\xi_0(t)\xi_0(0)]|dt < \infty$.

**(A.2)** There is a positive constant $B$ such that $P\{|\xi_0(t)| \geq B\} = 0$ for all $t$ in $\mathbb{R}_+$. Also, $\int_0^\infty E[\xi_0(t)\xi_0(t)dt = 1/2$.

To state the third requirement, let us define

$$\mathcal{F}_t := \sigma\{\xi_0(s) : s \leq t\} \quad \text{and} \quad \mathcal{F}^t := \sigma\{\xi_0(s) : s \geq t\}$$

for all $t$ in $\mathbb{R}_+$. Let us also define the mixing coefficient $\alpha$ as

$$\begin{aligned}\alpha(\tau) := \sup\{&E[\eta_1\eta_2] - E[\eta_1]E[\eta_2] : |\eta_1| \leq 1 \text{ and } |\eta_2| \leq 1 \ P\text{-a.s.},\\ &\eta_1 \text{ is } \mathcal{F}_t\text{-measurable and } \eta_2 \text{ is } \mathcal{F}^{t+\tau}\text{-measurable},\\ &\tau > 0\}\end{aligned}$$

for each $\tau > 0$. The third requirement on $\{\xi_0\}$ is that for some integer $k > 1$, we have

$(\mathbf{A.3})_\mathbf{k}$ The integral $\int_0^\infty \tau^{k-1}\alpha(\tau)d\tau < \infty$.

Note that under these assumptions and the assumption that $\sum_{i=0}^{\infty} |c_{i,\epsilon}| < \infty$ for each $\epsilon > 0$, we have that

$$\sup_{(t,x)\in[0,T]\times S^1} |\zeta^\epsilon(t,x)| \leq B\epsilon^{-1} \sum_{i=0}^{\infty} |c_{i,\epsilon}| < \infty.$$

Also, these assumptions are sufficient to imply that $\tilde{\xi}_i^\epsilon$ tends to $W_i$ weakly in $C^\gamma([0,T])$ for each $0 < \gamma < (k-1)/(2k)$, where

$$\tilde{\xi}_i^\epsilon(t) := \int_0^t \epsilon^{-1}\xi_i(s\epsilon^{-2})ds$$

for all $i$ and all $t$ in $[0,T]$ (see [3] and [8]).

The first class of boundary perturbations, which we shall denote by $\mathcal{C}_1$, consists of boundary perturbations of the form (4) where we have

**(C.1)** that $\sup_{i,\epsilon\geq 0} |c_{i,\epsilon}| < \infty$.

This class obviously allows that $c_{i,0} = 1$ for all $i$, which gives us a white noise field as the limiting Gaussian boundary perturbations. This class thus admits that the limiting Gaussian field be a generalized function both in time and in space. Calculations will be somewhat easier with the second class of boundary perturbations, $\mathcal{C}_2$, which consists of random fields of the form (4) where for some $0 < \beta \leq 1$, we have

$(\mathbf{C.2})_\beta$ that $\sup_{\epsilon \geq 0} \sum_{i=0}^{\infty} i^\beta |c_{i,\epsilon}| < \infty$.

We shall see that in this class of perturbations, the limiting Gaussian field will still maintain independent increments in the $t$ direction, but will be smooth in the $x$ direction; i.e., the limiting Gaussian field will be a generalized function in $t$ but a regular function in $x$.

Recalling in more detail the results stated earlier, we have that the finite-dimensional distributions of $\tilde{\xi}_i$ converge to those of $W_i$ for each $i$, and that for each $\gamma < (k-1)/(2k)$, there are nonnegative random variables $\{Y^{i,\epsilon,\gamma}\}$ with $\sup_{i,\epsilon \geq 0} E[Y^{i,\epsilon,\gamma}] < \infty$ for each $\gamma$ and such that $P$-a.s.

$$|\tilde{\xi}_i^\epsilon(t) - \tilde{\xi}_i^\epsilon(s)| \leq Y^{i,\epsilon,\gamma} |t-s|^\gamma$$

for all $0 \leq s \leq t \leq T$ and all $\epsilon \geq 0$. Thus $\tilde{\xi}_i \Rightarrow W_i$ in $C^\gamma([0,T])$ for each $0 < \gamma < (k-1)/(2k)$ (here $\Rightarrow$ denotes convergence in law in the relevant Polish space, which in this case is the space of real-valued functions on $[0,T]$ which are Hölder-continuous of exponent $\gamma$).

We can immediately use these results to consider the limiting law of $\zeta^\epsilon$ under the assumptions $(\mathbf{A.1})$–$(\mathbf{A.3})_\mathbf{k}$ and $(\mathbf{C.1})$ or $(\mathbf{C.2})_\beta$. We consider first boundary perturbations of the class $\mathcal{C}_2$, since the analysis is somewhat easier. Set

$$\tilde{\zeta}^\epsilon(t,x) := \int_0^t \zeta^\epsilon(s,x) ds \tag{6}$$

for each $\epsilon \geq 0$ and each $(t,x)$ in $[0,T] \times S^1$. By Prohorov's theorem to show the convergence of the laws of $\{\tilde{\zeta}^\epsilon\}$, we need to show that the laws are tight in the appropriate space and that all cluster points of these laws coincide. Let us first consider the tightness question. For each $0 < \gamma < (k-1)/(2k)$ and all $(t,x)$ and $(s,y)$ in $[0,T] \times S^1$, we have that

$$|\tilde{\zeta}^\epsilon(t,x) - \tilde{\zeta}^\epsilon(t,y) - \tilde{\zeta}^\epsilon(s,x) + \tilde{\zeta}^\epsilon(s,y)| \leq \sum_{i=0}^{\infty} |c_{i,\epsilon}| |\phi_i(x) - \phi_i(y)| Y^{i,\epsilon,\gamma} |t-s|^\gamma.$$

Using the fact that $|\phi_i(x) - \phi_i(y)| \leq \min\{2, ir(x,y)\}$ for all $i$ and all $x$ and $y$ (here $r(\cdot,\cdot)$ is the natural metric on $S^1$), we have that for each $(t,x)$ and $(s,y)$ in $[0,T] \times S^1$,

$$|\tilde{\zeta}^\epsilon(t,x) - \tilde{\zeta}^\epsilon(t,y) - \tilde{\zeta}^\epsilon(s,x) + \tilde{\zeta}^\epsilon(s,y)| \leq 2^{1-\beta} \sum_{i=0}^{\infty} i^\beta |c_{i,\epsilon}| Y^{i,\epsilon,\gamma} r^\beta(x,y) |t-s|^\gamma. \tag{7}$$

This indicates that we should define the Banach spaces $C_{\beta,\gamma}$ for each $0 < \gamma < 1$ and $0 < \beta' < 1$ as as the closure of $C^\infty([0,T] \times S^1)$ with respect to the norm

$$|||\varphi|||_{\beta,\gamma} := \sup_{(t,x) \in [0,T] \times S^1} |\varphi(t,x)| + \sup_{\substack{0 \leq s < t \leq T \\ x,y \in S^1, x \neq y}} \frac{|\varphi(t,x) - \varphi(s,x) - \varphi(t,y) + \varphi(s,y)|}{r^\alpha(x,y) |t-s|^\gamma}$$

on $C^\infty([0,T] \times S^1)$. The norm $|||\varphi|||_{\beta,\gamma}$ measures the variation of $\varphi$ over rectangles, in a way analogous to simple Hölder continuity. From the bound (7), we have that

$$E[|||\tilde{\zeta}^\epsilon|||_{\gamma,\beta}] \leq K \left( \sup_{\epsilon > 0} \sum_{i=0}^{\infty} i^\beta |c_{i,\epsilon}| \right) \left( \sup_{i,\epsilon > 0} E[Y^{i,\epsilon,\gamma}] \right)$$

for all $\epsilon > 0$, where $K$ is some constant independent of $\epsilon > 0$. For each $0 < \gamma < (k-1)/(2k)$. By standard embedding techniques and Arzela-Ascoli, clearly $C_{\gamma_1,\alpha_1}$ will be compactly embedded in $C_{\gamma_2,\alpha_2}$ whenever $\gamma_1 < \gamma_2$ and $\alpha_1 < \alpha_2$. This gives us the required tightness. It is easy to see, by projecting along a finite number of $\{\phi_i\}$, that all cluster points (in the weak topology) of the laws of $\{\tilde{\zeta}^\epsilon\}$ must be the law of the Gaussian random field

$$\tilde{\zeta}^0(t,x) := \sum_{i=0}^{\infty} c_{i,0}\phi_k(x)W_i(t).$$

This is enough to complete the proof of the following claim:

**Proposition 1.** *Assume that assumptions* **(A.1)**–$\mathbf{(A.3)_k}$ *and* $\mathbf{(C.2)}_\beta$ *hold. Then for all* $\gamma < (k-1)/(2k)$ *and all* $\beta' < \beta$, $\tilde{\zeta}^\epsilon \Rightarrow \tilde{\zeta}^0$ *in* $C_{\gamma,\beta'}$.

Now consider the assumptions **(A.1)**–$\mathbf{(A.3)_k}$ and **(C.1)**. We expect the limiting law to be that of $\zeta^0$ as in (5). As we earlier noted, the assumption **(C.1)** allows us to consider white noise on $[0,T]\times S^1$, so it is natural to now integrate $\zeta^\epsilon$ in both variables. In analogy to (6), let us set

$$\bar{\zeta}^\epsilon(t,x) := \int_0^t \int_{y\in\mathcal{E}(x)} \zeta^\epsilon(s,y)dsdy$$

for all $(t,x)$ in $[0,T]\times[0,2\pi]$, where $\mathcal{E}(x) := \{(\cos\theta,\sin\theta) : 0 \le \theta \le x\}$ for each $x$ in $[0,2\pi)$. To show the tightness, we can use the procedures of [5] to show that for any $(t,x)$ ands $(s,y)$ in $[0,T]\times[0,2\pi]$,

$$E[|\bar{\zeta}^\epsilon(t,x) - \bar{\zeta}^\epsilon(t,y) - \bar{\zeta}^\epsilon(s,x) + \bar{\zeta}^\epsilon(s,y)|^{2k}] \le K|t-s|^k|x-y|^k \tag{8}$$

(see the appendix) where $K$ is some constant independent of the $(t,x)$ and $(s,y)$. By using a natural extension of Garsia's estimate (see [4]), we then get that for each $0 < \gamma < (k-1)/(2k)$, there are nonnegative random variables $\{\bar{Y}^{\gamma,\epsilon}\}$ with $\sup_{\epsilon>0} E[\bar{Y}^{\gamma,\epsilon}] < \infty$ such that

$$|\bar{\zeta}^\epsilon(t,x) - \bar{\zeta}^\epsilon(t,y) - \bar{\zeta}^\epsilon(s,x) + \bar{\zeta}^\epsilon(s,y)| \le \bar{Y}^{\gamma,\epsilon}|t-s|^\gamma|x-y|^\gamma \tag{9}$$

for all $(t,x)$ and $(s,y)$ in $[0,T]\times[0,2\pi]$ (see Lemma 1 in the appendix). Once again, it is easy to show that all cluster points of the laws of $\{\bar{\zeta}^\epsilon\}$ must coincide with the law of

$$\bar{\zeta}^0(t,x) := \sum_{i=0}^{\infty}\left(\int_{y\in\mathcal{E}(x)} \phi_k(y)dy\right)W_i(t).$$

This gives us the result analogous to Proposition 1:

**Proposition 2.** *Assume that assumptions* **(A.1)**–$\mathbf{(A.3)_k}$ *and* **(C.1)** *hold. Then for all* $0 < \gamma < (k-1)/(2k)$, $\bar{\zeta}^\epsilon \Rightarrow \bar{\zeta}^0$ *in* $C_{\gamma,\gamma}$.

## 2. The linear problem.

We can directly use the results of the last section to study the limiting laws of the linear problem associated with (1); let $u_l^\epsilon$ be given by

$$\begin{aligned} \frac{\partial u_l^\epsilon}{\partial t} &= \Delta u_l^\epsilon \\ u_l^\epsilon(0,\cdot) &= \mathbf{0} \\ \left.\frac{\partial u_l^\epsilon}{\partial \nu}\right|_{[0,T]\times S^1} &= \zeta^\epsilon. \end{aligned} \tag{10}$$

If $p$ is the Green's function for (10); i.e., the solution of the generalized problem

$$\frac{\partial p}{\partial t} = \Delta_x p$$
$$p(0,\cdot,y) = \delta_y$$
$$\left.\frac{\partial_x p}{\partial \eta}\right|_{[0,T]\times S^1} = \mathbf{0},$$

(here the subscript $x$ is added to the Laplacian and normal derivation operators to emphasize that these operators are acting on the $x$-dependence of the arguments, and here $\delta_y$ is the Dirac delta generalized function at $y$) then $u_l^\epsilon$ of (10) may be represented as

$$u_l^\epsilon(t,x) = \int_0^t \int_{S^1} p(t,x,y)\zeta^\epsilon(s,y)dsdy \tag{11}$$

for all $(t,x)$ in $[0,T]\times D^2$ and all $\epsilon > 0$. (Under the assumption that $\sum_{i=0}^{\infty} |c_{i,\epsilon}| < \infty$ for each $\epsilon > 0$, $\zeta^\epsilon$ is a well-defined function, so classical results of existence and uniqueness hold.) The goal is then to write (11) as $u^\epsilon = B_1(\tilde{\zeta}^\epsilon)$ for $\{\zeta^\epsilon\}$ in class $\mathcal{C}_1$ and $u^\epsilon = B_2(\bar{\zeta}^\epsilon)$ for $\{\zeta^\epsilon\}$ in class $\mathcal{C}_2$, where $B_1$ and $B_2$ are bounded linear operators from the appropriate boundary-function spaces as introduced in Section 1 to the appropriate solution space.

The basis for the necessary estimates is the following calculation; consider the problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v \\ v(0,\cdot) &= \mathbf{0} \\ \left.\frac{\partial v}{\partial \nu}\right|_{[0,T]\times \partial H^2} &= g \end{aligned} \tag{12}$$

in the half space $H^2 := \{(x_1,x_2) : x_2 \geq 0\}$. Here $g$ is any continuous and bounded function on $[0,T]\times \partial H^2$ For the problem (12), the solution is given by

$$v(t,x) = \int_0^t \int_{\mathbb{R}} p_G(t-s,x,(y,0))g(s,(y,0))dsdy \tag{13}$$

for all $(t,x)$ in $[0,T]\times H^2$, where $p_G$ is the heat kernel

$$p_G(t,x,y) = (4\pi t)^{-1}\exp\left(\frac{\|x-y\|^2}{4\pi t}\right)$$

for all $t>0$ and all $x$ and $y$ in $\mathbb{R}^2$. Rewriting, we have that

$$\begin{aligned} v(t,(x_1,x_2)) &= -2\int_0^t \int_{\mathbb{R}} \frac{\partial p_G}{\partial t}(r,(x_1,x_2),(y,0))\left(\int_{t-r}^t g(s,(y,0))ds\right)drdy \\ &= -2\int_0^t \int_{\mathbb{R}} \frac{\partial p_G}{\partial t}(r,(x_1,x_2),(y,0)) \\ &\qquad \cdot\left\{\int_{t-r}^t g(s,(y,0))ds - \int_{t-r}^t g(s,(x_1,0))ds\right\}drdy. \end{aligned} \tag{14}$$

Using the easily-verified fact that there is a positive constant $K$ such that

$$\left|\frac{\partial p_G}{\partial t}(t,x,y)\right| \le Kt^{-2}\exp\left(-\frac{\|x-y\|^2}{3t}\right)$$

for all $t>0$ and all $x$ and $y$ in $I\!R^2$, we thus have that

$$\begin{aligned}|v(t,(x_1,x_2))| \le 2K\int_0^t\int_{I\!R} r^{-2}\exp\left(-\frac{|x_1-y|^2}{3r}\right)\exp\left(-\frac{|x_2|^2}{3r}\right)\\ \cdot\left|\int_{t-r}^t g(s,(y,0))ds - \int_{t-r}^t g(s,(x_1,0))ds\right| drdy.\end{aligned} \tag{15}$$

We then get, after transferring the calculation (15) back to the manifold $D^2$, that if $0<\gamma''<\gamma<(k-1)/(2k)$ and $0<\beta''<\beta'<\beta$, there is a positive constant $K$ such that

$$|u^\epsilon(t,x)| \le \begin{cases} K|||\tilde\zeta^\epsilon|||_{\gamma',\beta'} & \text{if } 2\gamma''+\beta''-1\ge 0;\\ K|||\tilde\zeta^\epsilon|||_{\gamma,\beta'}(\text{dist}(x,S^1))^{-(2\gamma''+\beta''-1)} & \text{if } 2\gamma''+\beta''-1<0\end{cases}$$

for all $\epsilon\ge 0$. Here $\text{dist}(x,S^1)$ is the distance between $x$ and $S^1$ for any point $x$ in $D^2$. The proper space in which to study the laws of $\{u^\epsilon\}$ for boundary perturbations of class $\mathcal{C}_2$ is thus the collection of spaces $C'_\eta$, where $C'_\eta$ is the closure of $C^\infty([0,T]\times D^2)$ in the norm

$$|||\varphi|||'_\eta := \begin{cases}\sup_{(t,x)\in[0,T]\times D^2}|\varphi(t,x)| & \text{if } \eta\ge 0\\ \sup_{(t,x)\in[0,T]\times D^{2,\circ}}|\varphi(t,x)|(\text{dist}(x,S^1))^{-\eta} & \text{if } \eta<0.\end{cases}$$

The calculations of (15) give us that for any $0<\gamma''<\gamma<(k-1)/(2k)$ and $0<\beta''<\beta'<\beta$, the mapping which takes $\tilde\zeta^\epsilon$ to $u^\epsilon$ is a continuous linear transformation from $C_{\gamma',\beta'}$ to $C'_{2\gamma''+\beta''-1-\eta}$, so the following result is true;

**Proposition 3.** *Under the assumptions* **(A.1)**–**(A.3)**$_{\mathbf{k}}$ *and* **(C.2)**$_\beta$*, for each* $\beta'<\beta$ *and each* $\gamma<(k-1)/(2k)$*,* $u^\epsilon\Rightarrow u^0$ *in* $C'_{2\gamma+\beta'-1}$.

Similar calculations hold for boundary perturbations of class $\mathcal{C}_2$. Instead of (15), we should rewrite (13) as

$$\begin{aligned}v(t,(x_1,x_2)) &= -2\int_0^t\int_{I\!R}\frac{\partial^2 p_G}{\partial t\partial y}(r,(x_1,x_2),(z,0))\left(\int_{t-r}^t\int_{y\ge z} g(s,(y,0))dsdy\right)drdz\\ &= -2\int_0^t\int_{I\!R}\frac{\partial^2 p_G}{\partial t\partial y}(r,(x_1,x_2),(z,0))\\ &\quad\cdot\left(\int_{t-r}^t\int_{y\ge z} g(s,(y,0))dsdy - \int_{t-r}^t\int_{y\ge x} g(s,(y,0))dsdy\right)drdz\end{aligned}$$

so that using the fact that there is a positive constant $K$ such that

$$\left|\frac{\partial^2 p_G}{\partial t\partial y}(t,x,y)\right| \le Kt^{-5/2}\exp\left(-\frac{\|x-y\|^2}{3t}\right)$$

for all $t>0$ and all $x$ and $y$ in $I\!R^2$, we thus have that

$$\begin{aligned}|v(t,(x_1,x_2))| \le 2K\int_0^t\int_{I\!R} r^{-5/2}\exp\left(-\frac{|x_1-y|^2}{3r}\right)\exp\left(-\frac{|x_2|^2}{3r}\right)\\ \cdot\left|\int_{t-r}^t\int_{y\ge z} g(s,(y,0))dsdy - \int_{t-r}^t\int_{y\ge x} g(s,(y,0))dsdy\right| drdz.\end{aligned} \tag{16}$$

Transferring this calculation to $D^2$, we get that if $0 < \gamma' < \gamma < (k-1)/(2k)$, then there is a constant $K$ such that

$$|u^\epsilon(t,x)| \leq \begin{cases} K|||\tilde{\zeta}^\epsilon|||_{\gamma',\gamma'} & \text{if } 3\gamma' - 2 \geq 0; \\ K|||\tilde{\zeta}^\epsilon|||_{\gamma,\beta'}(\mathrm{dist}(x,S^1))^{-(3\gamma'-2)} & \text{if } 3\gamma' - 2 < 0 \end{cases}$$

for all $\epsilon > 0$; i.e., for any $0 < \gamma' < \gamma < (k-1)/(2k)$ the mapping from $\bar{\zeta}^\epsilon$ to $u^\epsilon$ is a continuous linear transformation from $C_{\gamma,\gamma}$ to $C'_{3\gamma'-2}$. This gives us

**Proposition 4.** *Under assumptions* **(A.1)**–**(A.3)$_\mathbf{k}$** *and* **(C.1)**, *for each* $0 < \gamma < (k-1)/(2k)$, $u^\epsilon \Rightarrow u^0$ *in* $C'_{3\gamma-2}$.

## 3. The nonlinear problem.

The only remaining task before us is to use the results of the last section, i.e., the results about the linear equation (10) to show the corresponding results for the nonlinear problem. This is easily accomplished by considering the mapping from $\varphi \mapsto \hat{B}(\varphi)$ defined by the integral equation

$$(\hat{B}(\varphi))(t,x) = \int_{D^2} p(t,x,y)u_0(y)dy + \varphi(t,x) + \int_0^t \int_{D^2} p(t-s,x,y)f(y,(\hat{B}(\varphi))(s,y))dsdy \quad (17)$$

for all $(t,x)$ in $[0,T] \times D^2$ and all functions $\varphi$ on $[0,T] \times D^2$ such that (17) is well-defined. For each $\gamma$ in $I\!R$, the assumptions

**(D.1)$_\gamma$** There is a constant $\bar{F}$ and two exponents $\delta_1$ and $\delta_2 > 0$ satisfying $\delta_1 - \gamma\delta_2 < -1$, such that for all $x$ in $D^2$ and all $u$ in $I\!R$,

$$|f(x,u)| \leq \bar{F}(1 + (\mathrm{dist}(x,S^1))^{\delta_1}|u|^{\delta_2})$$

and

**(D.2)$_\gamma$** There is a constant $\bar{f}$ such that for all $x$ in $D^2$ and all $u$ and $v$ in $I\!R$,

$$|f(x,u) - f(x,v)| \leq \bar{f}(\mathrm{dist}(x,S^1))^\gamma|u-v|$$

are sufficient to ensure that $\hat{B}$ is a well-defined homeomorphism of $C'_\gamma$ to itself (see [9] or [10]). Note that if $0 < \gamma_1 < \gamma_2$, then assumptions **(D.1)$_{\gamma_2}$** and **(D.2)$_{\gamma_2}$** imply assumptions **(D.1)$_{\gamma_1}$** and **(D.2)$_{\gamma_1}$**. This gives us the complete results about (1);

**Theorem 1.** *Assume that assumptions* **(A.1)**–**(A.3)$_\mathbf{k}$** *and* $(C.1)_\beta$ *hold, and that assumption* $(D.1)_\gamma$ *and* $(D.2)_\gamma$ *hold for some* $0 < \gamma < 3(k-1)/(2k) - 2$. *Then* $u^\epsilon \Rightarrow u^0$ *in* $C'_\gamma$.

**Theorem 2.** *Assume that assumptions* **(A.1)**–**(A.3)$_\mathbf{k}$** *and* **(C.2)$_\beta$** *hold, and that assumption* **(D.1)$_\gamma$** *and* **(D.2)$_\gamma$** *hold for some* $0 < \gamma < (k-1)/k + \beta - 1$. *Then* $u^\epsilon \Rightarrow u^0$ *in* $C'_\gamma$.

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**Appendix.**

We now explain in greater detail how to get the estimates (8) and (9). Let us begin with by estimating

$$E\left[\left(\sum_{i=l_1}^{l_2} c_{i,\epsilon}\int_s^t \epsilon^{-1}\xi_i(r\epsilon^{-2})dr\int_{z\in\mathcal{E}(y)\sim\mathcal{E}(x)}\phi_i(z)dz\right)^{2k}\right] \tag{A.1}$$

for any fixed $0\leq s<t\leq t\leq T$, $0\leq x<y\leq 2\pi$ and $\epsilon>0$, and any two nonnegative integers $l_1$ and $l_2$. For convenience, let us define

$$\Xi_i := c_{i,\epsilon}\int_s^t \epsilon^{-1}\xi_i(r\epsilon^{-2})dr\int_{z\in\mathcal{E}(y)\sim\mathcal{E}(x)}\phi_i(z)dz$$

for each $i=0,1,\dots$. Using the notation $E:=\{l_1,l_1+1,\cdots,l_2\}$, we can write

$$E\left[\left(\sum_{i=l_1}^{l_2}\Xi_i\right)^{2k}\right] = \sum_{(i_1,i_2,\cdots i_{2k})\in E^{2k}} E\left[\Xi_{i_1}\Xi_{i_2}\cdots\Xi_{i_{2k}}\right].$$

The expression on the right may be rearranged so that the indices are increasing; let us define for each positive integer $p$ the set $E_p:=\{(i_1,i_2,\cdots,i_p)\in E^p : i_1\leq i_2\cdots i_p\}$. Then

$$E\left[\left(\sum_{i=l_1}^{l_2}\Xi_i\right)^{2k}\right] = (2k)!\sum_{(i_1,i_2,\cdots i_{2k})\in E_{2k}} E\left[\Xi_{i_1}\Xi_{i_2}\cdots\Xi_{i_{2k}}\right]. \tag{A.2}$$

Since the $\{\xi_i\}$ are i.i.d. with zero mean, we have that $E\left[\Xi_{i_1}\Xi_{i_2}\cdots\Xi_{i_{2k}}\right]=0$ for any $(i_1,i_2,\cdots i_{2k})$ in $E_{2k}$ such that $i_{p-1}<i_p<i_{p+1}$ for some integer $1\leq p<2k$; thus the sum in (A.2) consists of terms with *groups* of at least two equal indices. With this thought, we have

$$E\left[\left(\sum_{i=l_1}^{l_2}\Xi_i\right)^{2k}\right] = (2k)!\sum_{n=1}^{2k}\sum_{\substack{(j_1,j_2,\cdots,j_n)\in\mathbb{I}^n\\ j_p\geq 2\text{ all }p,\ \sum_{p=1}^n j_p=2k}}\ \sum_{(i_1,i_2,\cdots i_n)\in E_n} E[(\Xi_{i_1})^{j_1}]E[(\Xi_{i_2})^{j_2}]\cdots E[(\Xi_{i_n})^{j_n}] \tag{A.3}$$

where $\mathbb{I}:=\{1,2,\dots\}$. Consider now any such term $E[(\Xi_i)^p]$ for any nonnegative integers $i$ and $p$ with $p\geq 2$. We have that

$$E[(\Xi_i)^p] = c_{i,\epsilon}^p\langle\phi_i,\chi_A\rangle^p E\left[\left(\int_s^t\epsilon^{-1}\xi_i(r\epsilon^{-2})dr\right)^p\right].$$

By the results of Khas'minskii, we know that by assumption $(\mathbf{A.3})_\mathbf{k}$ implies that there is a constant $K$, which is independent of $s$, $t$, and $\epsilon$, and which we may assume to be greater than 1, such that

$$E\left[\left(\int_s^t\epsilon^{-1}\xi_i(r\epsilon^{-2})dr\right)^p\right]\leq K|t-s|^{p/2}.$$

(Khas'minskii gave the result for $p$ even in [5]; if $p$ is odd, we may simply use Lyapunov's inequality– see [6], P. 34). With this estimate and Cauchy-Schwartz, we have that

$$E[(\Xi_i)^p]\leq K(\sup_{q,\epsilon}|c_{q,\epsilon}|^p)\langle\phi_i,\chi_A\rangle^2\|\chi_A\|_{L^2(S^1)}^{p-2}|t-s|^{p/2}.$$

Inserting this into (A.3), we have (recall that we assumed that $K \geq 1$)

$$E\left[\left(\sum_{i=l_1}^{l_2} \Xi_i\right)^{2k}\right] \leq (2k)!K^{2k}\left(\sup_{p,\epsilon}|c_{p,\epsilon}|^{2k}\right)|t-s|^k$$

$$\cdot \sum_{n=1}^{2k} \|\chi_A\|_{L^2(S^1)}^{2k-2n} \sum_{\substack{(j_1,j_2,\cdots,j_n)\in I\!I^n \\ j_p\geq 2 \text{ all } p, \sum_{p=1}^n j_p=2k}} \sum_{(i_1,i_2,\cdots i_n)\in E_n} \prod_{q=1}^n \langle \phi_{i_q}, \chi_A\rangle^2.$$

The innermost sum is bounded from above by

$$\sum_{(i_1,i_2,\cdots i_n)\in E_n} \prod_{q=1}^n \langle \phi_{i_q}, \chi_A\rangle^2 \leq \sum_{(i_1,i_2,\cdots i_n)\in E^n} \prod_{q=1}^n \langle \phi_{i_q}, \chi_A\rangle^2 \leq \left(\sum_{i\in E} \langle \phi_{i_q}, \chi_A\rangle^2\right)^n$$

so since the cardinality of the set $\{(j_1, j_2, \cdots, j_n) \in I\!I^n : j_p \geq 2 \text{ all } p, \sum_{p=1}^n j_p = 2k\}$ is clearly less than $(2k-2)^n$, we have that

$$E\left[\left(\sum_{i=l_1}^{l_2} c_{i,\epsilon}\int_s^t \epsilon^{-1}\xi_i(r\epsilon^{-2})dr \int_{z\in\mathcal{E}(y)\sim\mathcal{E}(x)} \phi_i(z)dz\right)^{2k}\right]$$

$$\leq (2k)!K^{2k}\left(\sup_{p,\epsilon}|c_{p,\epsilon}|\right)^{2k}(2k)(2k-2)^{2k}|t-s|^k\|\chi_A\|_{L^2(S^1)}^{2k-2}\left(\sum_{i=l_1}^{l_2}\langle\phi_i,\chi_A\rangle^2\right)$$

$$\leq \tilde{K}|t-s|^k\|\chi_A\|_{L^2(S^1)}^{2k-2}\left(\sum_{i=l_1}^{l_2}\langle\phi_i,\chi_A\rangle^2\right).$$

It is not difficult to see from this that we may pass to the limit $l_1 = 0$ and $l_2 \to \infty$ in (A.1), and that (8) must hold.

Now we should indicate why (8) implies (9). We shall do this by adapting Garsia's celebrated result (see [1], [4], and [11]). Let $\{n_i;\ i = 1, 2, \ldots, l\}$ be a collection of positive integers, and let $\mathcal{C}_i$ be the unit cube in $I\!R^{n_i}$. Let $\Psi$ be a positive, convex, and continuous function on $[0, \infty)$ with $\lim_{x\to\infty}\Psi(x) = \infty$. Let for each $i = 1, 2, \ldots, l\}$, $p_i$ be a nonnegative and increasing function on $[0, \infty)$ with $p_i(0) = 0$. Let $\|x\|$ be the Euclidean distance of $x$, for any $x$ in $\bigcup_{i=1}^l I\!R^{n_i}$. Finally, if $\varphi$ is a mapping from $\bar{\mathcal{C}} := \mathcal{C}_{n_1} \times \mathcal{C}_{n_2} \cdots \mathcal{C}_{n_l}$ to $I\!R$, and $\bar{x}^1 = (x_1^1, x_2^1, \ldots, x_l^1)$ and $\bar{x}^2 = (x_1^2, x_2^2, \ldots, x_l^2)$ are in $\bar{\mathcal{C}}$, the define by an abuse of notation the variation of $\varphi$ over the rectangle $[\bar{x}^1, \bar{x}^2]$ as

$$\varphi([\bar{x}^1, \bar{x}^2]) := \sum_{(i_1,i_2,\ldots,i_l)\in\{1,2\}^l} (-1)^{|\{j:i_j=2\}|}\varphi(x_1^{i_1}, x_2^{i_2}, \ldots, x_l^{i_l}) \tag{A.4}$$

where for any set $A$, $|A|$ is its cardinality. Our theorem is

**Lemma 1.** *If $\varphi$ is a measurable function on $\bar{\mathcal{C}}$ such that*

$$B := \int_{\bar{x}^1=(x_1^1,x_2^1,\ldots,x_l^1)\in\bar{\mathcal{C}}}\int_{\bar{x}^2=(x_1^2,x_2^2,\ldots,x_l^2)\in\bar{\mathcal{C}}} \Psi\left(\frac{|\varphi([\bar{x}^1,\bar{x}^2])|}{\prod_{i=1}^l p_i(\|x_i^2 - x_i^1\|/\sqrt{n_i})}\right) d\bar{x}^1 d\bar{x}^2$$

*is finite, then there is a subset* $K$ *of* $\bar{\mathcal{C}}$ *of Lebesgue measure zero such that if* $\bar{x}^1 = (x_1^1, x_2^1, \ldots, x_l^1)$ *and* $\bar{x}^2 = (x_1^2, x_2^2, \ldots, x_l^2)$ *are in* $\bar{\mathcal{C}} \sim K$*, then*

$$|\varphi([\bar{x}^1, \bar{x}^2])| \leq 2^l \cdot 4 \int_{\substack{0 \leq u_i \leq \|x_i^2 - x_i^1\| \\ i=1,2,\ldots,l}} \Psi^{-1}\left(\frac{B}{\prod_{i=1}^l u^{2n_i}}\right) dp_1(u_1) dp_2(u_2) \cdots dp_l(u_l).$$

**Proof.** For any rectangle $Q$ in $\bigcup_{i=1}^l I\!R^{n_i}$, let $|Q|$ be its volume and $e(Q)$ the length of its edge. Let us consider a collection of rectangles $\{Q_i^1; i = 1, 2, \ldots, l\}$ with $Q_i^1$ contained in $\mathcal{C}_i$ for each $i = 1, 2, \ldots$. We allow the possibility of a degenerate rectangle, i.e., a single point. Let us define, by another abuse of notation, the averaged value of $\varphi$ as

$$\varphi(Q_1, Q_2, \ldots, Q_l) = \left(\prod_{i=1}^l \frac{1}{|Q_i|}\right) \int_{\substack{x_i \in Q_i \\ i=1,2,\ldots,l}} \varphi(x_1, x_2, \ldots, x_l) dx_1 dx_2 \ldots dx_l,$$

with averaginng being replaced by simple evaluation for any indices for which the rectangles are degenerate. Let us now two such collection of rectangles $\{Q_i^1; i = 1, 2, \ldots, l\}$ and $\{Q_i^2; i = 1, 2, \ldots, l\}$ with $Q_i^1 \supset Q_i^2$ and $p_i(e(Q_i^2)) = \frac{1}{2} p_i(e(Q_i^1))$ for all $i = 1, 2, \ldots, l$. The techniques of Garsia (see [1], [4], or [11]) then give us that

$$\left| \sum_{(j_1, j_2, \ldots, j_l) \in \{1,2\}^l} (-1)^{|\{r : j_r = 2\}|} \varphi(Q_1^{j_1}, Q_2^{j_2}, \cdots, Q_l^{j_l}) \right|$$
$$\leq 4 \int_{\substack{e(Q_i^1) \leq u_i \leq e(Q_i^2) \\ i=1,2,\ldots,l}} \Psi^{-1}\left(\frac{B}{\prod_{i=1}^l u^{2n_i}}\right) dp_1(u_1) dp_2(u_2) \cdots dp_l(u_l). \qquad (A.5)$$

Consider now a third such collection of rectangles $\{Q_i^1; i = 1, 2, \ldots, l\}$ with with $Q_i^2 \supset Q_i^3$ and $p_i(e(Q_i^3)) = \frac{1}{2} p_i(e(Q_i^2))$ for all $i = 1, 2, \ldots, l$. Consider the sum

$$\sum_{(k_1, k_2, \ldots, k_l) \in \{1,3\}^l} (-1)^{|\{q : k_q = 3\}|} \sum_{(j_1, j_2, \ldots, j_l) \in I(k_1, k_2, \ldots, k_l)} (-1)^{|\{r : j_r = 2\}|} \varphi(Q_1^{j_1}, Q_2^{j_2}, \ldots, Q_l^{j_l}) \qquad (A.6)$$

where $I(k_1, k_2, \ldots, k_l) = \{k_1, 2\} \times \{k_2, 2\} \cdots \{k_l, 2\}$ for all $(k_1, k_2, \ldots, k_l)$ in $\{1, 3\}^l$ (we are adding up terms of the type (A.5)). By rewriting (A.6) as

$$\sum_{(i_1, i_2, \ldots, i_l) \in \{1,2,3\}^l} (-1)^{|\{r : i_r = 2\}|} \varphi(Q_1^{i_1}, Q_2^{i_2}, \ldots, Q_l^{i_l})$$
$$\sum_{(k_1, k_2, \ldots, k_l) \in \{1,3\}^l} \sum_{(j_1, j_2, \ldots, j_l) \in I(k_1, k_2, \ldots, k_l)} \chi_{\{(j_1, j_2, \ldots, j_l) = (i_1, i_2, \ldots, i_l)\}} (-1)^{|\{q : j_q = 3\}|}$$

and considering each index $(i_1, i_2, \ldots, i_l)$ individually, we can show that in fact (A.6) reduces to

$$\sum_{(j_1, j_2, \ldots, j_l) \in \{1,3\}^l} (-1)^{|\{j : i_j = 3\}|} \varphi(Q_1^{j_1}, Q_2^{j_2}, \cdots, Q_l^{j_l}). \qquad (A.7)$$

Thus by summing up the bounds as in (A.5) according to (A.6), we can see that

$$\left| \sum_{(j_1,j_2,\ldots,j_l)\in\{1,3\}^l} (-1)^{|\{r:j_r=3\}|}\varphi(Q_1^{j_1}, Q_2^{j_2}, \cdots, Q_l^{j_l}) \right| \leq 4 \int_{\substack{e(Q_i^1)\leq u_i \leq e(Q_i^3) \\ i=1,2,\ldots,l}} \Psi^{-1}\left( \frac{B}{\prod_{i=1}^l u^{2n_i}} \right) dp_1(u_1)dp_2(u_2)\cdots dp_l(u_l).$$

By taking sequences of rectangles decreasing to $\bar{x}^1 = (x_1^1, x_2^1, \ldots, x_l^1)$ and $\bar{x}^2 = (x_1^2, x_2^2, \ldots, x_l^2)$ and by again using the calculations (A.6)–(A.7), we may, as in the proof of Garsia's theorem, conclude the result. ■

Using Lemma 1 we can easily derive some classical moduli of continuity (see [7] and the calculations of [11]).

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