HYDRODYNAMICAL LIMITS AND GEOMETRIC MEASURE THEORY: MEAN CURVATURE LIMITS FROM A THRESHOLD VOTER MODEL

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ABSTRACT. We consider hydrodynamical limits for a simple threshold voter model for a microscopically-evolving random interface. This model, which is a zero-temperature Ising model, was studied by Spohn in a 1+1 setting. The model leads to motion by a certain anisotropic mean curvature. Here we develop this model through some notions of geometric measure theory, dispensing with the 1+1 restriction.

§1. INTRODUCTION

There is currently much interest in rigorously deriving hydrodynamical limits from interacting particle systems (see [19] as a brief outline). The preponderance of this effort has gone to derivations of PDE's as the macroscopic limits. There is also a wealth of problems in the related area of interfacial dynamics. The first effort in this direction was by Spohn [17] (see also [18] and, in another direction, the papers of [11] and [12]), which contained an analysis of a threshold voter model (see also [3], [4], and [14]). Spohn's analysis (and that of [18]) essentially reduce the interface to a height function, which approximately solves a PDE. We here consider a slight generalization of the model of Spohn with the intent of developing some of the tools needed to carry out hydrodynamical limits in the framework of geometric measure theory.

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§2. The Model and the Results

Consider the configuration space $\mathscr{X} \overset{\text{def}}{=} \{0,1\}^{\mathbb{Z}^2}$; i.e., to each $x \in \mathbb{Z}^2$ we attach a spin of zero or one. Let $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ be, respectively, the L^1, L^2, and L^∞ norms on $\mathbb{R}^2$; i.e.,

$$\begin{aligned} \|(x,y)\|_1 &\overset{\text{def}}{=} |x| + |y| \\ \|(x,y)\|_2 &\overset{\text{def}}{=} \sqrt{x^2+y^2} \qquad\qquad (x,y) \in \mathbb{R}^2 \\ \|(x,y)\|_\infty &\overset{\text{def}}{=} \max\{|x|,|y|\}. \end{aligned}$$

For each configuration $\sigma \in \mathscr{X}$ and each $x \in \mathbb{Z}^2$, define

$$n_\sigma(x) \overset{\text{def}}{=} \sum_{y \in \mathbb{Z}^2 : \|y-x\|_1 = 1} |\sigma(y) - \sigma(x)|$$

as the number of neighbors of x with disagreeing spin. Also, for $\sigma \in \mathscr{X}$ and $x \in \mathbb{Z}^2$, define a new configuration σ^x by flipping the spin at x;

$$\sigma^x(y) \overset{\text{def}}{=} \begin{cases} 1 - \sigma(x) & \text{if } y = x \\ \sigma(y) & \text{else.} \end{cases}$$

We will say that the configuration σ will flip to σ^x with rate 1 if $\sigma(x)$ has at least two disagreeing neighbors; i.e., if $n_\sigma(x) \geq 2$. The generator of this process thus is

Definition 2.1 (Generator).

$$(\mathscr{L}f)(\sigma) \overset{\text{def}}{=} \sum_{x \in \mathbb{Z}^2} \chi_{\{n_\sigma(x) \geq 2\}} \{f(\sigma^x) - f(\sigma)\} \qquad \sigma \in \mathscr{X}$$

for all $f \in B(\mathscr{X})$ which depend on a finite number of coordinates (see [5] and [13]).

We note that this generator is the same as that of a zero-range process as long as the configuration is locally "monotone".

We define the region of 1's by filling in each square in the (dual) lattice whose center is a one; for each configuration σ, define the "island" of 1's as

$$\mathcal{I}_\sigma \overset{\text{def}}{=} \bigcup_{x \in \mathbb{Z}^2 : \, \sigma(z) = 1} \{y \in \mathbb{R}^2 : \|y - x\|_\infty \leq 1/2\}.$$

The task is then to track the evolution of $\partial\mathcal{I}_\sigma$ as σ evolves. We will do this via a *current*; as we will see, the dynamics preserve the slope of the interface, and currents will respect this. Our notation of currents is from [7] and [15]. For $m \in \{0,1,2\}$, we will let $\mathscr{D}^m$ be, respectively, the collection of C^∞ differential m-forms on $\mathbb{R}^2$ with compact support. The duals of the $\mathscr{D}^m$'s are the collection of m-dimensional currents. We will mainly be dealing with the collection of integral currents, which we will denote by $\mathscr{I}^1$. We track the evolution of $\mathcal{I}_\sigma$ by setting

$$T^n_\sigma \overset{\text{def}}{=} \partial(\mathcal{I}_\sigma/n); \tag{1}$$

note that we have homothetically scaled $\mathcal{I}_\sigma$ by $1/n$; we will consider diffusive scaling, which will in addition speed up time by n^2. By (1), we mean the following. Let $\langle\cdot,\cdot\rangle'$ be the pairing of vector spaces with their duals (in particular the pairing of multilinear forms and their duals). If we let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard basis of $\mathbb{R}^2$, then we let $\mathbf{e}^*_1$ and $\mathbf{e}^*_2$ be the basis of $\mathbb{R}^{2,*}$. For any subset F of $\mathbb{R}^2$ we can define a 2-dimensional current $\mathfrak{T}_F$ and then a 1-dimensional current $\partial\mathfrak{T}_F$ by by

$$\begin{aligned} \mathfrak{T}_F(\varphi) &\overset{\text{def}}{=} \int_{x\in\mathcal{I}_\sigma/n} \langle\varphi(x), \mathbf{e}^*_1\wedge\mathbf{e}^*_2\rangle'\, dx \qquad \varphi\in\mathscr{D}^2 \\ (\partial\mathfrak{T}_F)(\varphi) &\overset{\text{def}}{=} \mathfrak{T}_F(d\varphi). \qquad \varphi\in\mathscr{D}^1 \end{aligned} \tag{2}$$

We will define $T^n_\sigma \overset{\text{def}}{=} \partial\mathfrak{T}_{\mathcal{I}_\sigma/n}$ for all $\sigma\in\mathscr{X}$.

We now fix the initial distribution of our process. We assume that

Assumption 2.2 (Initial Data). Let $\{\pi^n_\circ;\, n\ge 1\}$ be a collection of probability measures on $\mathscr{X}$ such that there is an integral current $T_\circ\in\mathscr{I}^1$ where

$$\lim_{n\to\infty} \pi^n_\circ\left\{\sigma\in\mathscr{X} : |T^n_\sigma(\varphi) - T_\circ(\varphi)| \ge \eta\right\} = 0$$

for all $\varphi\in\mathscr{D}^1$ and all $\eta>0$. We assume that $T_\circ$ is the boundary current (defined as in (2) of some subset of $\mathbb{R}^2$). We also assume that there is an $L>0$ and a compact subset K of $\mathbb{R}^2$ such that

$$\begin{aligned} \pi^n_\circ\left\{\sigma\in\mathscr{X} : \|T^n_\sigma\|(\mathbb{R}^2)\le L\right\} &= 1 \\ \pi^n_\circ\left\{\sigma\in\mathscr{X} : \mathcal{I}_\sigma\subset K\right\} &= 1 \end{aligned} \tag{3}$$

for all $n\ge 1$.

Having both initial data and generator, we define our Markov process. We will use a canonical probability triple; our probability space will be $D_{\mathscr{X}}[0,\infty)$, the collection of $\mathscr{X}$-valued paths which are right-continuous with left-hand limits (see [5, Chapter 3]). The stochastic process is $\boldsymbol{\sigma}_t(\omega)\overset{\text{def}}{=}\omega(t)$ for all $t\ge 0$ and all $\omega\in D_{\mathscr{X}}[0,\infty)$. For each n, we let $\mathbb{P}_n\in\mathscr{P}(D_{\mathscr{X}}[0,\infty))$ be the solution of the martingale problem with generator $n^2\mathscr{L}$ and initial data $\pi^n_\circ$; let $\mathbb{E}_n$ be the associated expectation operator.

Our goal is to find a macroscopic description for the $\mathbb{P}_n$-law of $\{T^n_{\boldsymbol{\sigma}_t};\, t\ge 0\}$. This will require a bit more notation. First of all, we will convert to a probability measure on currents. Define $\hat{\mathbb{P}}_n\in\mathscr{P}(D_{\mathscr{I}^1}[0,\infty))$ by

$$\hat{\mathbb{P}}_n(A)\overset{\text{def}}{=}\mathbb{P}_n\{T^n_{\boldsymbol{\sigma}}\in A\} \qquad A\in\mathscr{B}(D_{\mathscr{I}^1}[0,\infty)) \tag{4}$$

for all n. For any $T \in \mathscr{I}^1$, we define $\overrightarrow{T}$ as the tangent vector field; i.e., such that $T = \|T\| \wedge \overrightarrow{T}$. Next, we will need an anisotropic Hausdorff measure. Define first of all

$$\varrho(x,y) \overset{\text{def}}{=} \begin{cases} \frac{xy}{|x|}\frac{\sqrt{x^2+y^2}}{|x|+|y|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \qquad (x,y) \in \mathbb{R}^2 \tag{5}$$

We would like to use ϱ to define a Finsler metric, but the set $\{z \in \mathbb{R}^2 : \varrho(z) \le 1\}$ is not convex (indeed, ϱ can be negative). Fortunately, it is possible to add any arbitrarily large multiple of Euclidean distance (see (9) below). We fix $\kappa > 0$ sufficiently large, and we let $\mathscr{H}^1_{\varrho,\kappa}$ be the be 1-dimensional Hausdorff measure with unit ball

$$\{z \in \mathbb{R}^2 : \varrho(z) + \kappa\|z\|_2 \le 1\}.$$

By taking κ sufficiently large, this set will be convex (the unit ball in the standard Euclidean norm is strictly convex, and perturbations of strictly convex sets are still convex). For any integral current T, we now define a new current by

$$\begin{aligned}\hbar_{\varrho,\kappa}(T)(\varphi) &\overset{\text{def}}{=} \left(\mathscr{H}^1_{\varrho,\kappa}\lfloor_{\operatorname{supp} T} \wedge \overrightarrow{T}\right)(\varphi) \\ &\overset{\text{def}}{=} \int_{x\in \operatorname{supp} T} \left\langle \varphi(x), \overrightarrow{T}(x)\right\rangle' \mathscr{H}^1_{\varrho,\kappa}(dx) \qquad \varphi \in \mathscr{D}^1\end{aligned}$$

Also, for any 1-form $\varphi = \varphi_1 \mathbf{e}_1^* + \varphi_2 \mathbf{e}_2^*$, $d * d\varphi$ is the 1-form defined as

$$d * d\varphi = \left(\frac{\partial^2\varphi_2}{\partial x^2} - \frac{\partial^2\varphi_1}{\partial x \partial y}\right)\mathbf{e}_1^* + \left(\frac{\partial^2\varphi_2}{\partial x\partial y} - \frac{\partial^2\varphi_1}{\partial y^2}\right)\mathbf{e}_2^*$$

Our claim is that $T^n_{\boldsymbol{\sigma}}$ converges to the solution of

$$\mathfrak{t}_t(\varphi) = T_\circ(\varphi) + \int_0^t \hbar_{\varrho,\kappa}(\mathfrak{t}_s)(d * d\varphi)ds. \qquad \varphi \in \mathscr{D}^1 \tag{6}$$

Our main theorem will be that $\hat{\mathbb{P}}_n$ will converge almost-surely to the (deterministic) law of the solution of (6), at least when it is unique. Not surprisingly, however, we need some more assumptions. Our first is a *density* assumption. The general framework of questions of convergence to geometric flows is that one needs to know *a priori* that the interface does not fatten (see [6]). We need a similar assumption. Define a set

$$\mathfrak{B} \overset{\text{def}}{=} \{z \in \mathbb{R}^2 : \|z\|_\infty < 1\}. \tag{7}$$

For any $T \in \mathscr{I}^1$, $\varepsilon > 0$, and $x \in \mathbb{R}^2$, we define

$$\theta_\varepsilon(x;T) \overset{\text{def}}{=} \frac{1}{2\varepsilon}\|T\|(x + \varepsilon\mathfrak{B}).$$

We make the following hypothesis;

Assumption 2.3 (Density). We assume that for $t > 0$,

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ \varepsilon^3 n^2 \mathbb{E}_n\left[\int_0^t \int_{z\in\mathbb{R}^2} (\theta_\varepsilon(z;T^n_{\boldsymbol{\sigma}_s})-1)^+ \right.$$
$$\left. \times \left\{1+\left(\theta_\varepsilon(z;T^n_{\boldsymbol{\sigma}_s})-1\right)^+\right\} \|T^n_\sigma\|(dz)\,ds\right] = 0.$$

The point of this is that if x is in the support of T^n_σ, then the interface is "mesoscopically" monotone if and only if $\theta_\varepsilon(x;T^n_\sigma)=1$ for ε small. Our second assumption is one of local "indecomposability"; we want to disallow a lot of tiny bubbles in a small region. We will define

$$\mathscr{F}(B) \overset{\text{def}}{=} \left\{T\in\mathscr{I}^1 : T\lfloor_B \text{ is indecomposable}\right\}. \tag{8}$$

This means that $T\lfloor_B$ is either zero, consists of a single bubble, or that it passes through B exactly once. Our assumption then is

Assumption 2.4 (Local Indecomposability). We assume that for $t > 0$, there is a $p\in(1,\infty)$ such that

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ \varepsilon^{1+2/p} n \mathbb{E}_n\left[\int_0^t \int_{z\in\mathbb{R}^2} \left\{1+\left(\theta_\varepsilon(z;T^n_{\boldsymbol{\sigma}_s})\right)^+\right\} \|T^n_{\boldsymbol{\sigma}_s}\|(dz) \right.$$
$$\left. \times \left(\mathscr{H}^2\left\{y\in\mathbb{R}^2 : T^n_{\boldsymbol{\sigma}_s}\not\in\mathscr{F}(\varepsilon\mathfrak{B}+y)\right\}\right)^{1/q} ds\right] = 0.$$

where $\frac{1}{p}+\frac{1}{q}=1$.

We need one more assumption, which turns out to be a technical one. We can't have any really sharp corners at the interface (this can be taken care of by other means; see [18]). Our assumption is

Assumption 2.5. We assume that

$$\overline{\lim_{n\to\infty}}\ \sup_{\substack{x\in\mathbb{R}^2\\ z\in\mathbb{Z}^2\\ i\in\{1,2\}}} \int_{\sigma\in\mathscr{X}} \left\{\|T^1_\sigma\|\left(n(x+\varepsilon D+y)\cap(z+\mathbb{R}\mathbf{e}_i)\right)\right\}^2 \pi^n_\circ(d\sigma) < \infty.$$

The meaning of this condition is that in any $x+\varepsilon D$, the jumps in the interface in either direction are bounded in the mean-square sense. Our main theorem is then

Theorem 2.6. *Assume that (6) has a unique solution on* $[0,T]$ *and that Assumptions 2.3, 2.4, and 2.5 hold. Then* $\hat{\mathbb{P}}_n|_{\mathscr{B}(D_{\mathscr{I}^1}[0,T])}$ *converges weakly to* $\delta_{\mathfrak{t}}|_{\mathscr{B}(D_{\mathscr{I}^1}[0,T])}$ *(the Dirac mass on* $D_{\mathscr{I}^1}[0,T]$*) centered on* $\mathfrak{t}$*).*

Proof. Given in the next section. □

We see now why $\kappa > 0$ need not be specified, as long as it is sufficiently large; for any other $\kappa' > 0$ sufficiently large, and any $T \in \mathscr{I}^1$ which is a boundary current,

$$\hbar_{\varrho,\kappa}(T)(d * d\varphi) - \hbar_{\varrho,\kappa'}(d * d\varphi) = (\kappa - \kappa')T(d * d\varphi) \\ = (\kappa - \kappa')\partial T(*d\varphi) = 0 \quad (9)$$

by an integration by parts (i.e., since $\partial T = 0$ which follows from T being a boundary current).

We also note that we can interpret (6) as mean curvature flow in an anisotropic Finsler metric (see [2]). To do so, first solve the PDE

$$\begin{aligned} \ddot{J}(m) &= \frac{1}{(m+1)^2(m^2+1)}J(m) \\ J(1) &= 1 \qquad\qquad\qquad\qquad m \in [0,\infty) \\ \dot{J}(1) &= \frac{1}{2} \end{aligned}$$

and define a unit ball

$$\mathfrak{D} \overset{\text{def}}{=} \left\{(x,y) \in \mathbb{R}^2 : |x|J(|y|/|x|) \leq 1\right\}.$$

Let $\mathscr{H}^1_{\mathfrak{D}}$ be 1-dimensional Hausdorff measure with unit ball $\mathfrak{D}$. Then we have

Proposition 2.7. *The equation (6) is equivalent to mean curvature flow in the* $\mathscr{H}^1_{\mathfrak{D}}$ *measure; i.e., the gradient flow of the length function.*

Proof. Given in Appendix A. □

Of course we have left a significant question unresolved: what are hypotheses on the initial condition under which Assumptions 2.3 and 2.4 will hold? Clearly the hydrodynamical limit holds when one can locally reduce the calculation to the one of Spohn; our goal here was to identify some quantitiative and verifiable assumptions which would ensure this; i.e., Assumptions 2.3 and 2.4. The next task is to look at the dynamics of the quantities involved in Assumptions 2.3 and 2.4. It is reasonable to hope that these quantities may behave as some sort of Lyapunov functions. We should note, however, that those calculations might involve a different collection of techniques. We expect that the generalization of the present analysis to higher-dimensional problems would be mainly technical, involving the usual difficulties of hydrodynamical limits. The analysis of the density and local indecomposibility, however, is nontrivial even in the continuum limit; this is related to questions of fattening, which are subtle (see [1], [8], and [9]). We thus would expect that the study of Assumptions 2.3 and 2.4 would involve some adaptation of the fine techniques of geometric motions to local configurational analysis. It should also be mentioned that one of the tools which has been exploited with such success in Euclidean mean curvature flow—the Huisken monotonicity formula (see [10])—has not yet been developed for general evolutions by anisotropic mean curvature.

§3. THE MAIN STEPS

We now need to write down the evolution of $T^n_{\boldsymbol{\sigma}_t}$ under $\mathbb{P}_n$. Fix a $\varphi = \varphi_1\mathbf{e}_1^* + \varphi_2\mathbf{e}_2^* \in \mathscr{D}^1$; we need to apply the generator $n^2\mathscr{L}$ to the map $\sigma \mapsto T^n_\sigma(\varphi)$. Let's first of all see what happens if we fix a $z \in \mathbb{Z}^2$ and replace σ by σ^z. Modulo rotations, if $\sigma(z) = 1$, the possible configurations are below, where z is the central point.

$$\begin{array}{c}\text{configuration a:}\\(n_\sigma(z) = 2, \sigma(z) = 1)\end{array} \quad \begin{array}{c}0\\011\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}0\\001\\1\end{array}$$

$$\begin{array}{c}\text{configuration b:}\\(n_\sigma(z) = 2, \sigma(z) = 1)\end{array} \quad \begin{array}{c}1\\010\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}1\\000\\1\end{array}$$

$$\begin{array}{c}\text{configuration c:}\\(n_\sigma(z) = 2, \sigma(z) = 0)\end{array} \quad \begin{array}{c}0\\001\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}0\\011\\1\end{array}$$

$$\begin{array}{c}\text{configuration d:}\\(n_\sigma(z) = 2, \sigma(z) = 0)\end{array} \quad \begin{array}{c}1\\000\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}1\\010\\1\end{array}$$

$$\begin{array}{c}\text{configuration e:}\\(n_\sigma(z) = 3, \sigma(z) = 1)\end{array} \quad \begin{array}{c}0\\010\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}0\\000\\1\end{array}$$

$$\begin{array}{c}\text{configuration f:}\\(n_\sigma(z) = 3, \sigma(z) = 0)\end{array} \quad \begin{array}{c}0\\101\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}0\\111\\1\end{array}$$

$$\begin{array}{c}\text{configuration g:}\\(n_\sigma(z) = 4, \sigma(z) = 1)\end{array} \quad \begin{array}{c}0\\010\\0\end{array} \quad \text{becomes} \quad \begin{array}{c}0\\000\\0\end{array}$$

$$\begin{array}{c}\text{configuration h:}\\(n_\sigma(z) = 4, \sigma(z) = 0)\end{array} \quad \begin{array}{c}1\\101\\1\end{array} \quad \text{becomes} \quad \begin{array}{c}1\\111\\1\end{array}$$

Define now the function

$$f_\varphi \overset{\text{def}}{=} *d\varphi \overset{\text{def}}{=} \frac{\partial\varphi_2}{\partial x} - \frac{\partial\varphi_1}{\partial y}$$

Then we have

Lemma 3.1. *For each $n \geq 1$ and $z \in \mathbb{Z}^2$,*

$$T^n_{\sigma^z}(\varphi) - T^n_\sigma(\varphi) = -\frac{1}{n^2} f_\varphi(z/n)(-1)^{\sigma(z)} + \frac{1}{n^4} E^n_\varphi(z;\sigma)$$

where there is a constant K which depends only on φ such that

$$|E^n_\varphi(z;\sigma)| \le K.$$

Proof. Shift things so that $z=0$. Then

$$T^n_\sigma(\varphi) = \int_{-1/(2n)}^{1/(2n)} \varphi_2(-1/(2n),s)ds + \int_{-1/(2n)}^{1/(2n)} \varphi_1(s,1/(2n))ds$$

$$T^n_{\sigma^z}(\varphi) = \int_{-1/(2n)}^{1/(2n)} \varphi_2(1/(2n),s)ds + \int_{-1/(2n)}^{1/(2n)} \varphi_1(s,-1/(2n))ds.$$

Thus

$$T^n_{\sigma^z}(\varphi) - T^n_\sigma(\varphi) = \int_{s=-1/(2n)}^{1/(2n)} \int_{r=-1/(2n)}^{1/(2n)} f_\varphi(r,s)dr\,ds.$$

This implies the result when $\sigma(z)=1$; note that these calculations are rotation-invariant. A negative sign should precede all quantities if $\sigma(z)=0$. □

We also have

Lemma 3.2. *For each $t \ge 0$,*

$$\mathbb{P}_n\left\{\sup_{0\le s\le t} \|T^n_{\boldsymbol{\sigma}_s}\| \le L\right\} = 1$$

$$\mathbb{P}_n\left\{\operatorname{supp} T^n_{\boldsymbol{\sigma}_s} \subset K\right\} = 1$$

where K and L are as in (2).

Proof. For each $n \ge 1$ and $z \in \mathbb{Z}^2$,

$$\|T^n_{\sigma^z}\| \le \|T^n_\sigma\|,$$

as the preceding diagrams show. The second claim is obvious. □

For $\sigma \in \mathscr{X}$, let's also define

$$\xi_\sigma(z) \overset{\text{def}}{=} (-1)^{\sigma(z)}\chi_{\{n_\sigma(z)\ge 2\}}. \qquad z \in \mathbb{Z}^2$$

Then we get that

$$T^n_{\boldsymbol{\sigma}_t}(\varphi) = T^n_{\boldsymbol{\sigma}_0}(\varphi) + \int_0^t A^n_\varphi(\boldsymbol{\sigma}_s)ds + M^{n,\varphi}_t + \frac{1}{n^2}\int_0^t \mathscr{E}^{1,n}_\varphi(\boldsymbol{\sigma}_s)ds$$

where

$$A^n_\varphi(\sigma) \overset{\text{def}}{=} -\sum_{z\in\mathbb{Z}^2} f_\varphi(z/n)\xi_\sigma(z)$$

$$\mathscr{E}^{1,n}_\varphi(\sigma) \overset{\text{def}}{=} \sum_{z\in\mathbb{Z}^2} E^n_\varphi(z;\sigma)\chi_{\{n_\sigma(z)\ge 2\}}$$

and where $M^{n,\varphi}$ is a $\mathbb{P}_n$-martingale with quadratic variation

$$\langle M^{n,\varphi}\rangle_t = n^2 \sum_{z\in\mathbb{Z}^2} \int_0^t \chi_{\{n_\sigma(z)\ge 2\}} \left\{T^n_{\boldsymbol{\sigma}^z_s}(\varphi) - T^n_{\boldsymbol{\sigma}_s}(\varphi)\right\}^2 ds.$$

We have

Proposition 3.3. *For all $\eta > 0$, and all $t > 0$,*

$$\overline{\lim_{n\to\infty}} \mathbb{P}_n \left\{ \sup_{0\le s\le t} |M^n_\varphi(s)| \ge \eta \right\} = 0$$
$$\overline{\lim_{n\to\infty}} \mathbb{P}_n \left\{ \sup_{0\le s\le t} \left| \int_0^s \mathscr{E}^{1,n}_\varphi(\boldsymbol{\sigma}_s)ds \right| \ge \eta \right\} = 0.$$

Proof. Fairly obvious. Note that

$$\langle M^{n,\varphi}\rangle_t \le \frac{1}{n^2} \sum_{z\in\mathbb{Z}^2} \int_0^t \chi_{\{n_\sigma(z)\ge 2\}} \left\{ f_\varphi(z/n)(-1)^{\boldsymbol{\sigma}_s(z)} + \frac{1}{n^2} E^n_\varphi(z;\boldsymbol{\sigma}_s) \right\}^2 ds.$$

Thus there is a constant K, which depends only on φ, such that

$$\langle M^n\rangle_t \le \frac{K}{n} \int_0^t \|T^n_{\boldsymbol{\sigma}_s}\|(\mathbb{R}^2)ds$$
$$\int_0^t \mathscr{E}^{1,n}_\varphi(\boldsymbol{\sigma}_s)ds \le \frac{K}{n} \int_0^t \|T^n_{\boldsymbol{\sigma}_s}\|(\mathbb{R}^2)ds$$

In light of Lemma 3.2, this implies the result. ☐

We now want to integrate by parts and show that A^n_φ is order 1 as n tends to infinity. We want to do this locally in the hope that locally the interface will be a graph and we can thus treat it by the methods of Spohn.

First of all, define a set

$$D \overset{\text{def}}{=} \{y \in \mathbb{R}^2 : \|y\|_\infty \le 1\}.$$

Let $\vartheta_1 \in C^\infty(\mathbb{R}^2;[0,1])$ be such that $\operatorname{supp}\vartheta_1 \subset\subset D$ and $\int_{y\in\mathbb{R}^2} \vartheta_1(y)dy = 1$. For each $\varepsilon > 0$ and z and y in $\mathbb{R}^2$, now define

$$F^{y,\varepsilon}_\varphi(z) \overset{\text{def}}{=} \frac{1}{\varepsilon} f_\varphi(z)\vartheta_1\left(\frac{z-y}{\varepsilon}\right).$$

Then

$$\sum_{z\in\mathbb{Z}^2} f_\varphi(z/n)\xi_\sigma(z) = \int_{y\in\mathbb{R}^2} \sum_{z\in\mathbb{Z}^2} F^{y,\varepsilon}_\varphi(z/n)\xi_\sigma(z)dy \tag{10}$$

For $B \subset \mathbb{R}^2$ convex, we define a locally "good" subset $G(B)$ of $\mathscr{I}^1$ as

$$G(B) \overset{\text{def}}{=} \bigcup_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\{\pm\mathbf{e}_1,\pm\mathbf{e}_2\}} \left\{ T \in \mathscr{F}(B) : \|T\|\{x \in B : \vec{T}(x) \notin \{\boldsymbol{\alpha},\boldsymbol{\beta}\}\} = 0 \right\};$$

recall the definition (8) of $\digamma(B)$. The point of $G(B)$ is that the interface is "increasing" when looked at from some direction. This allows us to do several things. First, it forces enough structure on the integrand in (10) that we can integrate by parts (Lemma 4.4). It also allows us to use local ergodicity of the zero-range process (Proposition 4.3). Local ergodicity allows us to replace averaged sums by expectations with respect to stationary measures. We can parametrize the stationary measures by *line* currents (17), and the relevant expectations can be denoted by changing the mass measure of these line currents to the $\mathscr{H}^1_{\rho,\kappa}$ via an operator ι_δ given below. This last comment and a knowledge of the stationary measures allows us to rewrite these line currents in a coordinate-free way and thence to proceed with the proof. Define

$$r(x) \overset{\text{def}}{=} \begin{cases} \frac{\|x\|_1\|x\|_2}{\varrho(x)+\kappa\|x\|_2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad x \in \mathbb{R}^2$$

$$\mathfrak{S} \overset{\text{def}}{=} \left\{x \in \mathbb{R}^2 \,:\, r(x) \le 1\right\};$$

recall the definition ϱ of (5). Now let $\vartheta_2 \in C^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{supp}\vartheta_2 \subset [0,1]$ and $\int_0^\infty \vartheta_2(s)ds = 1$. For $\delta > 0$ and $T \in \mathscr{I}^1$, define $\iota_\delta(T) \in \mathscr{I}^1$ as

$$(\iota_\delta(T))(\varphi) \overset{\text{def}}{=} \int_{x\in\mathbb{R}^2} \left\langle \varphi(x), \vec{T}(x)\right\rangle' \left\{\int_{y\in\mathbb{R}^2} \frac{1}{\delta}\vartheta_2\left(\frac{r(y-x)}{\delta}\right)\|T\|(dy)\right\}\|T\|(dx) \tag{11}$$

for all $\varphi \in \mathscr{D}^1$. We note that $\iota_\delta(T^n_\sigma)$ is well-defined whether or not T^n_σ is in a $G(\varepsilon D + y)$. We claim that

Proposition 3.4. *For each $\eta > 0$ and $t > 0$,*

$$\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\mathbb{P}_n\left\{\int_0^t\int_{y\in\mathbb{R}^2}\chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\left|\sum_{z\in\mathbb{Z}^2}F^{y,\varepsilon}_\varphi(z/n)\xi_{\boldsymbol{\sigma}_s}(z)\right.\right.$$
$$\left.\left. -\iota_\delta(T^n_{\boldsymbol{\sigma}_s})(dF^{y,\varepsilon}_\varphi)\right|dy\,ds \ge \eta\right\} = 0.$$

Proof. Given in the next section. □

This result stems from the local ergodic theorem. Not surprising, we need to make some finite approximations; this is where Assumption 2.5 comes in:

Proposition 3.5 (A Priori estimates). *We have that*

$$\varlimsup_{n\to\infty}\sup_{\substack{t\ge 0\\ x\in\mathbb{R}^2\\ z\in\mathbb{Z}^2\\ i\in\{1,2\}}}\mathbb{E}_n\left[\left\{\|T^1_{\boldsymbol{\sigma}_t}\|\left(n(x+\varepsilon D+y)\cap(z+\mathbb{R}\mathbf{e}_i)\right)\right\}^2\right] < \infty.$$

Proof. Given in Section 6. ☐

This means that the size of the jumps, in either direction, are locally bounded.

For $\varepsilon > 0$, $y \in \mathbb{R}^2$, and $n \geq 1$, we now define

$$R_\varphi^{y,\varepsilon,n}(\sigma) \overset{\text{def}}{=} \left| \sum_{z \in \mathbb{Z}^2} F_\varphi^{y,\varepsilon}(z/n)\xi_\sigma(z) \right| + \|dF_\varphi^{y,\varphi}\|. \qquad \sigma \in \mathscr{X}$$

We will need the following result:

Proposition 3.6. *If Assumptions 2.3 and 2.4 hold, then for each $\eta > 0$ and $t > 0$,*

$$\lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \mathbb{P}_n \left\{ \int_0^t \int_{y \in \mathbb{R}^2} \chi_{G^c(\varepsilon D + y)}(T^n_{\boldsymbol{\sigma}_s}) R_\varphi^{y,\varepsilon,n}(\boldsymbol{\sigma}_s) dy\, ds \geq \eta \right\} = 0.$$

Proof. Given in Section 5. ☐

We also claim

Proposition 3.7. *For each $\eta > 0$ and $t > 0$,*

$$\lim_{\delta \to 0} \varlimsup_{n \to \infty} \mathbb{P}_n \left\{ \int_0^t \int_{y \in \mathbb{R}^2} \left| \iota_\delta(T^n_{\boldsymbol{\sigma}_s})(d * d\varphi) - \hbar_{\varrho,\kappa}(T^n_{\boldsymbol{\sigma}_s})(d * d\varphi) \right| dy\, ds \geq \eta \right\} = 0.$$

Proof. Given in Appendix A. ☐

Finally, we get

Proof of Theorem 2.6. By Proposition 3.6, we have that for $\eta > 0$,

$$\lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \mathbb{P}_n \left\{ \int_0^t \int_{y \in \mathbb{R}^2} \chi_{G^c(\varepsilon D + y)}(T^n_{\boldsymbol{\sigma}_s}) \left| \sum_{z \in \mathbb{Z}^2} F_\varphi^{y,\varepsilon}(z/n)\xi_{\boldsymbol{\sigma}_s}(z) \right| dy\, ds \geq \eta \right\} = 0$$

and

$$\lim_{\varepsilon \to 0} \varlimsup_{\delta \to 0} \varlimsup_{n \to \infty} \mathbb{P}_n \left\{ \int_0^t \int_{y \in \mathbb{R}^2} \chi_{G^c(\varepsilon D + y)}(T^n_{\boldsymbol{\sigma}_s}) \left| \iota_\delta(T^n_{\boldsymbol{\sigma}_s})(dF_\varphi^{y,\varepsilon}) \right| dy\, ds \geq \eta \right\} = 0.$$

Combining this with Proposition 3.4, we see that for all $\eta > 0$,

$$\lim_{\varepsilon \to 0} \varlimsup_{\delta \to 0} \varlimsup_{n \to \infty} \mathbb{P}_n \left\{ \int_0^t \int_{y \in \mathbb{R}^2} \left| \sum_{z \in \mathbb{Z}^2} F_\varphi^{y,\varepsilon}(z/n)\xi_{\boldsymbol{\sigma}_s}(z) \right. \right.$$
$$\left. \left. - \iota_\delta(T^n_{\boldsymbol{\sigma}_s})(dF_\varphi^{y,\varepsilon}) \right| dy\, ds \geq \eta \right\} = 0.$$

We can now re-integrate with respect to y;

$$\int_{y\in\mathbb{R}^2} \iota_\delta(T^n_{\boldsymbol{\sigma}_s})(dF^{y,\varepsilon}_\varphi)dy = \iota_\delta(T^n_{\boldsymbol{\sigma}_s})(df_\varphi);$$

thus

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ \mathbb{P}_n\left\{\left|\int_0^t A^n_\varphi(\boldsymbol{\sigma}_s)ds - \int_0^t \iota_\delta(T^n_{\boldsymbol{\sigma}_s})(d*d\varphi)\right| dy\,ds \geq \eta\right\} = 0. \tag{12}$$

For notational convenience, we will now start to transfer things to the $\hat{\mathbb{P}}_n$ measures of (4). Let $\mathbf{t}$ be the canonical stochastic process on $D_{\mathscr{I}^1}[0,\infty)$; i.e., $\mathbf{t}_t(\omega) \overset{\text{def}}{=} \omega(t)$ for all $t\geq 0$ and all $\omega\in D_{\mathscr{I}^1}[0,\infty)$. Lemma 3.2 and (12) imply that the $\hat{\mathbb{P}}_n$'s are tight (use the compactness theorem for integral current; see [7, Section 4.2.17], [15, Chapter 5], and [20]. Note also that the collection of integral currents which have mass at most L and whose support is in K, where L and K are as in (3), is Polish). Thus

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ \hat{\mathbb{P}}_n\left\{\left|\mathbf{t}_t(\varphi) - \mathbf{t}_0(\varphi) - \int_0^t \iota_\delta(\mathbf{t}_s)(d*d\varphi)ds\right| \geq \eta\right\} = 0.$$

Proposition 3.7 implies that

$$\lim_{n\to\infty}\hat{\mathbb{P}}_n\left\{\left|\mathbf{t}_t(\varphi) - \mathbf{t}_0(\varphi) - \int_0^t \hbar_{\varrho,\kappa}(\mathbf{t}_s)(d*d\varphi)ds\right| \geq \eta\right\} = 0.$$

and since we have assumed that (6) has a unique solution, this completes the proof. □

§4. Ergodic measures and local ergodicity

Let's now construct the relevant collection of ergodic measures of $\mathscr{L}$ and use this knowledge to prove some of the claims we made in the last section. Although our model is sufficiently simple, we note that the structure of the invariant measures for more complicated models is not so simple; see [3], [4], and [14].

We start out with a "canonical" ergodic measure. For each $\sigma\in\mathscr{X}$ and $x\in\mathbb{Z}$, define

$$h_\sigma(x) \overset{\text{def}}{=} \sup\{y\in\mathbb{Z} : \sigma(x,y)=1\}.$$

Definition 4.1 (Ergodic measure). Fix $m\in[0,\infty)$ and let $\mu^m\in\mathscr{P}(\mathscr{X})$ be uniquely specified by the following requirements: for all $x\in\mathbb{Z}$,

$$\mu^m\Big\{\sigma\in\mathscr{X} : y\mapsto\sigma(x,y) \text{ is nondecreasing},$$
$$\lim_{y\to\infty}\sigma(x,y)=0 \text{ and } \lim_{y\to-\infty}\sigma(x,y)=1\Big\} = 1 \tag{13}$$

for all finite index subsets I of $\mathbb{Z}$ and all $\theta : I \to \mathbb{R}$,

$$\int_{\sigma \in \mathscr{X}} \exp\left[i \sum_{x \in nI} \theta(x) \{ h_\sigma(x+1) - h_\sigma(x) \} \right] \mu^m(d\sigma) = \prod_{x \in I} \frac{m}{m+1-m\exp[i\theta(x)]}. \tag{14}$$

$$h_\sigma(0,0) = 1, \quad \text{and} \quad h_\sigma(0,1) = 0. \tag{15}$$

The requirement of (13) is that $\partial\mathcal{I}_\sigma$ is the graph of a piecewice-constant function (with the jumps connected). The requirement of (14) is that the jumps in this graph are i.i.d. with

$$\mu^m\{\sigma \in \mathscr{X} \,:\, h_\sigma(x+1) - h_\sigma(x) = j\} = \begin{cases} \frac{m^j}{(m+1)^{j+1}} & \text{if } j = 0, 1 \dots \\ 0 & \text{else} \end{cases}$$

for all $x \in \mathbb{Z}$. The requirement (15) is that $\partial\mathcal{I}_\sigma$ passes through $\{0\} \times \mathbb{R}$ at exactly $(0, 1/2)$. Note that these specifications are still valid if $m = 0$. Note also that

$$\mu^m \left\{ \lim_{|x| \to \infty} \frac{h(x)}{x} = m \right\} = 1$$

$$\int_{\sigma \in \mathscr{X}} \{ h_\sigma(x+1) - h_\sigma(x) \} \mu^m(d\sigma) = m \tag{16}$$

$$\mu^m\{\sigma \in \mathscr{X} \,:\, h_\sigma(x+1) - h_\sigma(x) > 0\} = \frac{m}{m+1},$$

the last two equalities holding for all $x \in \mathbb{Z}$.

Now let $\mathscr{G}$ be the group of rotations of $\mathbb{R}^2$ by 90°, translations by elements of $\mathbb{Z}^2$, and reflections; in other words every $g \in \mathscr{G}$ can be written in the form

$$g(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n x + b$$

where $n \in \{0, 1, 2, 3\}$ and $b \in \mathbb{Z}^2$. Of particular importance is the case when $n = 0$; for $z \in \mathbb{Z}^2$, let $\tau_z \in \mathscr{G}$ be defined by

$$\tau_z(x) \overset{\text{def}}{=} x + z \qquad x \in \mathbb{Z}^2$$

For any $g \in \mathscr{G}$ and any configuration $\sigma \in \mathscr{X}$, we can define a new configuration $g\sigma$ by

$$(g\sigma)(x) \overset{\text{def}}{=} \sigma(g(x)). \qquad x \in \mathbb{Z}^2$$

We now want to define an equivalence between measures of this sort and *line* currents; i.e., currents of the form

$$\mathscr{I}^1_L \overset{\text{def}}{=} \left\{ \mathscr{H}^1 \lfloor_l \wedge \overrightarrow{t} \,:\, l \text{ is a line in } \mathbb{R}^2 \text{ and } \overrightarrow{t} \text{ is tangent to } l \right\}. \tag{17}$$

We will do this as follows. Define first of all

$$\mathscr{I}_L^{1,\prime} \overset{\text{def}}{=} \left\{ \mathscr{H}^1 \lfloor_{\{(s,ms):s\in\mathbb{R}\}} \wedge \frac{\mathbf{e}_1 + m\mathbf{e}_2}{\sqrt{1+m^2}} \in \mathscr{I}_L^1 \,:\, m \in [0,\infty) \text{ is rational} \right\}.$$

For any $T \in \mathscr{I}_L^{1,\prime}$, define

$$\tilde{\mu}_T \overset{\text{def}}{=} \lim_{N\to\infty} \frac{1}{2\,\mathrm{Card}\{x \in \mathbb{Z} : |x| \le N,\, mx \in \mathbb{Z}\}} \times \sum_{\substack{x\in\mathbb{Z} \\ |x|\le N,\, mx\in\mathbb{Z}}} \left\{ \tau_{(x,mx)}\mu^m + \tau_{(x,mx-1)}\mu^m \right\}.$$

The measures $\tilde{\mu}_T$ are translation-invariant along $\operatorname{supp} T$ and on average the interface passes through the origin.

Note that

$$\mathscr{I}_L^1 = \overline{\left\{ gT \,:\, g \in \mathscr{G},\, T \in \mathscr{I}_L^{1,\prime} \right\}},$$

where this closure is in the topology of currents. For any $T \in \mathscr{I}_L^1$, we thus define

$$\mu_T \overset{\text{def}}{=} \lim_{T'\to T,\, T'\in\mathscr{G}(\mathscr{I}_L^{1,\prime})} \tilde{\mu}_{T'}.$$

These are the ergodic measures.

We next state the local ergodic theorem. First, we will define $\mathscr{S} \subset B(\mathscr{X})$ as the collection of bounded and "local" functions which see only the interface; $\mathscr{S}$ is the collection of functions $\Phi \in B(\mathscr{X})$ such that there is a finite set $A \subset \mathbb{Z}^2$ such that

(1) $\Phi(\sigma) = \Phi(\sigma')$ for any configurations σ and σ' such that $\sigma\big|_A = \sigma'\big|_A$.
(2) $\Phi(\sigma) = 0$ if either $\sigma\big|_A \equiv 0$ or $\sigma\big|_A \equiv 1$.

The local ergodic theorem replaces averaged sums over small regions by expectations against the appropriate μ_T measure where $T \in \mathscr{I}_L^1$. We thus need a rule for selecting the appropriate line measure T.

Definition 4.2 (Average Current). Fix $B \subset \mathbb{R}$ convex. For any $T \in G(B)$, let $\underline{T}_B \in \mathscr{I}_L^1$ be the unique element of $\mathscr{I}_L^1$ such that

$$\partial(\underline{T}_B \lfloor_B) = \partial(T \lfloor_B).$$

The local ergodic theorem is:

Proposition 4.3 (Local Ergodicity). *Fix any $\Phi \in \mathscr{S}$. Then for any $t > 0$,*

$$\lim_{\varepsilon\to 0} \varlimsup_{\delta\to 0} \varlimsup_{n\to\infty} \mathbb{E}_n \left[\int_0^t \int_{y\in\mathbb{R}^2} \chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s}) \right.$$
$$\frac{1}{4\delta^2\varepsilon^2} \sum_{\substack{z/n\in\varepsilon D+y \\ z\in\mathbb{Z}^2}} \left| \frac{1}{n} \sum_{\substack{u/n\in\delta D+z/n \\ u\in\mathbb{Z}^2}} \left\{ \Phi(\tau_u \boldsymbol{\sigma}_s) \right.\right.$$
$$\left.\left.\left. - \int_{\sigma\in\mathscr{X}} \Phi(\tau_u\sigma)\mu_{\underline{T}^n_{\boldsymbol{\sigma}_s},\delta D+z/n}(d\sigma) \right\} \right| dy\, ds \right] = 0.$$

Proof. Given in Appendix B. □

We next develop some calculations and tools needed for the proof of Proposition 3.4. First, we need some notation. Define

$$H_1(\sigma) \overset{\text{def}}{=} \sum_{\substack{w\in\mathbb{Z}^2 \\ \|w\|_1=1}} w|\sigma(w)-\sigma(0)|. \qquad \sigma\in\mathscr{X}$$

For $w \in \mathbb{R}^2$ and $\sigma \in \mathscr{X}$, we define subsets $d_r(w)$ and $D_\sigma(w)$ of $\mathbb{R}^2$ as

$$\begin{aligned} d_r(w) &\overset{\text{def}}{=} \{x \in \operatorname{Span}(w) : \|w\|_\infty < r\} \qquad r > 0 \\ D_\sigma(w) &\overset{\text{def}}{=} \bigcup_{\substack{r>0 \\ d_r(w)\cap\operatorname{supp} T^1_\sigma=\emptyset}} d_r(w); \end{aligned}$$

note that $D_\sigma(w)$ contains only the shifted copy of the straight part of $\operatorname{supp} T^1_\sigma$ which passes by the origin in the direction w. We define

$$H_2(\sigma, w) \overset{\text{def}}{=} -\int_{x\in D_\sigma(w)} \frac{x}{\|x\|_1} \|T^1_\sigma\|(dx)$$

and then

$$H(\sigma, w) \overset{\text{def}}{=} H_1(\sigma)\chi_{\{n_\sigma(0)=2,\sigma(0)=1\}} + H_2(\sigma, w)\chi_{\{n_\sigma(z)=1,\sigma(z)=0\}}$$

for $w \in \mathbb{R}^2$ and $\sigma \in \mathscr{X}$.

Lemma 4.4. *For any $y \in \mathbb{R}^2$, $\varepsilon > 0$, $w \in \mathbb{Z}^2$ such that $\|w\|_1 = 1$ and $n \geq 1$ such that $T^n_\sigma \in G(\varepsilon D + y)$, we have that*

$$\sum_{z\in\mathbb{Z}^2} F^{y,\varepsilon}_\varphi(z/n)\xi_\sigma(z) = \frac{1}{n}\sum_{z\in\mathbb{Z}^2} \left\langle dF^{y,\varepsilon}_\varphi(z/n), H(\tau_z\sigma, w)\right\rangle' + \frac{1}{n^2} v^{y,\varepsilon}_n(\sigma, w)$$

where there is a universal constant K such that

$$|v_n^{y,\varepsilon}(\sigma,w)| \le \frac{K}{n^2}\|D^2F_\varphi^{y,\varepsilon}\| \sum_{\substack{x\in\mathbb{Z}^2\\ x/n\in\operatorname{supp}\varepsilon D+y\\ n_\sigma(x)=1,\,\sigma(x)=0}} \left\{\frac{1}{n}+|H_2(\tau_x\sigma,w)|^2\right\}.$$

Proof. In order to get a handle on things, we represent $T_\sigma^n\lfloor_{\varepsilon D+y}$ as a path integral. There is a continuous map $\gamma:\mathbb{R}_+\to\mathbb{R}^2$ which is piecewise-differentiable such that

(1) on every interval $\left(\frac{j}{n},\frac{j+1}{n}\right)$, $s\mapsto\gamma$ is affine and $\|\dot\gamma(s)\|_{\mathbb{R}^2}=1$,
(2) for every $\varphi\in\mathscr{D}^1$ with $\operatorname{supp}\varphi\subset\subset\varepsilon D+y$,

$$(T_\sigma^n\lfloor_{\varepsilon D+y})(\varphi)=\int_0^\infty\langle\varphi(\gamma(s)),\dot\gamma(s)\rangle'\,ds.$$

Since $T_\sigma^n\in G(\varepsilon D+y)$, there is an increasing sequence $0<s_1<t_1<s_2<\dots s_n<t_n<s_{n+1}\dots$ of nonnegative integers such that

$$\begin{aligned}\dot\gamma\left(\frac{s_j}{n}+\right)\ne\dot\gamma\left(\frac{s_j}{n}+\right)\quad&\text{and}\quad\dot\gamma\left(\frac{s_j}{n}+\right)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\dot\gamma\left(\frac{s_j}{n}-\right)\\ \dot\gamma\left(\frac{t_j}{n}+\right)\ne\dot\gamma\left(\frac{t_j}{n}+\right)\quad&\text{and}\quad\dot\gamma\left(\frac{t_j}{n}+\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\dot\gamma\left(\frac{t_j}{n}-\right);\end{aligned}$$

the s_j's correspond to left-hand turns and the t_j's correspond to right-hand turns; note that we can adjust γ as needed outside $\varepsilon D+y$ so that the first turn γ makes is to the left; we also will choose γ so that $\gamma(s_1/n)\notin\varepsilon D+y$. We then set

$$\begin{aligned}x_j&\overset{\text{def}}{=}\gamma\left(\frac{s_j}{n}\right)+\frac{1}{2n}\left\{\dot\gamma\left(\frac{s_j}{n}+\right)-\dot\gamma\left(\frac{s_j}{n}-\right)\right\}\\ \tilde x_j&\overset{\text{def}}{=}\gamma\left(\frac{t_j}{n}\right)+\frac{1}{2n}\left\{\dot\gamma\left(\frac{t_j}{n}+\right)-\dot\gamma\left(\frac{t_j}{n}-\right)\right\}\\ z_j&\overset{\text{def}}{=}\gamma\left(\frac{t_j}{n}\right)+\frac{1}{2n}\left\{\dot\gamma\left(\frac{t_j}{n}-\right)-\dot\gamma\left(\frac{t_j}{n}+\right)\right\}.\end{aligned}$$

It is then fairly easy to see that

$$\begin{aligned}\sum_{\substack{z\in\varepsilon D+y\\ z\in\mathbb{Z}^2}}F_\varphi^{y,\varepsilon}(z/n)\xi_\sigma(z)&=\sum_{j=0}^\infty\left\{F_\varphi^{y,\varepsilon}(x_j)-F_\varphi^{y,\varepsilon}(\tilde x_j)\right\}\\ &=\sum_{j=0}^\infty\left\{F_\varphi^{y,\varepsilon}(x_{j+1})-F_\varphi^{y,\varepsilon}(\tilde x_j)\right\};\end{aligned}$$

we will make a Taylor expansion of each summand. We note that

$$\dot{\gamma}\left(\frac{s_j}{n}+\right) = \dot{\gamma}\left(\frac{t_j}{n}-\right) \quad \text{and} \quad \dot{\gamma}\left(\frac{s_j}{n}-\right) = \dot{\gamma}\left(\frac{t_j}{n}+\right)$$

and so

$$\begin{aligned} x_j - \tilde{x}_j &= \left\{\gamma\left(\frac{s_j}{n}\right) - \gamma\left(\frac{t_j}{n}\right)\right\} + \frac{1}{n}\left\{\dot{\gamma}\left(\frac{t_j}{n}-\right) - \dot{\gamma}\left(\frac{t_j}{n}+\right)\right\} \\ x_j - \tilde{x}_j &= \left\{\gamma\left(\frac{s_{j+1}}{n}\right) - \gamma\left(\frac{t_j}{n}\right)\right\} + \frac{1}{n}\left\{\dot{\gamma}\left(\frac{t_j}{n}-\right) - \dot{\gamma}\left(\frac{t_j}{n}+\right)\right\}. \end{aligned}$$

It is easy to check from a picture that

$$\dot{\gamma}\left(\frac{t_j}{n}+\right) - \dot{\gamma}\left(\frac{t_j}{n}-\right) = H_1(\tau_{\tilde{x}_j}\sigma)$$

for all relevant j. Also,

$$\begin{aligned} \gamma\left(\frac{s_j}{n}\right) - \gamma\left(\frac{t_j}{n}\right) &= \frac{1}{n}H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n)) \\ \gamma\left(\frac{s_{j+1}}{n}\right) - \gamma\left(\frac{t_j}{n}\right) &= \frac{1}{n}H_2(\tau_{z_j}\sigma, \gamma(s_{j+1}/n) - \gamma(t_j/n)) \end{aligned}$$

for all relevant j; thus

$$\begin{aligned} x_j - \tilde{x}_j &= \frac{1}{n}H_1(\tau_{\tilde{x}_j}\sigma) + \frac{1}{n}H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n)) \\ x_{j+1} - \tilde{x}_j &= \frac{1}{n}H_1(\tau_{\tilde{x}_j}\sigma) + \frac{1}{n}H_2(\tau_{z_j}\sigma, \gamma(s_{j+1}/n) - \gamma(t_j/n)). \end{aligned}$$

Check that

$$\begin{aligned} \left\{\tilde{x}_j : \tilde{x}_j/n \in \operatorname{supp} F_\varphi^{y,\varepsilon}\right\} &= \left\{x \in \mathbb{Z}^2 : x/n \in \operatorname{supp} F_\varphi^{y,\varepsilon},\, n_\sigma(x) = 2,\, \sigma(x) = 1\right\} \\ \left\{z_j : z_j/n \in \operatorname{supp} F_\varphi^{y,\varepsilon}\right\} &= \left\{x \in \mathbb{Z}^2 : x/n \in \operatorname{supp} F_\varphi^{y,\varepsilon},\, n_\sigma(x) = 1,\, \sigma(x) = 0\right\}. \end{aligned}$$

We use all of this to prove the first representation result of the proposition. For any j,

$$\begin{aligned} &F_\varphi^{y,\varepsilon}(x_j) - F_\varphi^{y,\varepsilon}(\tilde{x}_j) \\ &\quad = \left\langle dF_\varphi^{y,\varepsilon}(\tilde{x}_j), H_1(\tau_{\tilde{x}_j})\right\rangle' + \left\langle dF_\varphi^{y,\varepsilon}(\tilde{x}_j), H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n))\right\rangle' + \frac{1}{n^2}v_j^1 \\ &\quad = \left\langle dF_\varphi^{y,\varepsilon}(\tilde{x}_j), H_1(\tau_{\tilde{x}_j})\right\rangle' + \left\langle dF_\varphi^{y,\varepsilon}(z_j), H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n))\right\rangle' + \frac{1}{n^2}v_j^2 \end{aligned}$$

where there is a constant K such that

$$|v_j^1| \le K\|D^2 F_\varphi^{y,\varepsilon}\| \left|H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n))\right|^2$$
$$|v_j^2| \le K\|D^2 F_\varphi^{y,\varepsilon}\| \left\{\frac{1}{n} + \left|H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n))\right|^2\right\}$$

where the second bound follows from the first with a redefinition of K. Note that either

$$H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n)) = H_2(\tau_{z_j}\sigma, \mathbf{e}_1)$$

or

$$H_2(\tau_{z_j}\sigma, \gamma(s_j/n) - \gamma(t_j/n)) = H_2(\tau_{z_j}\sigma, \mathbf{e}_2)$$

for all j. The proof of the second representation is similar. □

Our next task is to evaluate the expectation of H_1 and H_2 under the invariant measures. We have

Proposition 4.5. *Fix* $m \in [0, \infty)$. *Then*

$$\int_{\sigma\in\mathscr{X}} \sum_{k\in\mathbb{Z}} H_1(\tau_{(0,k)}\sigma)\chi_{\{n_\sigma(0,k)=2,\, \sigma(0,k)=1\}}\mu^m(d\sigma) = \frac{m}{m+1}\{-\mathbf{e}_1 + \mathbf{e}_2\}$$
$$\int_{\sigma\in\mathscr{X}} \sum_{k\in\mathbb{Z}} H_1(\tau_{(k,0)}\sigma)\chi_{\{n_\sigma(0,k)=2,\, \sigma(0,k)=1\}}\mu^m(d\sigma) = \frac{1}{m+1}\{-\mathbf{e}_1 + \mathbf{e}_2\}$$
$$\int_{\sigma\in\mathscr{X}} \sum_{k\in\mathbb{Z}} H_2(\tau_{(0,k)}\sigma, \mathbf{e}_2)\chi_{\{n_\sigma(0,k)=1, \sigma(0,k)=0\}} = -m\mathbf{e}_2$$
$$\int_{\sigma\in\mathscr{X}} \sum_{k\in\mathbb{Z}} H_2(\tau_{(k,0)}\sigma, \mathbf{e}_1)\chi_{\{n_\sigma(k,0)=1, \sigma(k,0)=0\}} = -\frac{1}{m}\mathbf{e}_1$$

Proof. Let E_m be the expectation operator associated with μ^m. First, we have

$$\sum_{\substack{k\in\mathbb{Z}\\ n_\sigma(0,k)=2\\ \sigma(0,k)=1}} H_1(\tau_{(0,k)}\sigma) = \{\mathbf{e}_2 - \mathbf{e}_1\}\chi_{\{h_\sigma(0)>h_\sigma(-1)\}};$$

and thus from (16),

$$E_m\left[\sum_{\substack{k\in\mathbb{Z}\\ n_\sigma(0,k)=2\\ \sigma(0,k)=1}} H_1(\tau_{(0,k)}\sigma)\right] = \{\mathbf{e}_2 - \mathbf{e}_1\}\mu^m\{\sigma \in \mathscr{X} : h_\sigma(0) > h_\sigma(-1)\}$$
$$= \frac{m}{m+1}\{-\mathbf{e}_1 + \mathbf{e}_2\}.$$

On the other hand, for any l,

$$E_m\left[\sum_{\substack{k\in\mathbb{Z}\\ n_\sigma(k,0)=1\\ \sigma(k,0)=1}} H_1(\tau_{(k,0)}\sigma)\big|h_\sigma(l)<0\le h_\sigma(l+1)\right]$$

$$=\{-\mathbf{e}_1+\mathbf{e}_2\}\mu_m\left\{h_\sigma(l+1)=0\big|h_\sigma(l)<0\le h_\sigma(l+1)\right\}$$

$$=\frac{1}{m+1}\{-\mathbf{e}_1+\mathbf{e}_2\}.$$

Partition $\mathscr{X}$ based on where the interface crosses level 0, and the second claim of the proposition follows. Next we observe that

$$\sum_{\substack{k\in\mathbb{Z}\\ n_\sigma(0,k)=1\\ \sigma(0,k)=0}} H_2(\tau_{(0,k)}\sigma,\mathbf{e}_2)=-\{h_\sigma(1)-h_\sigma(0)\}\mathbf{e}_2;$$

use again (16) and we get the third claim. Finally, for any l and any $j\ge 1$,

$$\mu^m\left\{\sum_{\substack{k\in\mathbb{Z}\\ n_\sigma(0,k)=1\\ \sigma(0,k)=0}} H_2(\tau_{(0,k)}\sigma,\mathbf{e}_2)=j\mathbf{e}_1\big|h_\sigma(l)<-1\le h_\sigma(l)\right\}$$

$$=\mu^m\left\{h_\sigma(l)=-1 \text{ and } h_\sigma(l+k)=-1 \text{ for } 1\le k\le j-1\right.$$

$$\left.\text{and } h_\sigma(l+j)>-1\big|h_\sigma(l)<-1\le h_\sigma(l)\right\}$$

$$=\frac{m}{(m+1)^{j+1}}$$

Thus

$$E_m\left[\sum_{\substack{k\in\mathbb{Z}\\ n_\sigma(0,k)=1\\ \sigma(0,k)=0}} H_2(\tau_{(0,k)}\sigma,\mathbf{e}_2)=j\big|h_\sigma(l)<-1\le h_\sigma(l)\right]=-\frac{1}{m}\mathbf{e}_1.$$

Again we partition and sum. □

We now can put things back together. Let's first write down some notation. For any $T\in\mathscr{I}$, we define an $\mathbb{R}^{2,*}$-valued sigma-finite measure ¥(T) as

$$¥(T)(A)\overset{\text{def}}{=}\int_{x\in A}\overrightarrow{T}(x)\|T\|(dx)$$

for all $A\subset\mathbb{R}^2$ such that $\|T\|(A)<\infty$. Then we have

Proposition 4.6. *Fix a line current $T = \mathscr{H}^1\lfloor_l \wedge \overrightarrow{t} \in \mathscr{I}^1_L$. Then*

$$\int_{\sigma\in\mathscr{X}} \sum_{k\in\mathbb{Z}} H(\tau_{(0,k)}\sigma, \mathbf{e}_1)\mu_T(d\sigma) = -\text{¥}(\hbar_{\varrho,\kappa}(T) - \kappa T)([-1/2,1/2]\times\mathbb{R})$$

and

$$\int_{\sigma\in\mathscr{X}} \sum_{k\in\mathbb{Z}} H(\tau_{(k,0)}\sigma, \mathbf{e}_1)\mu_T(d\sigma) = -\text{¥}(\hbar_{\varrho,\kappa}(T) - (\kappa+1)T)([-1/2,1/2]\times\mathbb{R}).$$

Proof. We will prove the result for $l = \{(s, ms) : s \in \mathbb{R}\}$ and $\overrightarrow{t} = \frac{\mathbf{e}_1 + m\mathbf{e}_2}{\sqrt{1+m^2}}$ where $m \in [0,\infty)$. The other cases follow from rotation and reflection. We note that

$$\frac{m}{m+1}\{\mathbf{e}_1 + m\mathbf{e}_2\} - m\mathbf{e}_2 = -\frac{m}{m+1}\{\mathbf{e}_1 + m\mathbf{e}_2\} = -\varrho(1,m)\overrightarrow{t}$$

and

$$\begin{aligned}\varrho(1,m) &= \varrho(1,m) + \kappa\sqrt{1+m^2} - \kappa\sqrt{1+m^2}\\ &= \mathscr{H}^1_{\varrho,\kappa}\lfloor_l([-1/2,1/2]\times\mathbb{R}) - \kappa\mathscr{H}^1\lfloor_l([-1/2,1/2]\times\mathbb{R}).\end{aligned}$$

On the other hand,

$$\frac{1}{m+1}\{-\mathbf{e}_1 + \mathbf{e}_2\} + \frac{1}{m}\mathbf{e}_2 = \frac{1}{m(m+1)}\{\mathbf{e}_1 + m\mathbf{e}_2\} = \frac{\sqrt{1+m^2}}{m(m+1)}\overrightarrow{t}$$

and

$$\begin{aligned}\frac{\sqrt{1+m^2}}{m(m+1)} &= \frac{1}{m}\left\{\varrho(1,m) + \kappa\sqrt{1+m^2}\right\} + \frac{\kappa+1}{m}\sqrt{1+m^2}\\ &= -\mathscr{H}^1_{\varrho,\kappa}\lfloor_l([-1/2,1/2]\times\mathbb{R}) + (\kappa+1)\mathscr{H}^1\lfloor_l([-1/2,1/2]\times\mathbb{R}).\end{aligned}$$

This is what we needed. ☐

Proposition 4.7. *Let T be a line current. Then for any fixed $\delta > 0$ and $\varphi \in \mathscr{D}^1$,*

$$\lim_{n\to\infty} \hbar_{\varrho,\kappa}(T^n_\sigma)(\varphi) - \iota_\delta(T^n_\sigma)(\varphi) = 0,$$

both μ_T-a.s. and in $L^1(\mu_T)$.

Proof. It is sufficient to prove the almost-sure result if $m \in [0,\infty)$ and $T = \mathscr{H}^1_l \wedge \overrightarrow{t}$ where $l = \{(s, ms) : s \in \mathbb{R}\}$ and $\overrightarrow{t} = \frac{\mathbf{e}_1 + m\mathbf{e}_2}{\sqrt{1+m^2}}$. First notice that μ^m-a.s.,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\delta}\int_{x\in\mathbb{R}^2}\vartheta_2\left(\frac{r(x)}{\delta}\right)\|T^n_\sigma\|(dx) &= \frac{1}{\delta}\int_{s\in\mathbb{R}}\vartheta_2\left(\frac{r(s,ms)}{\delta}\right)(1+m)ds\\ &= \frac{1}{\delta}\int_{s\in\mathbb{R}}\vartheta_2\left(s\frac{r(1,m)}{\delta}\right)ds(1+m) = \frac{1+m}{r(1,m)}.\end{aligned}$$

Thus, μ^m-a.s.

$$\lim_{m\to\infty} \iota_\delta(T^n_\sigma)(\varphi) = \int_{s\in\mathbb{R}} \langle \varphi(s,ms), \mathbf{e}_1 + m\mathbf{e}_2\rangle' \, \frac{1+m}{r(1,m)} ds$$

$$= \int_{s\in\mathbb{R}} \left\langle \varphi(s,ms), \overrightarrow{t}\right\rangle' \frac{(1+m)\sqrt{1+m^2}}{r(1,m)} ds.$$

On the other hand,

$$\hbar_{\varrho,\kappa}(T)(\varphi) = \int_{s\in\mathbb{R}} \left\langle \varphi(s,ms), \overrightarrow{t}\right\rangle' \{\rho(1,m) + \kappa\sqrt{1+m^2}\} ds.$$

Check that

$$\frac{(1+m)\sqrt{1+m^2}}{r(1,m)} = \rho(1,m) + \kappa\sqrt{1+m^2}.$$

□

Finally, we give the

Proof of Proposition 3.4. We fix $\eta > 0$ and $t > 0$. We first observe that we may restrict the y integral to a bounded set in view of Lemma 3.2. We note that $D^2 F^{y,\varepsilon}_\varphi$ is order ε^{-2}. We now claim that by Lemma 4.4,

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \, \mathbb{P}_n \left\{ \int_0^t \int_{y\in\mathbb{R}^2} \chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s}) \left| \sum_{z\in\mathbb{Z}^2} F^{y,\varepsilon}_\varphi(z/n)\xi_{\boldsymbol{\sigma}_s}(z) \right.\right.$$

$$\left.\left. - \frac{1}{n}\sum_{z\in\mathbb{Z}^2} \left\langle dF^{y,\varepsilon}_\varphi(z/n), H(\tau_z\boldsymbol{\sigma}_s, w)\right\rangle' \right| dy\, ds \geq \eta \right\} = 0$$

for $w \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. The term in the absolute value signs is on order

$$\frac{1}{n^2\varepsilon^3} \sum_{\substack{z\in\mathbb{Z}^2 \\ z/n\in\varepsilon D+y \\ n_{\boldsymbol{\sigma}_s}(z)\geq 1}} \left\{ \frac{1}{n} + |H_2(\tau_z\boldsymbol{\sigma}_s, w)|^2 \right\},$$

and this is order $\frac{1}{n\varepsilon^2}$ in expectation by Proposition 3.5. Next, since $F^{y,\varepsilon}_\varphi$ is slowly-varying (but of order ε^{-2}), we next claim that

$$\lim_{\varepsilon\to 0} \overline{\lim_{\varsigma\to 0}} \, \overline{\lim_{n\to\infty}} \, \mathbb{P}_n \left\{ \int_0^t \int_{y\in\mathbb{R}^2} \chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s}) \left| \frac{1}{n}\sum_{z\in\mathbb{Z}^2} \left\langle dF^{y,\varepsilon}_\varphi(z/n), \right.\right.\right.$$

$$\left.\left.\left. H(\tau_z\boldsymbol{\sigma}_s, w) - \frac{1}{4(n\varsigma)^2} \sum_{\substack{u/n\in\varsigma D+z/n \\ u\in\mathbb{Z}^2}} H(\tau_u\boldsymbol{\sigma}_s, w) \right\rangle' \right| ds\, dy \geq \eta \right\} = 0.$$

Here the term in the absolute value signs is of order $\frac{\varsigma}{\varepsilon^3}\frac{n\varepsilon}{n} = \frac{\varsigma}{\varepsilon^2}$.

We now use Proposition 4.3 with H in place of Φ. Of course to do so, we must first replace H by a local function; this can be done by standard methods in light of Proposition 3.5. Thus we have

$$\lim_{\varepsilon\to0}\overline{\lim}_{\varsigma\to0}\overline{\lim}_{n\to\infty}\mathbb{P}_n\left\{\int_0^t\int_{y\in\mathbb{R}^2}\chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\right.$$
$$\times\frac{1}{4\varsigma^2\varepsilon^2n^3}\sum_{\substack{z\in\mathbb{Z}^2\\ z/n\in\varepsilon D+y}}\left\|\sum_{\substack{u/n\in\varsigma D+z/n\\ u\in\mathbb{Z}^2}}H(\tau_u\boldsymbol{\sigma}_s,w)\right.$$
$$\left.\left.-\int_{\sigma\in\mathscr{X}}\sum_{\substack{u/n\in\varsigma D+z/n\\ u\in\mathbb{Z}^2}}H(\tau_u\sigma,w)\mu_{\underline{T}^n_{\boldsymbol{\sigma}_s},\varsigma D+z/n}(d\sigma)\right\|dy\,ds\geq\eta\right\}=0$$

Let's now assume that $w=\mathbf{e}_2$ (or equivalently $w=-\mathbf{e}_2$); the other case can be handled similarly. By Proposition 4.6, we have that

$$\lim_{\varepsilon\to0}\overline{\lim}_{\varsigma\to0}\overline{\lim}_{n\to\infty}\mathbb{P}_n\left\{\int_0^t\int_{y\in\mathbb{R}^2}\chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\right.$$
$$\times\frac{1}{4\varsigma^2\varepsilon^2n^2}\sum_{\substack{z\in\mathbb{Z}^2\\ z/n\in\varepsilon D+y}}\left\|\frac{1}{n}\sum_{\substack{u/n\in\varsigma D+z/n\\ u\in\mathbb{Z}^2}}H(\tau_u\boldsymbol{\sigma}_s,\mathbf{e}_2)\right.$$
$$-\yen(\hbar_{\varrho,\kappa}(\underline{T}^n_{\boldsymbol{\sigma}_s,\varsigma D+z/n})-\kappa\underline{T}^n_{\boldsymbol{\sigma}_s,\varsigma D+z/n})(\varsigma D+z/n)$$
$$\left.\left.\times\chi_{\{\mathrm{dist}_\infty(z/n,\mathrm{supp}\,T^n_{\boldsymbol{\sigma}_s})\leq\varsigma\}}\right\|dy\,ds\geq\eta\right\}=0$$

The error here is due to large jumps in the interface and due to u being within ς of both $\partial(\varepsilon D+y)$ and $\mathrm{supp}\,T^n_{\boldsymbol{\sigma}_s}$. This last type of error has been bounded by (25) in Appendix B and we refer the reader to those calculations. We next need to write things in terms of the ι_δ and use Proposition 4.7. We have that

$$\lim_{\varepsilon\to0}\overline{\lim}_{\varsigma\to0}\overline{\lim}_{\delta\to0}\overline{\lim}_{n\to\infty}\mathbb{P}_n\left\{\int_0^t\int_{y\in\mathbb{R}^2}\chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\right.$$
$$\times\frac{1}{4\varsigma^2\varepsilon^2n^2}\sum_{\substack{z\in\mathbb{Z}^2\\ z/n\in\varepsilon D+y\\ \mathrm{dist}_\infty(z/n,\mathrm{supp}\,T^n_{\boldsymbol{\sigma}_s})\leq\varsigma}}\left\|\yen(\hbar_{\varrho,\kappa}(\underline{T}^n_{\boldsymbol{\sigma}_s,\varsigma D+z/n})-\kappa\underline{T}^n_{\boldsymbol{\sigma}_s,\varsigma D+z/n})(\varsigma D+z/n)\right.$$
$$\left.\left.-\yen(\iota_\delta(T^n_{\boldsymbol{\sigma}_s})-\kappa T^n_{\boldsymbol{\sigma}_s})(\varsigma D+z/n)\right\|dy\,ds\geq\eta\right\}=0.$$

To see this, we can break the inner integral in (11) into a collection of boxes whose sides are order α/n where $\alpha \ll \delta$. Then use the ergodic theorem coupled with the two-block estimate. Let's collect everything together. We get that

$$\begin{aligned} \lim_{\varepsilon\to 0} \overline{\lim_{\varsigma\to 0}}\, \overline{\lim_{\delta\to 0}}\, \overline{\lim_{n\to\infty}}\, \mathbb{P}_n \Bigg\{ \int_0^t \int_{y\in\mathbb{R}^2} \chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s}) \Bigg| & \sum_{z\in\mathbb{Z}^2} F^{y,\varepsilon}_{\varphi}(z/n)\xi_{\boldsymbol{\sigma}_s}(z) \\ & -\frac{1}{n}\sum_{z\in\mathbb{Z}^2} \frac{1}{2\varsigma n}\chi_{\{\operatorname{dist}_\infty(z/n,\operatorname{supp} T^n_{\boldsymbol{\sigma}_s})<\varsigma\}} \\ & \times \left\langle dF^{y,\varepsilon}_{\varphi}(z/n), \frac{1}{2\varsigma}\yen(\iota_\delta(T^n_{\boldsymbol{\sigma}_s}) - \kappa T^n_{\boldsymbol{\sigma}_s})(\varsigma D + z/n)) \right\rangle' \Bigg| \, dy\, ds \ge \eta \Bigg\} = 0. \end{aligned}$$

We return to the fact that $dF^{y,\varepsilon}_{\varphi}$ is slowly-varying. We also recall the standard two-block estimate which is in the proof of the local ergodic theorem; this implies local regularity of the interface. We get that

$$\begin{aligned} \lim_{\varepsilon\to 0} \overline{\lim_{\delta\to 0}}\, \overline{\lim_{n\to\infty}}\, \mathbb{P}_n \Bigg\{ \int_0^t \int_{y\in\mathbb{R}^2} \chi_{G(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s}) \Bigg| & \sum_{z\in\mathbb{Z}^2} F^{y,\varepsilon}_{\varphi}(z/n)\xi_{\boldsymbol{\sigma}_s}(z) \\ & -\iota_\delta(T^n_{\boldsymbol{\sigma}_s})(dF^{y,\varepsilon}_{\varphi}) + \kappa T^n_{\boldsymbol{\sigma}_s}(dF^{y,\varepsilon}_{\varphi}) \big| \, dy\, ds \ge \eta \big\} = 0. \end{aligned}$$

A simple calculation using the fact that $T^n_{\boldsymbol{\sigma}_s}$ is a boundary current implies that

$$T^n_{\boldsymbol{\sigma}_s}(dF^{y,\varepsilon}_{\varphi}) = 0.$$

This completes the proof. □

§5. BAD SETS

Our goal here is to prove Proposition 3.6; i.e., to show that the "bad" set $G^c(\varepsilon D + y)$ can be neglected. We start out with

Lemma 5.1. *There is a $K > 0$ which depends only on φ such that for all $n \ge 1$, $\varepsilon > 0$, $y \in \mathbb{R}^2$, and $\sigma \in \mathscr{X}$,*

$$R^{y,\varepsilon,n}_{\varphi}(\sigma) \le K\left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z; T^n_\sigma)\right\}$$

for all $z \in 3\varepsilon\mathfrak{B} + y$.

Proof. Note that $\varepsilon D + y \subset 5\varepsilon\mathfrak{B} + z$. Thus

$$\begin{aligned} R^{y,\varepsilon,n}_{\varphi}(\sigma) &\le K\left\{\varepsilon^{-2} + \operatorname{Card}\left\{z \in \mathbb{Z}^2 : \frac{z}{n} \in \varepsilon D + y,\, n_\sigma(z) \ge 2\right\}\right\} \\ &\le K\left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z; T^n_\sigma)\right\} \end{aligned}$$

where the second bound follows from the first upon a redefinition of the constant K. □

Next we have

Lemma 5.2. *There is a* $K > 0$ *which depends only on* φ *such that for all* $n \geq 1$, $\varepsilon > 0$, $y \in \mathbb{R}^2$, *and* $\sigma \in \mathscr{X}$,

$$\begin{aligned}
&\chi_{G^c(\varepsilon D+y)}(T^n_\sigma) R^{y,\varepsilon,n}_\varphi(\sigma) \\
&\qquad \leq Kn \chi_{\digamma^c(10\varepsilon\mathfrak{B}+y)}(T^n_\sigma) \int_{z \in 3\varepsilon\mathfrak{B}+y} \left\{\varepsilon^{-2} + \varepsilon n \theta_{10\varepsilon}(z; T^n_\sigma)\right\} \|T^n_\sigma\|(dx) \\
&\qquad\qquad + Kn \int_{z \in 3\varepsilon\mathfrak{B}+y} (\theta_{5\varepsilon}(z; T^n_\sigma) - 1)^+ \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z; T^n_\sigma)\right\} \|T^n_\sigma\|(dz).
\end{aligned}$$

Proof. First, observe that

$$\begin{aligned}
\chi_{G^c(\varepsilon D+y)}(T^n_\sigma) R^{y,\varepsilon,n}_\varphi(\sigma) &\leq \chi_{\digamma^c(10\varepsilon\mathfrak{B}+y)\setminus G(\varepsilon D+y)}(T^n_\sigma) R^{y,\varepsilon,n}_\varphi(\sigma) \\
&\quad + \chi_{\digamma(10\varepsilon\mathfrak{B}+y)\setminus \digamma(\varepsilon D+y)}(T^n_\sigma) R^{y,\varepsilon,n}_\varphi(\sigma) \\
&\qquad + \chi_{\digamma(10\varepsilon\mathfrak{B}+y)\cap \digamma(\varepsilon D+y)\setminus G(\varepsilon D+y)}(T^n_\sigma) R^{y,\varepsilon,n}_\varphi(\sigma). \qquad (18)
\end{aligned}$$

If $T^n_\sigma \in \digamma(10\varepsilon\mathfrak{B} + y) \setminus \digamma(\varepsilon D + y)$, then the interface must enter and leave $\varepsilon D + y$ twice. For all points $z \in \operatorname{supp} T^n_\sigma \cap (3\varepsilon\mathfrak{B}+y)$, we must thus have $\theta_{5\varepsilon}(z; T^n_\sigma) \geq 1 + \frac{3}{2\varepsilon n}$; also $\|T^n_\sigma\|(3\varepsilon\mathfrak{B} + y) \geq K\varepsilon$ for some universal constant K. Thus

$$\begin{aligned}
&\chi_{\digamma(10\varepsilon\mathfrak{B}+y)\setminus \digamma(\varepsilon D+y)}(T^n_\sigma) R^{y,\varepsilon,n}_\varphi(\sigma) \\
&\qquad \leq \frac{2\varepsilon n}{3\varepsilon K} \int_{z \in 3\varepsilon\mathfrak{B}+y} (\theta_{5\varepsilon}(z; T^n_\sigma) - 1)^+ R^{y,\varepsilon,n}_\varphi(\sigma) \|T^n_\sigma\|(dz) \\
&\quad \leq K' n \int_{z \in 3\varepsilon\mathfrak{B}+y} (\theta_{5\varepsilon}(z; T^n_\sigma) - 1)^+ \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z; T^n_\sigma)\right\} \|T^n_\sigma\|(dz) \qquad (19)
\end{aligned}$$

for some constant K'. This bounds the second term on the right-hand side of (18). The last term in (18) corresponds to a "kink" in the interface inside $2\varepsilon D + y$. A bound similar to (19) holds for this case too, by a similar argument.

Finally, let's consider the first term on the right of (18). If $T^n_\sigma \notin G(\varepsilon D + y)$, then $\|T^n_\sigma\|(\varepsilon D + y) \geq 1/n$. Thus

$$\begin{aligned}
R^{y,\varepsilon}_\varphi(\sigma) &\leq Kn \int_{z \in \varepsilon D+y} \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z; T^n_\sigma)\right\} \|T^n_\sigma\|(dz) \\
&\qquad \leq Kn \int_{z \in 10\varepsilon\mathfrak{B}+y} \left\{\varepsilon^{-2} + \varepsilon n \theta_{10\varepsilon}(z; T^n_\sigma)\right\} \|T^n_\sigma\|(dz)
\end{aligned}$$

where the second bound follows from the first upon a redefinition of K. This gives the result. $\square$

We thus have the

Proof of Proposition 3.6. Fix $n \geq 1$, $\sigma \in \mathscr{X}$ and $\varepsilon > 0$. Then

$$\begin{aligned} n \int_{y\in\mathbb{R}^2} \int_{z\in 3\varepsilon\mathfrak{B}+y} & (\theta_{5\varepsilon}(z;T^n_\sigma)-1)^+ \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z;T^n_\sigma)\right\} \|T^n_\sigma\|(dz)\mathscr{H}^2(dy) \\ &= \varepsilon^2 n \mathscr{H}^2(3\mathfrak{B}) \int_{z\in\mathbb{R}^2} (\theta_{5\varepsilon}(z;T^n_\sigma)-1)^+ \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z;T^n_\sigma)\right\} \|T^n_\sigma\|(dz). \end{aligned}$$

We proceed by noting that

$$\varepsilon^2\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z;T^n_\sigma)\} \leq 1 + \varepsilon^3 n + \varepsilon^3 n\,(\theta_{5\varepsilon}(z;T^n_\sigma)-1)^+ \tag{20}$$

Recall that we let n grow first; thus effectively $\varepsilon^3 n \gg 1$; thus the integral of the first term on the right in the conclusion of Lemma 5.2 can effectively be bounded by the quantity

$$\varepsilon^3 n \int_{z\in\mathbb{R}^2} (\theta_{5\varepsilon}(z;T^n_\sigma)-1)^+ \left\{1 + (\theta_{5\varepsilon}(z;T^n_\sigma)-1)^+\right\} \|T^n_\sigma\|(dz). \tag{21}$$

Similarly,

$$\begin{aligned} n \int_{y\in\mathbb{R}^2} \int_{z\in 10\varepsilon\mathfrak{B}+y} & \chi_{\mathcal{F}^c(10\varepsilon\mathfrak{B}+y)}(T^n_\sigma) \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z;T^n_\sigma)\right\} \|T^n_\sigma\|(dz)\mathscr{H}^2(dy) \\ &= \int_{z\in\mathbb{R}^2} \left\{\varepsilon^{-2} + \varepsilon n \theta_{5\varepsilon}(z;T^n_\sigma)\right\} \\ &\quad \times \mathscr{H}^2\left\{(10\varepsilon\mathfrak{B}+z) \cap \{y\in\mathbb{R}^2 : T^n_\sigma \notin \mathcal{F}(10\varepsilon\mathfrak{B}+y)\}\right\} \|T^n_\sigma\|(dz). \end{aligned} \tag{22}$$

Note that by Hölder's inequality,

$$\begin{aligned} \mathscr{H}^2 & \left\{(10\varepsilon\mathfrak{B}+y) \cap \{y\in\mathbb{R}^2 : T^n_\sigma \notin \mathcal{F}(10\varepsilon\mathfrak{B}+y)\}\right\} \\ &\leq \varepsilon^{2/p} \left(\mathscr{H}^2(10\mathfrak{B})\right)^{1/p} \left\{\mathscr{H}^2\{y\in\mathbb{R}^2 : T^n_\sigma \notin \mathcal{F}(10\varepsilon B+y)\}\right\}^{1/q}, \end{aligned}$$

We can now make some calculations analogous to (20) and (21). We get that (22) is effectively bounded from above by

$$\begin{aligned} \varepsilon^{1+2/p} n \int_{z\in\mathbb{R}^2} & \left\{1 + (\theta_{5\varepsilon}(z;T^n_\sigma)-1)^+\right\} \|T^n_\sigma\|(dz) \\ &\times \left(\mathscr{H}^2\{y\in\mathbb{R}^2 : T^n_\sigma \notin \mathcal{F}(10\varepsilon B+y)\}\right)^{1/q}. \end{aligned}$$

This easily implies the conclusion of Proposition 3.6. □

§6. A PRIORI BOUNDS

We only need to give the

Proof of Proposition 3.5. We shall piggyback on the result for the simple exclusion process. We can cover $\partial\mathcal{I}_{\boldsymbol{\sigma}_0}$ with a finite number of charts $\{(\varphi_i, U_i);\ i = 1, 2 \dots J\}$ (we can assume that $\boldsymbol{\sigma}_0$ is deterministic, otherwise we randomize over the initial data). To each chart we can associate a copy of $\mathbb{Z}$. We can make the standard coupling arguments for the process projected through these charts to $\mathbb{Z}^J$. Motions of the interface at the edges of the charts corresponds to combinations of particles between the copies of $\mathbb{Z}$. This proves the result as long as $\partial\mathcal{I}_{\boldsymbol{\sigma}}$ stays within the charts. When it moves outside (say at some time τ, we simply choose new charts, transfer the estimates on the old charts at $\tau-$ to the new charts, and iterate our process. Note that since the dynamics cannot create new corners, the number of charts does not grow. $\square$

APPENDIX A

This appendix is dedicated to proving some deterministic results. We first give the

Proof of Proposition 2.7. First, we recall that $\mathscr{H}^1_{\mathfrak{D}}$ is defined as

$$\mathscr{H}^1_{\mathfrak{D}} \overset{\text{def}}{=} \lim_{\delta\to 0} \inf \left\{ \sum_{j=1}^{\infty} (2r_j) : A \subset \bigcup_{j=1}^{\infty} (r_j\mathfrak{D} + y_j),\ r_j \le \delta \right\}.$$

To prove that (6) is equivalent to motion by mean curvature, a local calculation suffices. Assume that locally

$$\mathfrak{t}_t(\varphi) = \int_{x\in\mathbb{R}} \left\{ \varphi_1(x, h(t,x)) + \varphi_2(x, h(t,x)) \frac{\partial h}{\partial x}(t,x) \right\} dx$$

for $\varphi(x,y) = \varphi_1(x,y)\mathbf{e}_1^* + \varphi_2(x,y)\mathbf{e}_2^*$ having support in some neighborhood N; i.e., $\operatorname{supp} \mathfrak{t}_t \cap N = \{(x, h(t,x));\ x \in N'\}$. Then

$$\dot{\mathfrak{t}}_t(\varphi) = \int_{x\in\mathbb{R}} \left\{ \frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x} \right\} (x, h(t,x)) \frac{\partial h}{\partial t}(t,x) dx;$$

an integration by parts is needed to prove this. On the other hand,

$$\begin{aligned} \hbar_{\varrho,\kappa}(\mathfrak{t}_s)(d * d\varphi) = \int_{x\in\mathbb{R}} \Bigg\{ &\left(\frac{\partial^2\varphi_2}{\partial x^2} - \frac{\partial^2\varphi_1}{\partial x\partial y} \right) (x, h(t,x)) \\ &+ \left(\frac{\partial^2\varphi_2}{\partial x\partial y} - \frac{\partial^2\varphi_1}{\partial y^2} \right) (x, h(t,x)) \Bigg\} \sigma\left(\frac{\partial h}{\partial x}(t,x) \right) dx \end{aligned}$$

where

$$\sigma(m) \overset{\text{def}}{=} \frac{\varrho(1,m)}{\sqrt{1+m^2}} + \kappa. \qquad m \in \mathbb{R}$$

An integration by parts shows that

$$\begin{aligned}
&\hbar_{\varrho,\kappa}(\mathfrak{t}_s)(d * d\varphi) \\
&\qquad = \int_{x\in\mathbb{R}} \left\{ \frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x} \right\} (x, h(t,x)) \left(\frac{\partial}{\partial x} \sigma \left(\frac{\partial h}{\partial x} \right) \right) (x, h(t,x)) dx
\end{aligned}$$

and thus

$$\frac{\partial h}{\partial t}(t,x) = \left(\frac{\partial}{\partial x} \sigma \left(\frac{\partial h}{\partial x} \right) \right) (x, h(t,x)). \tag{23}$$

On the other hand, for any C^∞ test function ψ with support in N,

$$\int_{z \in \operatorname{supp} \mathfrak{t}} \psi(z) \mathscr{H}^1_{\mathfrak{D}}(dz) = \int_{x\in\mathbb{R}} \psi(x, h(t,x)) J \left(\frac{\partial h}{\partial x}(t,x) \right) dx.$$

Fix $t > 0$ and $(x, h(t,x)) \in \operatorname{supp} \mathfrak{t}_t$; for $\mathfrak{t}$ to move by mean curvature in the $\mathscr{H}^1_{\mathfrak{D}}$ measure, we would need that $(x, h(t,x))$ have normal velocity

$$\begin{aligned}
\mathbf{v}(t,x) &= \left\{ -\frac{\partial h}{\partial x}(t,x)\mathbf{e}_1 + \mathbf{e}_2 \right\} \frac{\ddot{J}\left(\frac{\partial h}{\partial x}(t,x)\right) \frac{\partial^2 h}{\partial x^2}(t,x)}{J\left(\frac{\partial h}{\partial x}(t,x)\right)} \\
&= \left\{ -\frac{\partial h}{\partial x}(t,x)\mathbf{e}_1 + \mathbf{e}_2 \right\} \left\{ 1 + \left(\frac{\partial h}{\partial x}(t,x) \right)^2 \right\}^{-1} \dot{\sigma} \left(\frac{\partial h}{\partial x}(t,x) \right) \frac{\partial^2 h}{\partial x^2}(t,x).
\end{aligned}$$

The evolution of h would then be

$$\frac{\partial h}{\partial x}(t,x) = \left\langle -\frac{\partial h}{\partial x}(t,x)\mathbf{e}_1 + \mathbf{e}_2, \mathbf{v}(t,x) \right\rangle_{\mathbb{R}^2}$$

and this coincides exactly with (23). The only remaining step is to note that for all $m \in (0,\infty)$, $\rho(1,m) = m\rho(1/m, 1)$ and hence $J(m) = J(1/m)m$, and hence that

$$\dot{J}(1) = \frac{1}{2} J(1);$$

we can choose the value of $J(1)$ in any way we choose (the gradient flow is unaffected by premultiplying the length function); viz., we can choose $J(1) = 1$. □

We next give the

Proof of Proposition 3.7. Note that the closure of the $T^n_{\boldsymbol{\sigma}_s}$'s under $\mathbb{P}_n$ is the collection of integral currents T with mass measure

$$\|T\| = \mathscr{H}^1_{\mathfrak{B}} \lfloor_{\operatorname{supp} T}$$

where $\mathfrak{B}$ is given by (7) and $\mathscr{H}^1_{\mathfrak{B}}$ is the one-dimensional Hausdorff measure with this unit ball (the Density Assumption 2.3 implies that the multiplicity is always 1). Note that $T \mapsto \iota_\delta(T)$ is continuous in the weak topology (since it involves integrals against continuous functions). Although it is not so obvious that the mapping $T \mapsto \mathscr{H}^1_{\varrho,\kappa} \wedge \overrightarrow{T}$ is continuous, it is indeed continuous whenever the limit point has unit density (with respect to $\mathscr{H}^1_{\varrho,\kappa}$). Since T is rectifiable, it has a tangent plane for almost every x in its support. Thus it suffices to show that if

$$l = \{(s, ms) : \, s \in \mathbb{R}\} \subset \mathbb{R}^2,$$

then for any $\varphi \in C_c(\mathbb{R}^2)$,

$$\int_{z\in\mathbb{R}^2} \varphi(x) \left\{ \int_{y\in\mathbb{R}^2} \frac{1}{\delta} \vartheta_2 \left(\frac{r(y-x)}{\delta} \right) \mathscr{H}^1_{\mathfrak{B}} \lfloor_l (dy) \right\} \mathscr{H}^1_{\mathfrak{B}} \lfloor_l (dx) \\ = \int_{z\in\mathbb{R}^2} \varphi(x) \mathscr{H}^1_{\varrho,\kappa} \lfloor_l (dx).$$

First, note that

$$\frac{1}{\delta} \int_{y\in\mathbb{R}^2} \vartheta_2 \left(\frac{r(y)}{\delta} \right) \mathscr{H}^1_{\mathfrak{B}} \lfloor_l (dy) = \frac{1}{\delta} \int_{s\in\mathbb{R}} \vartheta_2 \left(s \frac{r(1,m)}{\delta} \right) (1+m) ds \\ = \frac{1+m}{r(1,m)}.$$

We can translate this calculation to any $x \in l$ and see that

$$\int_{z\in\mathbb{R}^2} \varphi(x) \left\{ \int_{y\in\mathbb{R}^2} \frac{1}{\delta} \vartheta_2 \left(\frac{r(y-x)}{\delta} \right) \mathscr{H}^1_{\mathfrak{B}} \lfloor_l (dy) \right\} \mathscr{H}^1_{\mathfrak{B}} \lfloor_l (dx) \\ = \int_{s\in\mathbb{R}} \varphi(s, ms) \frac{1+m}{r(1,m)} (1+m) ds \\ = \int_{s\in\mathbb{R}} \varphi(s, ms) \varrho(1,m) + \kappa \sqrt{1+m^2} ds = \int_{z\in\mathbb{R}^2} \varphi(x) \mathscr{H}^1_{\varrho,\kappa} \lfloor_l (dx).$$

This is what we needed. $\square$

APPENDIX B. PROOF OF THE LOCAL ERGODIC THEOREM

We now give the

Proof of Proposition 4.3. Fix $\eta > 0$ and $t > 0$. For any $y \in \mathbb{R}^2$, there are exactly twelve ways that $T^n_{\boldsymbol{\sigma}_t}$ can enter and exit $\varepsilon D + y$, if we consider only the side of entrance and the side of exit (we should make an appropriate rule if $T^n_{\boldsymbol{\sigma}_t}$ enters or exits exactly at a corner); i.e.,

$$G(\varepsilon D + y) = \cup_{j=1}^{12} G_i(\varepsilon D + y)$$

which is a disjoint union. Thus it suffices to prove the result if we replace $G(\varepsilon D+y)$ by each $G_i(\varepsilon D+y)$; by rotating it is sufficient to prove the result in two cases;

$$\begin{aligned}G_1(\varepsilon D+y) \overset{\text{def}}{=} \Big\{T \in G(\varepsilon D+y) :\ \partial\left(T\lfloor_{\varepsilon D+y}\right) = \mathscr{H}^0\lfloor_{x_{out}} - \mathscr{H}^0\lfloor_{x_{in}},\\ \overrightarrow{T}(x_{out}) = \overrightarrow{T}(x_{out}) = \mathbf{e}_1\Big\}\end{aligned}$$

and

$$\begin{aligned}G_2(\varepsilon D+y) \overset{\text{def}}{=} \Big\{T \in G(\varepsilon D+y) :\ \partial\left(T\lfloor_{\varepsilon D+y}\right) = \mathscr{H}^0\lfloor_{x_{out}} - \mathscr{H}^0\lfloor_{x_{in}},\\ \overrightarrow{T}(x_{in}) = \mathbf{e}_1,\ \overrightarrow{T}(x_{out}) = \mathbf{e}_2\Big\}.\end{aligned}$$

We shall prove the result in the first case; the second case is similar. Our first step is to replace the two-dimensional summation by a one-dimensional sum. For $\sigma \in \mathscr{X}$, $\varepsilon > 0$, and $y = (y_1, y_2) \in \mathbb{R}^2$, define

$$\hat{\Phi}_{\varepsilon D+y}(\sigma) \overset{\text{def}}{=} \sum_{\substack{u_2\in\mathbb{Z}\\ |u_2-y_2|\le\varepsilon}} \Phi(\tau_{(0,u_2)}\sigma).$$

We first claim that

$$\begin{aligned}\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\mathbb{P}_n\Bigg\{\int_0^t\int_{y=(y_1,y_2)\in\mathbb{R}^2}\chi_{G_1(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\\ \Bigg|\frac{1}{4\delta^2\varepsilon^2 n^3}\sum_{\substack{z/n\in\varepsilon D+y\\ z=(z_1,z_2)\in\mathbb{Z}^2}}\ \sum_{\substack{u_1\in\mathbb{Z}^2\\ |u_1-z_1|\le\delta n}}\Bigg|\sum_{\substack{u_2\in\mathbb{Z}^2\\ |u_2-z_2|\le\delta n}}\Phi(\tau_{(0,u_2)}\tau_{(u_1,0)}\boldsymbol{\sigma}_s)\\ -\sum\hat{\Phi}_{\varepsilon D+y}(\tau_{(u_1,0)}\boldsymbol{\sigma}_s)\chi_{\{\operatorname{dist}_\infty(z,\operatorname{supp}T^1_{\boldsymbol{\sigma}_s})<\delta n\}}\Bigg|\,dy\,ds \ge \eta\Bigg\} = 0.\end{aligned}$$

To see this, note that for any $z = (z_1, z_2) \in \mathbb{Z}^2$ with $z \in \varepsilon D + y$,

$$\sum_{\substack{u_2\in\mathbb{Z}\\ |u_2-z_2|\le\delta n}} \Phi(\tau_{(0,u_2)}\sigma) \approx \hat{\Phi}_{\varepsilon D+y}(\boldsymbol{\sigma}_s)\chi_{\{\operatorname{dist}_\infty(z,\operatorname{supp}T^1_{\boldsymbol{\sigma}_s})<\delta n\}} \tag{24}$$

where $\operatorname{dist}_\infty$ is the distance function in the $\|\cdot\|_\infty$ norm. The errors occur if the interface has a big step or if z is within distance δ of both $\partial(\varepsilon D+y)$ and of $\operatorname{supp} T^n_{\boldsymbol{\sigma}_s}$. Due to Proposition 3.5, we can neglect the errors due to the big steps. Thus the term inside the integral in (24) can be bounded in expectation by a term with order

$$\frac{1}{\delta^3\varepsilon n^3}\sum_{\substack{z\in\mathbb{Z}^2\\ z/n\in\varepsilon D+y\\ \operatorname{dist}_\infty(z/n,\partial(\varepsilon D+y))\le\delta\\ \operatorname{dist}_\infty(z/n,\operatorname{supp}T^n_{\boldsymbol{\sigma}_s})\le\delta}} (\delta n) \tag{25}$$

and this quantity is on order $\frac{1}{\delta^2\varepsilon^2 n^3}(\delta n)^2 = \frac{1}{\varepsilon^2 n}$. We can next explicitly sum over z_2 and claim that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\mathbb{P}_n\Bigg\{\int_0^t\int_{y=(y_1,y_2)\in\mathbb{R}^2}&\chi_{G_1(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\\
\frac{1}{4\delta^2\varepsilon^2 n^3}\sum_{\substack{z_1\in\mathbb{Z}\\|z_1-y_1|\le\varepsilon n}}&\Bigg|\sum_{\substack{z_2\in\mathbb{Z}\\|z_2-y_2|\le\varepsilon n}}\;\sum_{\substack{u_1\in\mathbb{Z}^2\\|u_1-z_1|\le\delta n}}\hat{\Phi}_{\varepsilon D+y}\big(\tau_{(u_1,0)}\boldsymbol{\sigma}_s\big)\chi_{\{\operatorname{dist}_\infty(z,\operatorname{supp}T^1_{\boldsymbol{\sigma}_s})<\delta n\}}\\
&\qquad-(2\delta n)\sum_{\substack{u_1\in\mathbb{Z}^2\\|u_1-z_1|\le\delta n}}\hat{\Phi}_{\varepsilon D+y}\big(\tau_{(u_1,0)}\boldsymbol{\sigma}_s\big)\Bigg|\,dy\,ds\ge\eta\Bigg\}=0.
\end{aligned}$$

The error here is due to boundary terms which occur again when z/n is within distance δ of both $\operatorname{supp}T^n_{\boldsymbol{\sigma}_s}$ and $\partial(\varepsilon D+y)$. Thus the error is again on order of (25). We now can apply the standard 1-dimensional local ergodic theorem to get that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\mathbb{E}_n\Bigg[\int_0^t\int_{y=(y_1,y_2)\in\mathbb{R}^2}&\chi_{G_1(\varepsilon D+y)}(T^n_{\boldsymbol{\sigma}_s})\\
\frac{1}{2\varepsilon^2 n}\sum_{\substack{z_1\in\mathbb{Z}\\|z_1-y_1|\le\varepsilon n}}&\Bigg|\frac{1}{2\delta n}\sum_{\substack{u_1\in\mathbb{Z}^2\\|u_1-z_1|\le\delta n}}\Big\{\hat{\Phi}_{\varepsilon D+y}\big(\tau_{(u_1,0)}\sigma\big)\\
&\qquad-\int_{\sigma\in\mathscr{X}}\hat{\Phi}_{\varepsilon D+y}(\sigma)\mu_{\underline{T}^n_{\boldsymbol{\sigma}_s},(\varepsilon D+y)\cap S_\delta(z_1/n)}\Big\}\Bigg|\,dy\,ds\Bigg]=0
\end{aligned}$$

where, for simplicity, we have defined $S_\delta(z_1/n)\overset{\text{def}}{=}[z_1/n-\delta, z_1/n+\delta]\times\mathbb{R}$. To apply the 1-dimensional local ergodic theorem, we invertibly map the interface h_σ onto the collection of jumps $\{h_\sigma(x)-h_\sigma(x-1); x\in\mathbb{Z}\}$. A local function of the configuration is then mapped into a local function of the jumps. We can then appeal to [16]. We now reverse our sequence of arguments. □

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