TRAVELLING WAVES FOR THE KPP EQUATION WITH NOISE

C. MUELLER AND R. SOWERS

1. Introduction

Let me begin with a disclaimer. This is a report on work in progress, so not everything is in its final form.

Travelling waves are a common phenomenon in partial differential equations, and many hundreds of papers have been written on the subject. It is generally understood that all physical systems are subject to random noise, and that this noise will influence the behavior of the system. On the other hand, there has not been much work on the qualitative behavior of stochastic partial differential equations (SPDE). In particular, there has only been one paper about travelling waves for SPDE, namely Tribe (1993). He studies the equation

$$\begin{aligned} u_t &= u_{xx} + \theta u - u^2 + \varepsilon u^{\frac{1}{2}} \dot{W} \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.1}$$

where $\dot{W} = \dot{W}(t,x)$ is 2-parameter white noise. We assume that there is a constant $a \in \mathbf{R}$ such that $u_0(x)$ satisfies

$$\begin{aligned} u_0(x) &= 1 \quad \text{for } x < -a \\ u_0(x) &= 0 \quad \text{for } x > a \end{aligned} \tag{1.2}$$

As explained in Mueller and Tribe (1993), (1.1) arises as the limit of a long-range contact process studied by Bramson, Durrett, and Swindle (1989). Let $F(t)$ denote the wavefront. We will give precise definitions later. In fact, Tribe uses arguments from oriented percolation to show that for small ε,

(1) $lim_{t\to\infty} F(t)/t$ exists a.s. and is nonrandom.

1991 *Mathematics Subject Classification.* Primary 60H15; secondary 35R60.

Supported by NSA grant MDA904-93-H-3036 and by the US Army Research Office, through the Mathematical Sciences Institute of Cornell University, contract DAAL03-91-C-0027.

Supported by ONR grant N00014-91-J-1526 and by an NSF postdoctoral fellowship.

The detailed version of this paper will be submitted for publication elsewhere.

(2) The law of $u(t, x+F(t))$ tends to a stationary distribution.

See Durrett (1984) for a survey of oriented percolation. It is not hard to prove statement (1) from (2). Statement (2) is proved using compactness arguments. Since the proof is abstract, it carries over to many other equations.

The Kolmogorov-Petrovskii-Piscuinov equation (KPP) was one of the first equations known to have a travelling wave solution:

$$u_t = u_{xx} + u - u^2. \tag{1.3}$$

Various authors have shown that for a fairly general class of initial conditions, $u(t, x)$ converges to a translation of the travelling wave solution. The KPP equation (1.3) arose as a model for the spread of a gene through a geographically spread population. Solutions to (1.1) are qualitatively different than solutions to (1.3); they do not remain in the interval $[0, 1]$. Thus, (1.1) is not a good model for random effects in the KPP equation.

In this work, we study

$$\begin{aligned} u_t &= u_{xx} + u - u^2 + \varepsilon\sqrt{u(1-u)}\dot{W} \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.4}$$

We assume that $u_0(x)$ satisfies (1.2), and in addition, that $0 \leq u_0(x) \leq 1$. This initial condition assures that for all $t > 0$, $x \in \mathbf{R}$, solutions $u(t, x)$ take values in $[0, 1]$. Therefore, (1.4) bears a closer relationship to the KPP equation than (1.1). For (1.4), Tribe's analysis will go through. In particular, it follows that for small ε, the wavefront has a limiting speed, and that the shape of the wave, measured from the wavefront, tends toward a stationary distribution.

Our main goal is to study the spread of the wavefront. As mentioned above, the shape of the wavefront has a stationary distribution, but in principle the distribution could be degenerate in a number of ways. By using techniques of Mueller and Perkins (1992), we can show that with probability 1, $u(t, x) = 0$ or 1 outside of a compact region. We wish to show that the size of this region $I(t)$, which we call the intermediate region, tends toward a stationary distribution. This shows that the random travelling wave solutions are not too different from the deterministic ones, since the travelling wave does not spread out and lose its identity.

Finally, T. Shiga has pointed out that there is a dual process for (1.4), which is a system of branching Brownian motions, in which particles coalesce at a Poisson rate, measured according to the local time beween pairs of particles. We do not give a more precise description. Our goal is to give a method which should carry over to many other SPDE, and we can only rarely expect such processes to have a dual. Therefore, we avoid using duality arguments.

2. Outline of Tribe's argument

Here we summarize the argument of Tribe (1993), which shows that the travelling wave has a limiting speed and approaches a limiting shape. Tribe's work is itself not completely written up, so this summary may not do it justice.

Here is the definition of the wavefront:

$$F(t) = \log\left(\int_{\mathbf{R}} e^x u(t,x)dx\right).$$

Note that if $u(t,x) = 1(x < a)$ then $F(t) = a$, so this definition corresponds to our intuition about where the wavefront should be. Also note that the mass beyond $F(t)$ must decline exponentially. That is, if $M(t,x) = \int_{F(t)+x}^{\infty} u(t,z)dz$, then $M(t,x) \le ce^{-x}$.

Fix $M > 0$ and let $\alpha \in (0,1/2)$. Fix $t > 0$. Standard arguments from Walsh (1986) show if C is large enough, then with high probability, $u(t,x)$ is Holder continuous of order α with constant C, for $F(t)-M < x < F(t)+M$. That is, with high probability,

$$|u(t,x) - u(t,y)| \le C|x-y|^\alpha$$

for $F(t) - M < x < F(t) + M$. Next, due to the influence of the $-u^2$ term in (1.1), one can show that with high probability,

$$\sup_{F(t)-M<x<F(t)+M} u(t,x) \le C.$$

Furthermore, one can show that the high probability mentioned above does not depend on t, at least if t is not close to 0.

These facts show that $u(t,F(t)+x)$ induces a tight sequence of probability measures $\mathcal{P}_\sqcup$ on the probability space, in the topology of uniform convergence on compact intervals. Then it is not hard to show that the $\mathcal{P}_\sqcup$ converge to a stationary distribution $\mathcal{P}$.

But this argument also gives the existence of a limiting speed, in the same manner as in the contact process. Suppose that $w_t(x) = u(t,F(t)+x)$, so that $w_t(x)$ is the shape of the wavefront at time t. Intuitively, one imagines that conditioned on w_t, the wavefront $F(t)$ has an average velocity $v(w_t) = \lim_{\varepsilon\to 0} E(F(t+\varepsilon)-F(t))/\varepsilon$ depending only on w_t. As time goes on, w_t should cycle through the ensemble of possible shapes, and its empirical distribution should approach the stationary distribution $\mathcal{P}$. Then the limiting velocity $v = \lim_{t\to\infty} F(t)/t$

should be the average of the velocities $v(w)$, that is

$$v = \int v(w)\mathcal{P}(\lceil\sqsupseteq).$$

3. Methods of Proof

We will outline the ideas in our proof, while noting that these ideas are by no means fixed or in final form.

First we show that the length $L(t)$ of the intermediate region $I(t)$, where $u(t,x)$ is strictly between 0 and 1, has a stationary distribution. The main idea is to relate $u(t,x)$ to the super-Brownian motion, which is known to have compact support. The super-Brownian motion $X(t,dx)$ is a measure-valued process which has a density $v(t,x) = X(t,dx)/dx$ if $x \in \mathbf{R}$; the density satisfies the equation

$$v_t = v_{xx} + v^{\frac{1}{2}}\dot{W}. \tag{3.1}$$

Dawson (1993) is a good reference for the super-Brownian motion; Konno and Shiga (1988) showed that the density satisfies (3.1) in the one-dimensional case. The super-Brownian motion is known to have compact support, provided $X(0,dx)$ has compact support. Our goal is to relate (1.4) and (3.1). There are 2 proofs of compact support for the super-Brownian: by Iscoe and by Dawson and Perkins. We follow the strategy of Dawson and Perkins, relying on a particle model for our process $u(t,x)$. The method of Iscoe seems less suitable, since it relies on properties of the Laplace functional, which only hold in the case of independent particles. For our equation, the particles cannot be independent. In fact, we use a method of Mueller and Perkins (1992), who considered the equation

$$u_t = u_{xx} + u^{\gamma}\dot{W} \tag{3.2}$$

and showed that compact support holds if $1/2 \leq \gamma < 1$, provided $u(0,x)$ has compact support.

To show compact support for (1.4), it is more convenient to consider the equation

$$\begin{aligned} u_t &= u_{xx} + u + \varepsilon\sqrt{u(1-u)}\dot{W} \\ u(0,x) &= u_0(x). \end{aligned} \tag{3.3}$$

We can couple solutions $u(t,x)$ to (1.4) and $v(t,x)$ to (3.3) such that with probability 1, $u(t,x) \leq v(t,x)$ for all (t,x). We will not prove this assertion here. Observe, however that the only difference between (1.4) and (3.3) is that (1.4) has the extra term $-u^2$.

Therefore, we need only consider solutions $u(t,x)$ to (3.3), and to show compact support for this equation.

Here we outline the strategy, explaining it first for the superprocess $v(t,x)$ which satisfies (3.1). Suppose that the process starts out with compact support on $[a,b]$, but at some small time s later the support of $v(s,x)$ has a maximum far to the right of b. Because the superprocess arises as a limit of a particle system, we can separate $v(t+s,x)$ into 2 pieces $v_1(t+s,x)+v_2(t+s,x)$, where $v_2(t+s,x)$ arises from the mass which is far beyond the point b at time s. In the above, we regard s as fixed and consider $t \geq 0$. Now it is not hard to show that if s is small, then with high probability $M(t) \equiv \int_{\mathbf{R}} v_2(s+t,x)ds$ is small for $t=0$. Furthermore, one can show that $M(t)$ satisfies the Feller equation:

$$dM = M^{\frac{1}{2}}dB$$

with initial condition $M(0)$. But the Feller diffusion dies out quickly, with high probability, if the initial mass $M(0)$ is small. Thus, the mass which is far beyond the initial support dies out quickly. By making this argument quantitative, Mueller and Perkins (1992) showed compact support for solutions of (3.2), and hence also for (3.1).

A similar argument works for (3.3). Here, instead of the Feller diffusion, we deal with the diffusion

$$\begin{aligned} dY &= Y dt + Y^{\frac{1}{2}}dB \\ Y(0) &= M(t,x). \end{aligned}$$

Again, if $Y(0)$ is small, then with high probability, $Y(t)$ dies out quickly. The whole argument goes through in the same way, except the the mass represented by $M(t,x)$ no longer evolves independently of the rest of the solution. The trouble comes because the term $v^{1/2}\dot{W}$ which appears in (3.1) is replaced by $\sqrt{u(1-u)}\dot{W}$ in (3.3). However, if u is small, then $\sqrt{u(1-u)}$ behaves like $\sqrt{u}$, and the previous argument will work. We must show that for x much larger than $F(t)$, $u(t,x)$ is likely to be small. We already know that the mass $M(t,x)$ in this region is exponentially small in x. Thus, if $u(t,x)$ is not small, and $x >> F(t)$, then there must be a very thin peak at position x. Now we use a large deviations argument of the Wentzell-Freidlin type. This sort of reasoning has been used by many people. Since the noise term in (3.3) is multiplied by ε, for small times $u(t,x)$ tends to behave like a solution of

$$u_t = u_{xx} + u.$$

For this equation, thin peaks are brought down very quickly. This kind of reasoning gives us what we need, that thin peaks do not occur very often in the region $x >> F(t)$. Thus, the term $\sqrt{u(1-u)}\dot{W}$ behaves like $\sqrt{u}\dot{W}$ in the region $x >> F(t)$, and Mueller and Perkins' argument shows that $u(t,x) = 0$ for $x >> F(t)$.

Next, we take up the proof of the stationary distribution for $L(t)$, the length of the intermediate region $I(t)$. Since the noise term $\varepsilon\sqrt{u(1-u)}\dot{W}$ in (1.4) is small, we can get some feeling for the solution by using Wentzell and Freidlin's theory of large deviations. In a nutshell, we use the fact that for limited regions of space and time, $u(t,x)$ evolves almost like a solution of the KPP equation (1.3). Also, we use the fact, mentioned earlier, that if $u(t,x)$ is very close to 0 or 1 on some interval $x \in (a,b)$, then with high probability, $u(t+\varepsilon, x) = 0$ or 1, respectively, for $x \in (a+\varepsilon_0, b+\varepsilon_0)$.

To be more precise, we identify 6 possible configurations in the intermediate region. We only give an intuitive description, leaving precise definitions for our paper.

(1) $sup_{a<x<b} u(t,x)$ is close to 0.
(2) $inf_{a<x<b} u(t,x)$ is close to 1.
(3) $\int_a^b u(t,x)$ is close to 0, but $\sup_{a<x<b} u(t,x)$ is close to 1.
(4) $\int_a^b u(t,x)$ is close to $b-a$, but $\inf_{a<x<b} u(t,x)$ is close to 0.
(5) $\int_a^b u(t,x)$ is close to $(b-a)/2$, and for $x \in (a,b)$, $u(t,x)$ never approaches 0 or 1 very closely.
(6) $\int_a^b u(t,x)$ is close to $(b-a)/2$, but for $x \in (a,b)$, $u(t,x)$ sometimes approaches either 0 or 1.

We have already described our strategy for cases (1) and (2), so we will move on to the other cases. Cases (3) and (4) are similar, so we will only deal with case (3). Because the noise term in (1.4) contains the term $\sqrt{u(1-u)}$, we do not expect the noise to decrease $u(t,x)$ very much if $u(t,x)$ is close to 1. On the other hand, large deviation estimates show that for small time, $u(t,x)$ behaves like a solution of

$$u_t = u_{xx} + u. \tag{3.4}$$

However, in case (3), solutions to (3.4) quickly approach 0. Therefore, with high probability, $\sup_{a<x<b} u(t,x)$ approaches 0, and we can use the strong Markov property to reduce to case (1). In case (5), a common strategy is to consider integrals such as $M(t) = \int_a^b u(t,x)dx$. Sometimes a test function is included in the integral, but since this is only an intuitive treatment, let us stick to $M(t)$ as defined. If we ignore boundary terms, and formally integrate (1.4) over $a < x < b$, we find that $M(t)$ is a submartingale with a strong drift upward due to the term $u - u^2$ from (1.4). This drift term remains strong as long as $u(t,x)$ remains strictly between 0 and 1. Whenever this fails to hold, we are in one of the other cases. On the other hand, the upward drift of $M(t)$ means that with high probability, we do not stay in case (5) for long. It remains to deal with case (6). Consider the region in (a, b) in which $u(t, x)$ is close to 1. The analysis for $u(t, x)$ close to 0 would be similar. If this region has small measure, then by the same argument as in case (2), $\sup_{a<x<b} u(t, x)$ would quickly decrease, reducing us to one of the other cases. Suppose that the region where $u(t, x)$ is close to 1 has large measure. Then the drift term $u - u^2$ may not be very large, so some other mechanism must be found to drive the solution toward 1. Here we use Wentzell-Freidlin estimates similar to those used in Mueller and Tribe. If ε is small, then $u(t, x)$ almost satisfies the KPP equation (1.3). However, the KPP equation with initial conditions described in case (6) tends to 1. The region where $u(t, x)$ is close to 1 acts as a nucleus for the travelling wave, which then spreads out to cover all of (a, b).

Now we have dealt with all cases, although only on the intuitive level. This reasoning can be used to show that there is a tendency for the intermediate region $I(t)$ to fill up. That is, on $x \in I(t)$, $u(t, x)$ tends to increase to 1. But at the same time, the wavefront is moving forward, which tends to increase the size of $I(t)$. One needs to show that the second effect does not swamp the first.

Here we give an even sketchier idea of the argument. From the above reasoning, we know that the intermediate region $I(t)$ tends to fill in at a certain rate r. As the wavefront moves forward at rate s, and the size of $I(t)$ is increased, this new region also tends to fill in. The following argument gives some notion of the equilibrium size of $I(t)$. Suppose a point $x = p$ is at the wavefront. It will get "filled in" at about time $1/r$ later. (The rate r is thought of as a percentage, so the time necessary to fill in 100% is about $1/r$). But by that time, the wave will have moved forward a distance s/r. Therefore, the width of $I(t)$ should be about s/r.

References

[1] Bramson, M., Durrett, R. and Swindle, G. (1989). Statistical Mechanics of Crabgrass. *Ann. Prob.* **17**, (2), 444-481.

[2] Dawson, D.A. (1993). Measure-Valued Markov Processes, in *Ecole d'Ete de Probabilites de Saint Flour XXI-1991*, P.L. Hennequin, editor, Springer-Verlag, New York, Berlin.

[3] Durrett, R. (1984). Oriented percolation in two dimensions. *Ann. Prob.* **12**, 999-1040

[4] Konno, N. and Shiga, T. (1988). Stochastic partial differential equations for some measure-valued diffusions. *Prob. Th. Rel. Fields* **79**, 201-225.

[5] Mueller, C. and Perkins, E. (1992). The compact support property for solutions to the heat equation with noise. *Prob. Th. Rel. Fields*, **93**, 325-358.

[6] Mueller, C. and Tribe, R. (1993). A stochastic PDE arising as the limit of a long-range contact process, and its phase transition. Submitted to *Prob. Th. Rel. Fields.*

[7] Tribe, R. (1993). A travelling wave theorem for stochastic PDE. In preparation.

[8] Walsh, J.B. (1986). An Introduction to Stochastic Partial Differential Equations, in *Ecole d'Ete de Probabilites de Saint Flour XIV, Lecture Notes in Math. 1180*, edited by P.L. Hennequin, Springer-Verlag, New York, Berlin.

Dept. of Mathematics, University of Rochester, Rochester, NY 14627

E-mail address: cmlr@troi.cc.rochester.edu

Dept. of Mathematics, University of Maryland, College Park, MD 20742

E-mail address: rbs@math.umd.edu