

LARGE DEVIATIONS FOR THE INVARIANT MEASURE OF A REACTION-DIFFUSION EQUATION WITH NON-GAUSSIAN PERTURBATIONS

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ABSTRACT

In this paper we establish a large deviations principle for the invariant measure of the non-Gaussian stochastic partial differential equation (SPDE) $\partial_t v^\epsilon = \mathcal{L}v^\epsilon + f(x, v^\epsilon) + \epsilon\sigma(x, v^\epsilon)\ddot{W}_{tx}$. Here \mathcal{L} is a strongly-elliptic second-order operator with constant coefficients; $\mathcal{L}h := DH_{xx} - \alpha h$, and the space variable x takes values on the unit circle S^1 . The functions f and σ are of sufficient regularity to ensure existence and uniqueness of a solution of the stochastic PDE, and in particular we require that $0 < m \leq \sigma \leq M$ where m and M are some finite positive constants. The perturbation W is a Brownian sheet. It is well-known that under some simple assumptions, the solution v^ϵ is a $C^\kappa(S^1)$ -valued Markov process for each $0 \leq \kappa < 1/2$, where $C^\kappa(S^1)$ is the Banach space of real-valued continuous functions on S^1 which are Hölder-continuous of exponent κ . We prove, under some further natural assumptions on f and σ which imply that the zero element of $C^\kappa(S^1)$ is a globally exponentially stable critical point of the unperturbed equation $\partial_t v^0 = \mathcal{L}v^0 + f(x, v^0)$, that v^ϵ has a unique stationary distribution $\nu^{\kappa, \epsilon}$ on $(C^\kappa(S^1), \mathcal{B}(C^\kappa(S^1)))$ when the perturbation parameter ϵ is small enough. Some further calculations show that as ϵ tends to zero, $\nu^{\kappa, \epsilon}$ tends to $\nu^{\kappa, 0}$, the point mass centered on the zero element of $C^\kappa(S^1)$. The main goal of this paper is to show that in fact $\nu^{\kappa, \epsilon}$ is governed by a large deviations principle (LDP). Our starting point in establishing the LDP for $\nu^{\kappa, \epsilon}$ is the LDP for the process v^ϵ , which has been shown in an earlier paper. Our methods of deriving the LDP for $\nu^{\kappa, \epsilon}$ based on the LDP for v^ϵ are slightly non-standard compared to the corresponding proofs for finite-dimensional stochastic differential equations, since the state space $C^\kappa(S^1)$ is inherently infinite-dimensional.

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1. Introduction.

In this paper we study the long-term behavior of the stochastic partial differential equation (SPDE)

$$\partial_t v^\epsilon = \mathcal{L}v^\epsilon + f(x, v^\epsilon) + \epsilon\sigma(x, v^\epsilon)\ddot{W}_{tx} \quad (1)$$

where \mathcal{L} is a time- and space-invariant second-order elliptic operator, f and σ are functions of sufficient regularity, and where W is a Brownian sheet. As the parameter ϵ tends to zero, the solutions v^ϵ of (1) will tend to solutions of

$$\partial_t v^0 = \mathcal{L}v^0 + f(x, v^0). \quad (2)$$

We shall find that, for some natural specifications of the boundary behavior of (1) and under some natural conditions on the function f , v^ϵ is a Markov process in the proper state-space, and that v^ϵ possesses a unique stationary distribution ν^ϵ . We can also impose conditions such that $v^0 \equiv 0$ is a unique asymptotically stable equilibrium point of (2). Then by continuity, we would expect that ν^ϵ tends to ν^0 , the unique stationary distribution of v^0 , which is concentrated on the identically zero function. The goal of this paper is to establish a large deviations result for ν^ϵ as a perturbation of ν^0 , i.e., the first term in a logarithmic expansion, as ϵ tends to 0, of $\nu^\epsilon(A)$, where A is a given set in the phase space of v^ϵ . This result is a natural extension of the large deviations result proved in [16].

To place the problem in the proper setting, we consider the following. The space variable x we assume to be in $S^1 := \{e^{i\theta} : \theta \in \mathbb{R}\}$; the solutions to (1) will in general not be well-defined functions if the space variable is in \mathbb{R}^n for $n \geq 2$ (see [18]), and we enforce periodicity in order to avoid specifying boundary conditions. The differential operator \mathcal{L} we take to be $\mathcal{L}h := Dh_{xx} - \alpha h$ where D and α are positive constants and where differentiation is with respect to the natural metric on S^1 (see Section 2). The class of admissible initial conditions we take to be $C(S^1)$. In order to ensure the existence and uniqueness both of the solution of (1) (with a fixed initial condition) and of the stationary distribution of v^ϵ , we require f to satisfy the following conditions:

(A.1) For all x in S^1 , $f(x, 0) = 0$. Also, there are positive constants \bar{F} and \bar{f} such that

$$|f(x, u)| \leq \bar{F} \quad \text{and} \quad |f(x, u) - f(x, v)| \leq \bar{f}|u - v|$$

for all x in S^1 and all u and v in \mathbb{R} .

Similarly, we shall also assume the following about σ ;

(A.2) The function σ is continuous as a function of both arguments and furthermore there are positive constants m , M , and $\bar{\sigma}$ such that

$$m \leq \sigma(x, u) \leq M \quad \text{and} \quad |\sigma(x, u) - \sigma(x, v)| \leq \bar{\sigma}|u - v|$$

for all x in S^1 and all u and v in \mathbb{R} .

Under these assumptions, we consider the equation (1). The random perturbation \ddot{W}_{tx} is to be interpreted as the formal derivative of a Brownian sheet W on $\mathbb{R}_+ \times S^1$, W being defined on some underlying (complete) probability triple (Ω, \mathcal{F}, P) . By a Brownian sheet on $\mathbb{R}_+ \times S^1$, we mean a random set function W on the Borel sets of $\mathbb{R}_+ \times S^1$ such that

- i)* for A a Borel subset of $\mathbb{R}_+ \times S^1$, $W(A)$ is a zero-mean Gaussian random variable
- ii)* for A and B Borel subsets of $\mathbb{R}_+ \times S^1$, $E[W(A)W(B)] = \text{Leb}(A \cap B)$, Leb being the natural Lebesgue measure on $(\mathbb{R}_+ \times S^1, \mathcal{B}(\mathbb{R}_+ \times S^1))$. (see [18]).

We can make a natural identification of S^1 with the interval $[0, 2\pi]$, and upon doing so, the random field $\bar{W}(t, x) := W([0, t] \times \{e^{i\theta} : 0 \leq \theta \leq x\})$ is a regular Brownian sheet on $\mathbb{R}_+ \times [0, 2\pi]$. Given the Brownian sheet W , stochastic integration against \ddot{W}_{tx} follows in the expected way.

With the above in mind, we may use [18] to see that there is a unique solution to (1), that is, to the SPDE

$$\begin{aligned} \partial_t v^\epsilon &= \mathcal{L}v^\epsilon + f(x, v^\epsilon) + \epsilon\sigma(x, v^\epsilon)\ddot{W}_{tx} \\ v^\epsilon[0] &= \zeta, \end{aligned} \quad t \geq 0 \quad (3)$$

for each fixed initial condition ζ in $C(S^1)$. By a solution to (3), we mean a solution in the weak sense; a random field, which is in $L^2([0, T'] \times S^1 \times \Omega)$ for each $T' > 0$ (the solution will in fact be in $C(\mathbb{R}_+ \times S^1)$) for which P -a.s.

$$\begin{aligned} \int_{S^1} v^\epsilon(t, x) \varphi(x) dx &= \int_{S^1} \zeta(x) \varphi(x) dx + \int_0^t \int_{S^1} v^\epsilon(s, x) \mathcal{L}\varphi(x) ds dx \\ &\quad + \int_0^t \int_{S^1} f(x, v^\epsilon(s, x)) \varphi(x) ds dx + \epsilon \int_0^t \int_{S^1} \varphi(x) \sigma(x, v^\epsilon(s, x)) W(ds, dx) \end{aligned}$$

for every φ in $C^2(S^1)$ and for all $t \geq 0$. When we wish to emphasize the dependence of v^ϵ on the initial condition ζ of (3), we shall write v_ζ^ϵ . In a similar way, by v_ζ^0 we shall naturally mean the solution to (2) for a fixed condition ζ in $C(S^1)$.

2. Notation.

In order to completely understand the goal of this paper, we shall need to develop some notation for the topological function spaces in which the solutions to (3) exist. The estimates found in [18] give us that the solution of (3) with $f \equiv 0$, $\sigma \equiv 1$, and $\zeta = \mathbf{0}$ is P -a.s. Hölder-continuous jointly in t and x with exponent $0 < \kappa < 1/4$, and for each $t \geq 0$, it is P -a.s. Hölder-continuous of exponent $0 < \kappa < 1/2$ as a function of x . Given these facts, it is natural that we carry out our investigations in a collection of Hölder spaces.

To fix the notation, let r be the standard metric on S^1 . For each $0 < \kappa < 1$, we shall define $[\cdot]_\kappa$ to be the Hölder seminorm of exponent κ for real-valued functions on S^1 , that is,

$$[\varphi]_\kappa := \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{r^\kappa(x, y)} : x, y \in S^1, x \neq y \right\}$$

for each mapping $\varphi : S^1 \rightarrow \mathbb{R}$. Also let $\|\cdot\|_{C(S^1)}$ be the supremum norm on real-valued functions of S^1 . We then define the norms $\|\cdot\|_0 := \|\cdot\|_{C(S^1)}$, and for each $0 < \kappa < 1$, $\|\cdot\|_\kappa := \|\cdot\|_{C(S^1)} + [\cdot]_\kappa$. For any $0 \leq \kappa < 1$ we let $C^\kappa(S^1)$ be the vector space of those mappings $\varphi : S^1 \rightarrow \mathbb{R}$ which are continuous and for which $\|\varphi\|_\kappa$ is finite. For any $0 \leq \kappa < 1$ we also denote by ρ_κ the metric on $C^\kappa(S^1)$ defined by the norm $\|\cdot\|_\kappa$. Finally, for any $T_1 \leq T_2$ in \mathbb{R} , define the pseudometric ρ^{T_1, T_2} on the class of real-valued functions whose domain contains $[T_1, T_2] \times S^1$ by $\rho^{T_1, T_2}(\varphi_1, \varphi_2) := \sup_{(t, x) \in [T_1, T_2] \times S^1} |\varphi_1(t, x) - \varphi_2(t, x)|$ for any such functions φ_1 and φ_2 .

3. Statement of the Large Deviations Principle.

Having established the notational prerequisites, we now continue our thoughts of Section 1, now being able to state precisely the goal of this paper. We wish to investigate the behavior of the stationary distribution of v^ϵ as ϵ tends to zero; the following statement is thus necessary:

Theorem 1. *For every $0 \leq \kappa < 1/2$ and $\epsilon > 0$ the process v^ϵ is a Markov process in $C^\kappa(S^1)$. For every $0 \leq \kappa < 1/2$, this Markov process has unique stationary distribution $\nu^{\kappa, \epsilon}$ for $\epsilon > 0$ small enough. For such a sufficiently small $\epsilon > 0$, $\nu^{\kappa, \epsilon}$ is the weak limit of the family $\{\nu^{\kappa, \epsilon, T}; T \geq 0\}$ of probability measures on $(C^\kappa(S^1), \mathcal{B}(C^\kappa(S^1)))$ given by*

$$\nu^{\kappa, \epsilon, T}(\Gamma) := \frac{1}{T} \int_0^T P \{v_0^\epsilon[t] \in \Gamma\} dt \quad \Gamma \in \mathcal{B}(C^\kappa(S^1)) \quad (4)$$

for every $T \geq 0$.

We shall prove this theorem in Section 10. We note that by the uniqueness result, if $0 \leq \kappa < \kappa' < 1/2$ and $\epsilon > 0$ is small enough that both $\nu^{\kappa, \epsilon}$ and $\nu^{\kappa', \epsilon}$ exist, then $\nu^{\kappa', \epsilon}$ is equal to the restriction of $\nu^{\kappa, \epsilon}$ to $\mathcal{B}(C^{\kappa'}(S^1))$. One can also show that in fact v^ϵ is a strong Markov process (see Lemma 3.3 in [13]). Also, one may relax the requirement that ϵ be small in order to ensure a unique invariant measure (see [12]).

The large deviations principle for $\nu^{\kappa, \epsilon}$ as a measure on $(C^\kappa(S^1), \mathcal{B}(C^\kappa(S^1)))$, for any $0 \leq \kappa < 1/2$, thus requires us to find an action functional V from $C^\kappa(S^1)$ to $[0, \infty]$ such that the following assertions hold;

(B.1) The level sets of V , i.e.

$$\Phi(s) := \{\xi \in C^\kappa(S^1) : V(\xi) \leq s\}$$

are compact in the ρ_κ -topology.

(B.2) For ξ^* in $C^\kappa(S^1)$ and positive numbers δ and γ ,

$$\nu^{\kappa, \epsilon} \{\xi \in C^\kappa(S^1) : \rho_\kappa(\xi, \xi^*) < \delta\} > \exp\left(-\frac{V(\xi^*) + \gamma}{\epsilon^2}\right)$$

for $\epsilon > 0$ small enough.

(B.3) For positive numbers s , δ , and γ ,

$$\nu^{\kappa, \epsilon} \{\xi \in C^\kappa(S^1) : \rho_\kappa(\xi, \Phi(s)) \geq \delta\} < \exp\left(-\frac{s - \gamma}{\epsilon^2}\right)$$

Of course it is not too hard, reviewing the literature of large deviations for stochastic ordinary differential equations, to guess the form of the action functional V . We should use the action functional for the process v^ϵ (see [16]); for any fixed times $T_1 < T_2$ and any real-valued function φ with domain containing $[T_1, T_2] \times S^1$ such that the restriction $\varphi|_{[T_1, T_2] \times S^1}$ is continuous, we define

$$S^{T_1, T_2}(\varphi) := \begin{cases} \frac{1}{2} \int_{[T_1, T_2] \times S^1} \left| \frac{\partial_t \varphi - \mathcal{L}\varphi - f(\cdot, \varphi)}{\sigma(\cdot, \varphi)} \right|^2 (t, x) dt dx & \text{if } \varphi|_{[T_1, T_2] \times S^1} \in W_2(T_1, T_2) \\ \infty & \text{if } \varphi|_{[T_1, T_2] \times S^1} \in C([T_1, T_2] \times S^1) \sim W_2(T_1, T_2) \end{cases}$$

where $W_2(T_1, T_2)$ is the closure of $C^\infty([T_1, T_2] \times S^1)$ in the norm

$$\|\varphi\|_{W_2(T_1, T_2)} = \left\{ \int_{[T_1, T_2] \times S^1} |\varphi|^2 + |\dot{\varphi}_t|^2 + |\dot{\varphi}_x|^2 + |\ddot{\varphi}_{xx}|^2 dt dx \right\}^{1/2}.$$

We claim that the action functional for $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$ will be

$$V(\xi) := \inf \{S^{0, T}(\varphi) : \varphi \in C([0, T] \times S^1), T > 0, \varphi[0] = \mathbf{0}, \varphi[T] = \xi\} \quad (5)$$

for each ξ in $C^\kappa(S^1)$. Note that in view of the uniqueness of action functionals and the above comments concerning the uniqueness of the invariant measures, the action functional for $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$ and $\{\nu^{\kappa', \epsilon}; \epsilon > 0\}$ must agree on $C^{\kappa'}(S^1)$, for $0 \leq \kappa < \kappa' < 1/2$. This allows us to ignore the index κ in writing the action functional V .

In the case where σ is an identically constant mapping, v^ϵ will be a continuous transformation of a Gaussian field, so that $\nu^{\kappa, \epsilon}$ may be described in terms of a Gaussian measure on $(C^\kappa(S^1), \mathcal{B}(C^\kappa(S^1)))$ as in [4] or [19]. But here the dependence of σ on v^ϵ precludes such an approach. We could also consider the approach of [5], Theorem 4.4.3, in representing the invariant measure for v^ϵ , but then we would need to develop the theory of [7] for infinite-dimensional diffusions. Instead of these two approaches, we shall present a proof of **(B.1)**–**(B.3)** which almost directly follows from the large deviations principle for v^ϵ .

This paper is organized into ten sections. In the next two sections, Sections 4 and 5, we review some basic representational results and bounds for equations of the form (3). Then, in Sections 6 through 8, we prove the assertions **(B.1)**–**(B.3)**. We do not, however, directly argue these assertions for an arbitrary $0 \leq \kappa < 1/2$, but only for $\kappa = 0$; it is slightly simpler to argue **(B.1)**–**(B.3)** from the large deviations principle for v^ϵ when $\kappa = 0$. Then in Section 9 we shall use some exponential tightness, introduced in Section 5, to transfer the large deviations principle from the case $\kappa = 0$ to the complete results for $0 \leq \kappa < 1/2$. Finally, in Section 10, we shall prove Theorem 1—the existence and uniqueness of a stationary distribution for v^ϵ ; because of the technicality of our arguments, we delay this task to the end of the paper.

4. Stochastic RDE's—the Green's function and some representation results.

We shall here quickly review how to represent the solution of (3) using the Green's function of the equation $\partial_t u = \mathcal{L}u$; this notation will be needed in our calculations.

Let $\{\phi_k; k = 1, 2, \dots\}$ be an orthonormal basis of $L^2(S^1)$ consisting of eigenfunctions of \mathcal{L} and let $\{\lambda_k; k = 1, 2, \dots\}$ be the corresponding eigenvalues. For concreteness, we shall take

$$\phi_{2k-1}(x) := \frac{1}{\pi^{1/2}} \Re(x^k) \quad \text{and} \quad \phi_{2k}(x) := \frac{1}{\pi^{1/2}} \Im(x^k); \quad x \in S^1, k = 1, 2, \dots$$

with $\phi_0 \equiv \frac{1}{(2\pi)^{1/2}}$, where $\Re(x)$ and $\Im(x)$ are the real and imaginary parts of any element of S^1 . The corresponding eigenvalues are then

$$\lambda_{2k-1} = \lambda_{2k} = Dk^2 + \alpha \quad k = 1, 2, \dots$$

with $\lambda_0 = \alpha$. The Green's function for the equation $\partial_t u = \mathcal{L}u$ is

$$G_t(x, y) := \chi\{t \geq 0\} \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y) \quad (t, x, y) \in \mathbb{R}_+ \times S^1 \times S^1 \quad (6)$$

where for any set A , χA is the indicator function of the set A . For ζ in $L^2(S^1)$, set \mathcal{T}_ζ to be

$$\mathcal{T}_\zeta(t, x) := \int_{S^1} G_t(x, y) \zeta(y) dy. \quad (t, x) \in \mathbb{R}_+ \times S^1$$

If we have any f' in $C(\mathbb{R}_+ \times S^1)$, the unique solution of

$$\begin{aligned} \partial_t u &= \mathcal{L}u + f' \\ u[0] &= \zeta \end{aligned} \quad (t, x) \in \mathbb{R}_+ \times S^1 \quad (7)$$

can then be represented as

$$u(t, x) = \mathcal{T}_\zeta(t, x) + \int_0^t \int_{S^1} G_{t-s}(x, y) f'(s, y) dy ds. \quad (t, x) \in \mathbb{R}_+ \times S^1$$

Of course there is another representation of the solution to (7) using the heat kernel. The solution to (7) can also be written as

$$u(t, e^{ix}) = \int_{\mathbb{R}} e^{-\alpha t} \frac{\exp\left[-\frac{(x-z)^2}{4Dt}\right]}{\sqrt{4\pi Dt}} \zeta(e^{iz}) dz + \int_0^t \int_{\mathbb{R}} e^{-\alpha(t-s)} \frac{\exp\left[-\frac{(x-z)^2}{4D(t-s)}\right]}{\sqrt{4\pi D(t-s)}} f'(s, e^{iz}) dz ds \quad (8)$$

for all $t \geq 0$ and e^{ix} in S^1 .

If we now take σ' in $L^\infty(\Omega \times \mathbb{R}_+ \times S^1)$ such that σ' is P -a.s. continuous as a function of (t, x) in $\mathbb{R}_+ \times S^1$, and such that for each (t, x) in $\mathbb{R}_+ \times S^1$, $\sigma'(t, x)$ is measurable with respect to

$$\mathcal{F}_t := \sigma\{W(A) : A \in \mathcal{B}([0, t] \times S^1)\}, \quad t \in \mathbb{R}_+ \quad (9)$$

then we may represent the solution of

$$\begin{aligned} \partial_t u &= \mathcal{L}u + f' + \sigma' \ddot{W}_{tx} \\ u[0] &= \zeta \end{aligned} \quad (t, x) \in \mathbb{R}_+ \times S^1 \quad (10)$$

as

$$u(t, x) = \mathcal{T}_\zeta(t, x) + \int_0^t \int_{S^1} G_{t-s}(x, y) f'(s, y) dy ds + \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma'(s, y) W(ds, dy) \quad (t, x) \in \mathbb{R}_+ \times S^1 \quad (11)$$

up to P -a.s. uniqueness. From this we see that the solution of (3) must P -a.s. satisfy the stochastic integral equation

$$\begin{aligned} v_\zeta^\epsilon(t, x) &= \mathcal{T}_\zeta(t, x) + \int_0^t \int_{S^1} G_{t-s}(x, y) f(y, v_\zeta^\epsilon(s, y)) dy ds \\ &\quad + \epsilon \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma(y, v_\zeta^\epsilon(s, y)) W(ds, dy) \end{aligned} \quad (12)$$

We note without further elaboration that this integral equation naturally defines a Picard iteration of \mathcal{F}_t -predictable elements of $L^2(\mathbb{R}_+ \times [0, T] \times \Omega)$ for each $T > 0$, and that standard procedures may be used to show the existence and uniqueness of an \mathcal{F}_t -predictable solution in $L^2([0, T] \times S^1 \times \Omega)$ for each $T > 0$ (see [18] Chapter 3 or [9] Sec. 5.1). See [18] and [11] for general discussions of stochastic parabolic PDE's.

5. Some Estimates of Continuity.

In this section we investigate, using the representation results of the previous section, the continuity of the solutions v^0 and v_ζ^ϵ of (2) and (3). Although some of our estimates, namely those of Propositions 1 through 4 take us slightly far afield of our goal of proving assertions **(B.1)**–**(B.3)**, we feel that these estimates and techniques, in addition to being necessary for our purposes, are of independent interest for theory of the continuity of non-Gaussian random fields. The reader eager to proceed with the proof of assertions **(B.1)**–**(B.3)** may jump to Section 6 and refer to this section as needed.

Let us start with \mathcal{T}_ζ , for some ζ in (S^1) . Using (8), it is easy to see that for all $t \geq 0$ and all x and y in S^1 ,

$$\|\mathcal{T}_\zeta[t]\|_{C(S^1)} \leq e^{-\alpha t} \|\zeta\|_{C(S^1)} \quad \text{and} \quad |\mathcal{T}_\zeta(t, x) - \mathcal{T}_\zeta(t, y)| \leq e^{-\alpha t} \omega(r(x, y)), \quad (13)$$

where ω is the modulus of continuity of ζ . Thus if ζ is in $C^\kappa(S^1)$ for some $0 \leq \kappa < 1$, then for all $t \geq 0$, $\mathcal{T}_\zeta[t]$ is also in $C^\kappa(S^1)$, with $\|\mathcal{T}_\zeta[t]\|_\kappa \leq e^{-\alpha t} \|\zeta\|_\kappa$. Now let us consider u solving the deterministic PDE (7), where f' is in $C(\mathbb{R}_+ \times S^1)$ and ζ is in $C(S^1)$. Using (8), we can easily show that

$$\begin{aligned} \|u[t]\|_{C(S^1)} &\leq e^{-\alpha t} \|\zeta\|_{C(S^1)} + \int_0^t e^{-\alpha(t-s)} \|f'[s]\|_{C(S^1)} ds \\ &\leq e^{-\alpha t} \|\zeta\|_{C(S^1)} + \frac{1}{\alpha} \|f'\|_0^t. \end{aligned} \quad (14)$$

We also have that, using the mean value theorem,

$$|u(t, x) - u(t, y)| \leq e^{-\alpha t} \omega(r(x, y)) + r(x, y) (\pi D\alpha)^{-1/2} \int_0^\infty s^{-1/2} e^{-s} ds \|f'\|_0^t$$

for all $t \geq 0$ and all x and y in S^1 , where again ω is the modulus of continuity of ζ . Thus we find that if ζ is in $C^\kappa(S^1)$ for some $0 \leq \kappa < 1$, then $u[t]$, where u solves (7), is also in $C^\kappa(S^1)$ for all $t \geq 0$, with

$$\|u[t]\|_\kappa \leq e^{-\alpha t} \|\zeta\|_\kappa + A_\kappa \|f'\|_{C([0, t] \times S^1)} \quad (15)$$

where

$$A_\kappa := \frac{1}{\alpha} + \sqrt{\frac{\pi^{1-2\kappa}}{D\alpha}} \int_0^\infty s^{-1/2} e^{-s} ds.$$

By a similar calculation using (8), we have that that for all $t \geq 0$ and all ζ in $C^\kappa(S^1)$, for any $0 \leq \kappa < 1$,

$$\|v_\zeta^0[t]\|_\kappa \leq e^{-\alpha t} \|\zeta\|_\kappa + \bar{f} \int_0^t e^{-\alpha(t-s)} \|v_\zeta^0[s]\|_\kappa ds$$

under assumption **(A.1)** so that by Gronwall's inequality

$$\|v_\zeta^0[t]\|_\kappa \leq \exp[-(\alpha - \bar{f})t] \|\zeta\|_\kappa \quad (16)$$

for all $t \geq 0$ and all ζ in $C^\kappa(S^1)$, for any $0 \leq \kappa < 1$.

Let us now consider the stochastic PDE

$$\begin{aligned} \partial_t \Xi &= \mathcal{L}\Xi + \sigma' \ddot{W}_{tx} \\ \Xi[0] &= \mathbf{0}, \end{aligned} \quad (t, x) \in \mathbb{R}_+ \times S^1 \quad (17)$$

or equivalently solutions of the integral equation

$$\Xi(t, x) = \int_{\mathbb{R}_+ \times S^1} G_{t-s}(x, y) \sigma'(s, y) W(ds, dy), \quad (t, x) \in \mathbb{R}_+ \times S^1 \quad (18)$$

where σ' is in $L^\infty(\Omega \times \mathbb{R}_+ \times S)$ such that σ' is P -a.s. continuous as a function of (t, x) in $\mathbb{R}_+ \times S^1$, and such that $\sigma'(t, x)$ is measurable with respect to \mathcal{F}_t as given in (9) for each t in \mathbb{R}_+ . We shall first demonstrate some relevant deterministic results concerning the Green's function G of (6), and then use these to estimate the continuity of Ξ solving (17). All of our estimates revolve around the following estimate, given in [6] Lemma 1, [18] Theorem 1.1, and [1] Lemma 3.3.3:

Proposition 1. *Let Ψ be a continuous and positive function from \mathbb{R} to \mathbb{R} which is convex increasing, symmetric about 0, and such that $\lim_{x \rightarrow \infty} \Psi(x) = \infty$. Let p be an increasing function from \mathbb{R}_+ to \mathbb{R}_+ such that $p(0) = 0$. Let C^n denote any n -dimensional cube in \mathbb{R}^n . Then if g is a measurable real-valued function on C^n such that*

$$B := \int_{x \in C^n} \int_{y \in C^n} \Psi \left(\frac{g(x) - g(y)}{p(|x - y|/\sqrt{n})} \right) dx dy$$

is finite, then there is a set $K \subset C^n$ of Lebesgue measure 0 such that for all x and y in $C^n \sim K$,

$$|g(x) - g(y)| \leq 8 \int_{u=0}^{|x-y|} \Psi^{-1}(B/u^{2n}) dp(u).$$

Our first result is an enhanced version of Lemma 3.9 of [18].

Proposition 2. *For $0 < \kappa < 1/2$ there is a positive number $L_{1,\kappa}$ such that for all $0 \leq t < \infty$ and all x and y in S^1 ,*

$$\left\{ \int_{\mathbb{R}_+ \times S^1} |G_r(x, z) - G_r(y, z)|^2 dr dz \right\}^{1/2} \leq L_{1,\kappa} r(x, y)^\kappa.$$

Proof. Take x and y in S^1 . Then

$$\int_{\mathbb{R}_+ \times S^1} |G_r(x, z) - G_r(y, z)|^2 dr dz = \sum_{k=0}^{\infty} \frac{1}{2\lambda_k} |\phi_k(x) - \phi_k(y)|^2.$$

Clearly for $k = 0, 1, \dots$,

$$|\phi_k(x) - \phi_k(y)| \leq \frac{2}{\pi^{1/2}}$$

and by the mean value theorem also

$$|\phi_{2k-1}(x) - \phi_{2k-1}(y)| \leq \frac{kr(x, y)}{\pi^{1/2}} \leq \frac{\lambda_{2k-1}^{1/2} r(x, y)}{(\pi D)^{1/2}}$$

and

$$|\phi_{2k}(x) - \phi_{2k}(y)| \leq \frac{kr(x, y)}{\pi^{1/2}} \leq \frac{\lambda_{2k}^{1/2} r(x, y)}{(\pi D)^{1/2}}.$$

We then calculate

$$\begin{aligned} \int_{\mathbb{R}_+ \times S^1} |G_{t-r}(x, z) - G_{t-r}(y, z)|^2 dr dz &\leq \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} |\phi_k(x) - \phi_k(y)|^{2\kappa} |\phi_k(x) - \phi_k(y)|^{2-2\kappa} \\ &= L_{1,\kappa}^2 r(x, y)^{2\kappa} \end{aligned}$$

where $L_{1,\kappa} := \{(2^{1-2\kappa}/(\pi D^\kappa)) \sum_{k=1}^{\infty} (1/\lambda_k^{1-\kappa})\}^{1/2}$ which is finite since $2(1-\kappa) > 1$. ■

Similarly,

Proposition 3. *We have that*

$$L_2 := \left\{ \sup_{x \in S^1} \int_0^\infty \int_{S^1} |G_s(x, y)|^2 ds dy \right\}^{1/2}$$

is finite.

Proof. Obvious from the previous proof. ■

We now use these two propositions and Proposition 1 to establish the following estimate of the Hölder norm of $\Xi[t]$ of (17) for any fixed $t \geq 0$. This estimate will be heavily used.

Proposition 4. *For each $0 < \kappa < 1/2$ there are positive constants K_κ^0 and K_κ^1 depending only on κ such that for all $t \geq 0$,*

$$P\{\|\Xi[t]\|_\kappa \geq L\} \leq \exp\left(-\left(L/\|\sigma'\|_{L^\infty(\Omega \times \mathbb{R}_+ \times S^1)}\right)^2 K_\kappa^1\right)$$

for all $L > 0$ such that $L/\|\sigma'\|_{L^\infty(\Omega \times \mathbb{R}_+ \times S^1)} \geq K_\kappa^0$.

Proof. Fixing a $0 \leq \kappa < 1/2$, let $\kappa' := (\kappa + 1/2)/2$. Fix also a $t \geq 0$, and let x^* be any element of S^1 . Then

$$\|\Xi[t]\|_\kappa \leq |\Xi(t, x^*)| + (1 + \pi^\kappa) [\Xi[t]]_\kappa. \quad (19)$$

We shall separately analyze the two terms in this sum in order to prove the lemma. For convenience, set $\sigma'_\infty := \|\sigma'\|_{L^\infty(\Omega \times \mathbb{R}_+ \times S^1)}$ and normalize Ξ by setting $\tilde{\Xi} := \Xi/\sigma'_\infty$.

Consider first the second term in (19); we proceed as in the proof of Proposition A.2 of [16]. Define $p(u) := L_{1,\kappa'} u^{\kappa'}$ for all $u \geq 0$, and $\Psi(x) := \exp(x^2/4)$ for all $x \in \mathbb{R}$. Define also

$$B := \int_{S^1} \int_{S^1} \Psi \left(\frac{\tilde{\Xi}(t, x) - \tilde{\Xi}(t, y)}{p(r(x, y))} \right) dx dy.$$

Let us estimate $E[B]$. Set

$$g(r, z) := \frac{G_{t-r}(x, z) - G_{t-r}(y, z)}{p(r(x, y))} \frac{\sigma'(r, z)}{\sigma'_\infty} \quad (r, z) \in \mathbb{R}_+ \times S^1$$

and let \tilde{M}_u be the right-continuous \mathcal{F}_t -martingale with associated quadratic variation

$$\tilde{M}_u := \int_0^u \int_{S^1} g(r, z) W(dr, dz) \quad \text{and} \quad \langle \tilde{M} \rangle_u := \int_0^u \int_{S^1} (g(r, z))^2 dr dz. \quad u \geq 0$$

By standard results, there then is a Brownian motion B , perhaps on an augmented probability triple, such that $\tilde{M}_u = B_{\langle \tilde{M} \rangle_u}$ for all $u \geq 0$. We now can use Proposition 2 to see that $\sup_{u \geq 0} \langle \tilde{M} \rangle_u < 1$ P -a.s., so

$$\begin{aligned} E \left[\Psi \left(\frac{\tilde{\Xi}(t, x) - \tilde{\Xi}(s, y)}{p(r(x, y))} \right) \right] &\leq E \left[\Psi \left(\tilde{M}_\infty^2 / 4 \right) \right] \\ &\leq E \left[\exp \left\{ \frac{1}{4} \left(\max_{0 \leq u \leq 1} |B_u| \right)^2 \right\} \right]. \end{aligned}$$

But it is not hard to prove that

$$E \left[\exp \left\{ \frac{1}{4} \left(\max_{0 \leq u \leq 1} |B_u| \right)^2 \right\} \right] \leq 4\sqrt{2};$$

one may, for example, apply Doob's maximal inequality to the submartingale $\exp[(B_u)^2/8]$ (see [3], Proposition 2.2.16(b)). We consequently know that

$$E[B] \leq (2\pi)^2 \cdot 4 \cdot 2^{1/2}. \quad (20)$$

Using Proposition 1, we have that for all x and y in S^1 ,

$$\begin{aligned} |\tilde{\Xi}(t, x) - \tilde{\Xi}(t, y)| &\leq 8 \int_0^{r(x, y)} \Psi^{-1} \left(\frac{B}{u^2} \right) dp(u) \\ &\leq \tilde{\eta}_2 \left((\ln_+ B)^{1/2} + \tilde{\eta}_3 \right) r(x, y)^\kappa \end{aligned}$$

where

$$\tilde{\eta}_2 := 8 \cdot 2^{3/2} L_{1, \kappa'} \pi^{\kappa' - \kappa} \quad \text{and} \quad \tilde{\eta}_3 := (\kappa' / \kappa) \pi^{\kappa - \kappa'} \sup_{u > 0} \frac{(\ln_+ u^{-2})^{1/2}}{u^{-(1/2 - \kappa)/2}}.$$

Thus

$$[\Xi[t]]_\kappa \leq \sigma'_\infty \tilde{\eta}_2 \left((\ln_+ B)^{1/2} + \tilde{\eta}_3 \right)$$

and so for $L > 0$,

$$\begin{aligned} P \{ [\Xi[t]]_\kappa \geq L \} &\leq P \left\{ (\ln_+ B)^{1/2} \geq \frac{L}{\sigma'_\infty \tilde{\eta}_2} - \tilde{\eta}_3 \right\} \\ &\leq \left\{ (2\pi)^2 \cdot 4 \cdot 2^{1/2} + 1 \right\} \exp \left(- \left(\frac{L}{\sigma'_\infty \tilde{\eta}_2} - \tilde{\eta}_3 \right)^2 \right), \end{aligned} \quad (21)$$

where we have used (20) and the calculation

$$E[\exp(\ln_+ B)] \leq E[\exp(\ln B) + \chi\{B < 1\}] \leq (2\pi)^2 \cdot 4 \cdot 2^{1/2} + 1.$$

Now we consider the first term of (19). We have that $\tilde{\Xi}(t, x^*)/L_2 = \tilde{M}_\infty^*$ where the \mathcal{F}_t -martingale \tilde{M}_u^* is defined as

$$\tilde{M}_u^* := \int_0^u \int_{S^1} \frac{G_{t-s}(x^*, z)}{L_2} \frac{\sigma'(s, z)}{\sigma'_\infty} W(ds, dz).$$

A proof similar to that of (20) will give us that

$$E \left[\Psi \left(\frac{\tilde{\Xi}(t, x^*)}{L_2} \right) \right] \leq E \left[\exp \left((\tilde{M}_\infty^*)^2 / 4 \right) \right] \leq 4 \cdot 2^{1/2}.$$

The representation

$$\Xi(t, x^*) = \sigma'_\infty L_2 \frac{\tilde{\Xi}(t, x^*)}{L_2}$$

then implies by Chebychev's exponential inequality that for every $L > 0$

$$P \{ |\Xi(t, x^*)| \geq L \} \leq 4 \cdot 2^{1/2} \exp \left(- \left(\frac{L}{2\sigma'_\infty L_2} \right)^2 \right). \quad (22)$$

Combining (21) and (22), for all $L > 0$,

$$\begin{aligned} P\{\|\Xi[t]\|_\kappa \geq L\} &\leq P\{\|\Xi[t]\|_\kappa \geq L/(2(1+\pi^\kappa))\} + P\{|\Xi(t, x^*)| \geq L/2\} \\ &\leq \left\{ (2\pi)^2 \cdot 4 \cdot 2^{1/2} + 1 \right\} \exp\left(-\left(\frac{L}{2(1+\pi^\kappa)\sigma'_\infty \tilde{\eta}_2} - \tilde{\eta}_3\right)^2\right) \\ &\quad + 4 \cdot 2^{1/2} \exp\left(-\left(\frac{L}{4\sigma'_\infty L_3}\right)^2\right). \end{aligned}$$

This clearly will give us the required result. Note that $\tilde{\eta}_2$ and $\tilde{\eta}_3$ do not depend on t . \blacksquare

We can now make a useful decomposition of the SPDE (3) into a coupled pair of equations, the first of which is of the form (7) and the second of which is of the form (17). From (12), for any ζ in $C(S^1)$ and any $\epsilon > 0$, we write that

$$v_\zeta^\epsilon = u_\zeta^\epsilon + \epsilon \psi_\zeta^\epsilon \quad (23)$$

where u_ζ^ϵ is

$$u_\zeta^\epsilon(t, x) = \mathcal{T}_\zeta(t, x) + \int_0^t \int_{S^1} G_{t-s}(x, y) f(y, v_\zeta^\epsilon(s, y)) ds dy$$

and

$$\psi_\zeta^\epsilon(t, x) = \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma(y, v_\zeta^\epsilon(s, y)) W(ds, dy)$$

for all (t, x) in $\mathbb{R}_+ \times S^1$. Comparing these equations with (10) and (11), we see that u_ζ^ϵ is the solution of

$$\begin{aligned} \partial_t u_\zeta^\epsilon &= \mathcal{L}u_\zeta^\epsilon + f(\cdot, v_\zeta^\epsilon) \\ u_\zeta^\epsilon[0] &= \zeta \end{aligned} \quad (t, x) \in \mathbb{R}_+ \times S^1$$

and ψ_ζ^ϵ solves

$$\begin{aligned} \partial_t \psi_\zeta^\epsilon &= \mathcal{L}\psi_\zeta^\epsilon + \sigma(\cdot, v_\zeta^\epsilon) \ddot{W}_{tx} \\ \psi_\zeta^\epsilon[0] &= \mathbf{0}. \end{aligned} \quad (t, x) \in \mathbb{R}_+ \times S^1$$

This decomposition is significant to our efforts in that u_ζ^ϵ may be analyzed for almost every fixed ω in Ω by using deterministic representations such as (8), while ψ_ζ^ϵ may be estimated by Proposition 4. Note that as a result of this decomposition, example, for every ζ in $C(S^1)$, for every $0 \leq \kappa < 1/2$, every $\epsilon > 0$ and every $t > 0$, we have from (15) and (23), that

$$\|v_\zeta^\epsilon[t]\|_\kappa \leq \|\mathcal{T}_\zeta[t]\|_{C^\kappa(S^1)} + A_\kappa \bar{F} + \epsilon \|\psi_\zeta^\epsilon[t]\|_\kappa. \quad (24)$$

This immediately gives us the following important result which shall be crucial to our calculations of Sections 7 through 10.

Proposition 5. *For any fixed $0 \leq \kappa < 1/2$ and any fixed ζ in $C(S^1)$,*

$$P\{\|v_\zeta^\epsilon[t]\|_\kappa \geq L\} \leq \exp\left[-\left(\frac{L - A_\kappa \bar{F} - \|\mathcal{T}_\zeta[t]\|_\kappa}{\epsilon M}\right)^2 K_\kappa^1\right] \quad (25)$$

for all $\epsilon > 0$, $L > 0$, and $t > 0$ such that

$$\frac{L - A_\kappa \bar{F} - \|\mathcal{T}_\zeta[t]\|_\kappa}{\epsilon M} \geq K_\kappa^0.$$

This implies in particular that for each fixed $0 \leq \kappa < 1/2$ and each fixed $\epsilon > 0$, the family of probability measures $\{\nu^{\kappa, \epsilon, T}; T > 0\}$ given by (4) are tight in $C^\kappa(S^1)$. Secondly, for any fixed $0 \leq \kappa < \kappa' < 1/2$,

$$\nu^{\kappa, \epsilon}\{\xi \in C^\kappa(S^1) : \|\xi\|_{\kappa'} > L\} \leq \exp\left[-\left(\frac{L - A_{\kappa'} \bar{F}}{\epsilon M}\right)^2 K_{\kappa'}^1\right] \quad (26)$$

for any $\epsilon > 0$ and $L > 0$ such that $\nu^{\kappa, \epsilon}$ exists and is unique and such that

$$\frac{L - A_{\kappa'} \bar{F}}{\epsilon M} \geq K_{\kappa'}^0.$$

Thus the family of probability measures $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$ is exponentially tight.

Recall that the family $\{P_\epsilon; \epsilon > 0\}$ on a Polish space \mathcal{X} with Borel sigma-field $B(\mathcal{X})$ is said to be *exponentially tight* if for each $R > 0$ there is a compact subset K_R of \mathcal{X} such that

$$\limsup_{\epsilon \rightarrow 0} \epsilon^2 \ln P_\epsilon(\sim K_R) \leq -R.$$

See [2], Chapter 2.

Proof. The bound (25) comes from combining (24) and Proposition 4. Since $C^{\kappa'}(S^1)$ is compactly embedded in $C^\kappa(S^1)$ for any $0 \leq \kappa < \kappa' < 1/2$ and using the fact that $\zeta = \mathbf{0} \in \cap_{0 \leq \kappa < 1/2} C^\kappa(S^1)$ in (4), we immediately get tightness of $\{\nu^{\kappa, \epsilon, T}; T > 0\}$. By weak convergence, we then arrive at the bound (26). The exponential tightness of $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$ comes from the bound (26) and the fact that $C^{\kappa'}(S^1)$ is compactly embedded in $C^\kappa(S^1)$. \blacksquare

6. Compactness of the level sets for $\kappa = 0$.

Let us for a moment recall the large deviations theory for the invariant measure of a stochastic differential equation; see for example Chapter 4 of [5]. In that case we have that the action functional for the invariant measure is continuous in the topology of state space, so the compactness of the level sets follows if one can show that these level sets are totally bounded. If we try to proceed in a like manner for our SPDE, we immediately are faced with the problem that V as in (5) cannot be continuous in $C^\kappa(S^1)$ for any $0 \leq \kappa < 1/2$. If $V(\xi)$ is finite, then there must be a $T > 0$ and a $\varphi \in W_2(0, T)$ with $\varphi[0] = \mathbf{0}$ and $\varphi[T] = \xi$, and from this we can show that ξ must be in $\cap_{0 \leq \kappa < 1/2} C^\kappa(S^1)$; that is,

$$\{\xi \in C(S^1) : V(\xi) < \infty\} \subset \bigcap_{0 \leq \kappa < 1/2} C^\kappa(S^1)$$

and thus the action functional V is necessarily discontinuous in the $C^\kappa(S^1)$ topology for each $0 \leq \kappa < 1/2$. Indeed, if $\sigma \equiv 1$, then one can find an explicit formula for V and see that $V(\xi)$ is finite only for those ξ which have a square-integrable generalized derivative (see [4]). Given these remarks, it is not surprising that our arguments will naturally be more detailed than in the finite-dimensional case.

Instead of directly proving the $C(S^1)$ -compactness of the sets $\Phi(s)$ of assertion **(B.1)**, we shall prove a deeper result from which **(B.1)** will naturally follow. As a point of departure for our proof, we first recall the compactness result for the process-level $C(S^1)$ -large deviations principle (see [16]):

(C.1) For each $0 \leq T_1 < T_2$, each $s \geq 0$, and each compact set K in $C(S^1)$, the set

$$\Phi_K(s) := \{\varphi \in C([T_1, T_2] \times S^1) : S^{T_1, T_2}(\varphi) \leq s, \varphi[0] \in K\}$$

is a compact set in $C([T_1, T_2] \times S^1)$.

We now introduce the following notation. Let $W_2^{1,2,e}$ be the space of those elements φ of $C((-\infty, 0] \times S^1)$ which have one generalized time derivative and two generalized space derivatives such that for every $T < 0$,

$$\int_{[T, 0] \times S^1} |\varphi|^2 + |\dot{\varphi}_t|^2 + |\dot{\varphi}_x|^2 + |\ddot{\varphi}_{xx}|^2 dt dx < \infty.$$

Let the action functional $S^{-\infty, 0}$ be given by

$$S^{-\infty, 0}(\varphi) := \begin{cases} \frac{1}{2} \int_{(-\infty, 0] \times S^1} \left| \frac{\partial_t \varphi - \mathcal{L}\varphi - f(\cdot, \varphi)}{\sigma(\cdot, \varphi)} \right|^2 (t, x) dt dx & \text{if } \varphi \in W_2^{1,2,e} \\ \infty & \text{if } \varphi \in C((-\infty, 0] \times S^1) \sim W_2^{1,2,e} \end{cases}$$

for φ in $C((-\infty, 0] \times S^1)$, and define the level sets

$$\Phi^{-\infty, 0}(s) := \left\{ \varphi \in C((-\infty, 0] \times S^1) : S^{-\infty, 0}(\varphi) \leq s, \lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0 \right\}$$

for each $s \geq 0$. Let us now endow $C((-\infty, 0] \times S^1)$ with the topology \mathcal{S} generated by open sets of the form

$$\{ \varphi \in C((-\infty, 0] \times S^1) : \rho^{T, 0}(\varphi, \varphi^*) < \eta \}$$

where φ^* runs over all of $C((-\infty, 0] \times S^1)$, T runs over all nonpositive reals, and η over all positive reals. We then claim

Proposition 6. *The set $\Phi^{-\infty, 0}(s)$ is compact in the \mathcal{S} topology.*

Proof. Our approach is similar to the proof of Lemma 3.3 in [17]. It is not difficult to see that the topological space $(C((-\infty, 0] \times S^1), \mathcal{S})$ satisfies the second axiom of countability; the Stone-Weierstrass theorem implies this. Thus, since compactness is equivalent to the Bolzano-Weierstrass property, we should take $\{\varphi_n\}_n$ in $\Phi^{-\infty, 0}(s)$ and show that there is a φ^* in $\Phi^{-\infty, 0}(s)$ to which some subsequence $\{\varphi_{n'}\}_{n'}$ of $\{\varphi_n\}_n$ converges in the \mathcal{S} topology.

In our proof we shall naturally consider compact subsets of $C(S^1)$, so by fixing a $0 < \kappa < 1/2$, we may pass to considerations of boundedness in $C^\kappa(S^1)$ to investigate compactness in $C(S^1)$.

We need the following lemma which we shall prove at the end of this section:

Lemma 1. *There is a positive number B such that if φ is in $\Phi^{-\infty, 0}(s)$, then for all $t \leq 0$, $\|\varphi[t]\|_\kappa \leq B$.*

Now define the set

$$K := \{ \zeta \in C(S^1) : \|\zeta\|_\kappa \leq B \}$$

and consider the sets $\Phi_K^{-n, 0}(s)$ for each positive integer n . By assertion **(C.1)** we can find a subsequence $\{\varphi_{n_1}\}_{n_1}$ of $\{\varphi_n\}_n$ and a φ^1 in $C([-1, 0] \times S^1)$ such that the restriction $\varphi_{n_1}|_{[-1, 0] \times S^1}$ tends to φ^1 in the supremum norm for $C([-1, 0] \times S^1)$, and by **(C.1)** we must have $S^{-1, 0}(\varphi^1) \leq s$. We can then find a subsequence $\{\varphi_{n_2}\}_{n_2}$ of $\{\varphi_{n_1}\}_{n_1}$ which is convergent in the supremum norm for $C([-2, 0] \times S^1)$. Proceeding by means of this diagonal argument, we arrive at a subsequence $\{\varphi_{n'}\}_{n'}$ of $\{\varphi_n\}_n$ and a φ^* in $C((-\infty, 0] \times S^1)$ such that $\{\varphi_{n'}\}_{n'}$ converges to φ^* in the \mathcal{S} topology and such that $S^{-\infty, 0}(\varphi^*) \leq s$. A simple argument gives us that for each $t \leq 0$, $\|\varphi^*[t]\|_\kappa \leq B$; that is, $\varphi^*[t]$ is in the compact set K for each $t \leq 0$.

To complete the proof of Proposition 6, we now need only to demonstrate that $S^{-\infty, 0}(\varphi^*) < \infty$ implies that $\lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0$. Suppose not; then there is a $0 < \eta \leq B$ and a sequence $\{t_n\}_n$ of negative numbers tending to $-\infty$ such that $\|\varphi^*[t_n]\|_{C(S^1)} \geq \eta$ for all n . A standard argument (see [5], Lemma 4.2.2, or [17], Lemma 5.4) gives us the following.

Lemma 2. *There is a $T_0 > 0$ and a $\beta > 0$ such that if $\|\varphi^*[t]\|_{C(S^1)} \geq \eta$, then $S^{t-T_0, t}(\varphi^*) \geq \beta$.*

Proof. By (16), we can always find a T_0 such that $\|v_\zeta^0[T_0]\|_{C(S^1)} \leq \eta/2$ for all ζ in K . Define the set

$$H := \{ \varphi \in C([0, T_0] \times S^1) : \varphi[0] \in K \text{ and } \|\varphi[T_0]\|_{C(S^1)} \geq \eta \}.$$

Clearly H is closed, and by our choice of T_0 , it cannot contain any trajectory of (2). It follows from the lower semicontinuity of S^{0, T_0} that $\beta := \inf\{S^{0, T_0}(\varphi) : \varphi \in H\}$ is strictly positive. \blacksquare

We can thus take a subsequence $\{t_{n_i}\}_{n_i}$ of $\{t_n\}_n$ such that $t_{n_{k+1}} \leq t_{n_k} - T_0$; then for all L ,

$$S^{-\infty, 0}(\varphi^*) \geq \sum_{l=1}^L S^{t_{n_{l+1}}, t_{n_l}}(\varphi^*) \geq \sum_{l=1}^L \beta = L\beta,$$

contradicting the fact that $S^{-\infty, 0}(\varphi^*)$ is finite. Thus we have shown that necessarily

$$\lim_{t \rightarrow -\infty} \rho_0(\varphi^*[t], \mathbf{0}) = 0,$$

which concludes the proof that $\Phi^{-\infty,0}(s)$ is \mathcal{S} -compact. ■

We now begin to prove the compactness of $\Phi(s)$. We start with

Proposition 7. For ξ in $C(S^1)$,

$$V(\xi) = \min \left\{ S^{-\infty,0}(\varphi) : \varphi \in C((-\infty, 0] \times S^1), \lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0, \varphi[0] = \xi \right\}. \quad (27)$$

Proof. An easy argument using the \mathcal{S} -compactness of the level sets $\Phi^{-\infty,0}(s)$ for each $s \geq 0$ shows that the infimum on the right hand side of (27) is attained, so that the minimum exists.

Take any φ in $C([0, T] \times S^1)$ with $T > 0$ such that $\varphi[0] = \mathbf{0}$ and $\varphi[T] = \xi$. Then define

$$\tilde{\varphi}[t] := \begin{cases} \mathbf{0} & \text{if } t < -T \\ \varphi[t+T] & \text{if } -T \leq t \leq 0. \end{cases}$$

Clearly

$$S^{0,T}(\varphi) = S^{-\infty,0}(\tilde{\varphi}) \geq \min \left\{ S^{-\infty,0}(\varphi) : \lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0, \varphi[0] = \xi \right\}$$

so

$$V(\xi) \geq \min \left\{ S^{-\infty,0}(\varphi) : \lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0, \varphi[0] = \xi \right\}. \quad (28)$$

To prove the other direction of inequality, let φ be the minimizer of the right-hand side of (27). For each $n \geq 1$, let φ_n in $W_2(-n, 0)$ be given by

$$\varphi_n[t] := \begin{cases} (t+n)\varphi[t] & \text{if } -n \leq t < -n+1 \\ \varphi[t] & \text{if } -n+1 < t \leq 0. \end{cases}$$

Note that $\varphi_n[-n] = \mathbf{0}$ and $\varphi_n[0] = \xi$. Let us calculate $S^{-n,0}(\varphi_n)$. We clearly have $S^{-n+1,0}(\varphi_n) = S^{-n+1,0}(\varphi) \leq S^{-\infty,0}(\varphi)$. We can also calculate that for $-n \leq t \leq -n+1$,

$$\begin{aligned} \partial_t \varphi_n[t] - \mathcal{L}\varphi_n[t] - f(\cdot, \varphi_n[t]) &= (t+n)(\partial_t \varphi[t] - \mathcal{L}\varphi[t] - f(\cdot, \varphi[t])) \\ &\quad + \varphi[t] + (t+n)f(\cdot, \varphi[t]) - f(\cdot, (t+n)\varphi[t]) \end{aligned}$$

so that by a simple bound using the fact that $|t+n| \leq 1$ for $-n \leq t \leq -n+1$, we have

$$\begin{aligned} S^{-n,-n+1}(\varphi_n) &\leq \frac{1}{m^2} \int_{[-n,-n+1] \times S^1} |(\partial_t \varphi - \mathcal{L}\varphi - f(\cdot, \varphi))(t, x)|^2 dt dx \\ &\quad + \frac{1}{m^2} \int_{[-n,-n+1] \times S^1} |1 + 2\bar{f}|^2 |\varphi(t, x)|^2 dt dx \\ &\leq 2 \frac{M^2}{m^2} S^{-n,-n+1}(\varphi) + 2\pi m^{-2} (1 + 2\bar{f})^2 (\|\varphi\|_{-n}^{-n+1})^2. \end{aligned}$$

Thus

$$S^{-n,0}(\varphi_n) \leq S^{-\infty,0}(\varphi) + 2 \frac{M^2}{m^2} S^{-n,-n+1}(\varphi) + 2\pi m^{-2} (1 + 2\bar{f})^2 (\|\varphi\|_{-n}^{-n+1})^2;$$

it is easy to see that $S^{-n,0}(\varphi_n) \geq V(\xi)$, so

$$S^{-\infty,0}(\varphi) \geq V(\xi) - 2 \frac{M^2}{m^2} S^{-n,-n+1}(\varphi) - 2\pi m^{-2} (1 + 2\bar{f})^2 \|\varphi\|_{-n}^{-n+1} \quad (29)$$

for all $n \geq 1$. But since $S^{-\infty,0}(\varphi)$ is finite (otherwise the opposite direction of inequality (28) would be trivially true), necessarily $\lim_n S^{-n,-n+1}(\varphi) = 0$, and since $\lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0$, also $\lim_n \|\varphi\|_{-n}^{-n+1} = 0$.

Thus we conclude the opposite direction of inequality (28) by passing to the limit in (29), completing the proof of Proposition 7. \blacksquare

Finally, we can prove that $\Phi(s)$ is compact. Again since $C(S^1)$ with the normal topology is separable, this is equivalent to showing the Bolzano-Weierstrass property. Take $\{\xi_n\}_n$ in $\Phi(s)$; then by Proposition 7, for each n there is a φ_n in $\Phi^{-\infty,0}(s)$ with $\varphi_n[0] = \xi_n$ and $S^{-\infty,0}(\varphi_n) = V(\xi_n)$. By the compactness of $\Phi^{-\infty,0}(s)$, $\{\varphi_n\}_n$ has an accumulation point φ (in the \mathcal{S} topology) which is also in $\Phi^{-\infty,0}(s)$. Now if φ_n tends to φ in the \mathcal{S} topology, then $\xi_n = \varphi_n[0]$ must tend to $\xi := \varphi[0]$ in $C(S^1)$, and by Proposition 7, we must also have $V(\xi) \leq S^{-\infty,0}(\varphi) \leq s$, so ξ is in $\Phi(s)$, completing the proof. \blacksquare

We now check Lemma 1.

Proof of Lemma 1. Fix t in $(-\infty, 0]$ and take $T < t$. Note that we can decompose φ as

$$\varphi[t] = \mathcal{T}_{\varphi[T]}[t - T] + \varphi_1[t] + \varphi_2[t],$$

where φ_1 and φ_2 respectively obey the PDE's

$$\begin{aligned} \partial_t \varphi_1 &= \mathcal{L}\varphi_1 + f(x, \varphi) \\ \varphi_1[T] &= \mathbf{0} \end{aligned} \quad T \leq t \leq 0$$

and

$$\begin{aligned} \partial_t \varphi_2 &= \mathcal{L}\varphi_2 + (\partial_t \varphi - \mathcal{L}\varphi - f(x, \varphi)) \\ \varphi_2[T] &= \mathbf{0}. \end{aligned} \quad T \leq t \leq 0$$

Now from (14), we have that $\|\varphi_1[t]\|_{\kappa} \leq A_{\kappa}\bar{F}$ and from Propositions 2 and 3, it is simple to check that

$$[\varphi_2[t]]_{\kappa} \leq L_{1,\kappa}M\sqrt{2s} \quad \text{and} \quad \|\varphi_2[t]\|_{C(S^1)} \leq L_2M\sqrt{2s}$$

for all $t \leq 0$. Thus in view of (13) and (14), we see that

$$\|\varphi[t]\|_{C(S^1)} \leq \|\varphi[T]\|_{C(S^1)} + A_{\kappa}\bar{F} + L_2M\sqrt{2s}$$

and

$$|\varphi(t, x) - \varphi(t, y)| \leq 2\|\varphi[T]\|_{C(S^1)} + (A_{\kappa}\bar{F} + L_{1,\kappa}M\sqrt{2s})r(x, y)^{\kappa}$$

for every x and y in S^1 . Letting T tend to $-\infty$, the result immediately follows, since $t \leq 0$ was arbitrarily chosen. \blacksquare

7. Lower bound for $\kappa = 0$.

Our proof almost directly follows from the lower large deviations bound for v^{ϵ} (see [16]);

(C.2) For any $T > 0$ and any positive numbers δ , γ , and s_0 , there is an $\epsilon_0 > 0$ such that

$$P \{ \rho^{0,T}(v_{\zeta}^{\epsilon}, \varphi) < \delta \} > \exp \left(-\frac{S^{0,T}(\varphi) + \gamma}{\epsilon^2} \right)$$

for all $0 < \epsilon < \epsilon_0$, ζ in $C(S^1)$, and φ in $\Phi_{\zeta}^{0,T}(s_0)$.

Of course we need only prove the assertion **(B.2)** for ξ^* in $C(S^1)$ such that $V(\xi^*) < \infty$.

By the definition of $V(\xi^*)$, there is a $T > 0$ and a φ in $C([0, T] \times S^1)$ with $\varphi[0] = \mathbf{0}$, $\varphi[T] = \xi^*$, and $S^{0,T}(\varphi) \leq V(\xi^*) + \gamma/3$. Fix a $0 < \kappa < 1/2$ and define the $C(S^1)$ -compact set K by

$$K := \{ \xi \in C(S^1) : \|\xi\|_{\kappa} \leq 2A_{\kappa}\bar{F} \}.$$

Proposition 5 then gives us that

$$\nu^{0,\epsilon}(\sim K) < \exp \left(-(A_{\kappa}\bar{F}/M\epsilon)^2 K_{\kappa}^1 \right) \quad (30)$$

when $0 < \epsilon < (A_\kappa \bar{F}) / (MK_\kappa^0)$ and $\epsilon > 0$ is small enough that $\nu^{0,\epsilon}$ exists. We shall need the following claim in our proof.

Lemma 3. *There is a $T' > 0$ such that for every ζ in K , there is a corresponding $\bar{\varphi}_\zeta$ in $C([0, T'] \times S^1)$ such that $\bar{\varphi}_\zeta[0] = \zeta$, $\bar{\varphi}_\zeta[T'] = \mathbf{0}$, and $S^{0,T'}(\bar{\varphi}) \leq \gamma/3$.*

Proof. Fixing a $T' > 0$, consider the function

$$\bar{\varphi}_\zeta[t] := \frac{T' - t}{T'} v_\zeta^0[t]. \quad 0 \leq t \leq T'$$

Then we find that

$$(\partial_t \bar{\varphi}_\zeta)[t] - (\mathcal{L} \bar{\varphi}_\zeta)[t] - f(\cdot, \bar{\varphi}_\zeta)[t] = \frac{-1}{T'} v_\zeta^0[t] + \frac{T' - t}{T'} f(\cdot, v_\zeta^0[t]) - f\left(\cdot, \frac{T' - t}{T'} v_\zeta^0[t]\right)$$

for all $0 \leq t \leq T'$, so that, in view of (16), a simple bound for $S^{0,T'}(\bar{\varphi}_\zeta)$ is given by the calculation

$$\begin{aligned} S^{0,T'}(\bar{\varphi}_\zeta) &\leq \frac{1}{2m^2(T')^2} \int_{t=0}^{T'} \int_{x \in S^1} (1 + 2\bar{f}t)^2 |v_\zeta^0(t, x)|^2 dt dx \\ &\leq \|\zeta\|_{C(S^1)}^2 \frac{2\pi}{2m^2(T')^2} \int_{t=0}^{T'} (1 + 2\bar{f}t)^2 e^{-2(\alpha - \bar{f})t} dt \\ &\leq \|\zeta\|_{C(S^1)}^2 \frac{\pi}{m^2(T')^2} \int_0^\infty (1 + 2\bar{f}t)^2 e^{-2(\alpha - \bar{f})t} dt. \end{aligned}$$

Then by taking T' large enough that

$$(2A_\kappa \bar{F})^2 \frac{\pi}{m^2(T')^2} \int_0^\infty (1 + 2\bar{f}t)^2 e^{-2(\alpha - \bar{f})t} dt \leq \gamma/3.$$

we have the lemma. ■

We then do the following. For each ζ in K , define $\tilde{\varphi}_\zeta$ in $C([0, T' + T] \times S^1)$ by

$$\tilde{\varphi}_\zeta[t] := \begin{cases} \bar{\varphi}_\zeta[t] & \text{if } 0 \leq t \leq T' \\ \varphi[t - T'] & \text{if } T' < t \leq T' + T. \end{cases}$$

This gives us that $S^{0,T'+T}(\tilde{\varphi}_\zeta) \leq V(\xi^*) + 2\gamma/3$ for every ζ in K . By using (30) and the lower bound (C.2), we then have for $\epsilon > 0$ small enough,

$$\begin{aligned} \nu^{0,\epsilon}\{\xi \in C(S^1) : \rho_0(\xi, \xi^*) < \delta\} &= \int_{C(S^1)} P\{\rho_{0,S^1}(v_\zeta^\xi[T' + T], \xi^*) < \delta\} \nu^{0,\epsilon}(d\zeta) \\ &\geq \int_{C(S^1)} P\{\rho^{0,T'+T}(v_\zeta^\xi, \tilde{\varphi}_\zeta) < \delta\} \nu^{0,\epsilon}(d\zeta) \\ &\geq \nu^{0,\epsilon}(K) \inf_{\zeta \in K} P\{\rho^{0,T'+T}(v_\zeta^\xi, \tilde{\varphi}_\zeta) < \delta\} \\ &\geq \nu^{0,\epsilon}(K) \exp\left(-\frac{S^{0,T'+T}(\tilde{\varphi}_\zeta) + \gamma/3}{\epsilon^2}\right) \\ &\geq \frac{1}{2} \exp\left(-\frac{V(\xi^*) + \gamma}{\epsilon^2}\right) \end{aligned}$$

which completes the proof. ■

8. Upper bound for $\kappa = 0$.

We with to argue (B.3) from the following process-level large deviations bound:

(C.3) For any $T > 0$ and any positive numbers δ , γ , and s_0 , there is an $\epsilon_0 > 0$ such that

$$P \{ \rho^{0,T} (v_\zeta^\epsilon, \Phi_\zeta(s)) \geq \delta \} < \exp \left(-\frac{s-\gamma}{\epsilon^2} \right)$$

for all $0 < \epsilon < \epsilon_0$, ζ in $C(S^1)$, and $0 \leq s \leq s_0$.

We begin with the fact that for any set Γ in $\mathcal{B}(C(S^1))$,

$$\nu^{0,\epsilon}(\Gamma) = \int_{C(S^1)} P \{ v_\zeta^\epsilon[t] \in \Gamma \} \nu^{0,\epsilon}(d\zeta)$$

for all $t \geq 0$ and all $\epsilon > 0$ small enough that $\nu^{0,\epsilon}$ is well-defined and unique. In particular, for all $t \geq 0$ and all small enough $\epsilon > 0$,

$$\nu^{0,\epsilon} \{ \xi \in C(S^1) : \rho_0(\xi, \Phi(s)) \geq \delta \} = \int_{C(S^1)} P \{ \rho_0(v_\xi^\epsilon[t], \Phi(s)) \geq \delta \} \nu^{0,\epsilon}(d\zeta). \quad (31)$$

Let us now fix a $0 < \kappa < 1/2$ and define the $C(S^1)$ -compact set

$$K := \{ \xi \in C(S^1) : \|\xi\|_\kappa \leq d \}$$

where $d := M(s/K_\kappa^1)^{1/2} + A_\kappa \bar{F}$. For the moment, we need only the constant d ; our usage of the set K will appear later. Nevertheless, let us now note a natural connection of K with the upper bound (C.3). Namely, that for $0 < \epsilon < (s/K_\kappa^1)^{1/2}/K_\kappa^0$ and $\epsilon > 0$ small enough that $\nu^{0,\epsilon}$ exists and is unique,

$$\nu^{0,\epsilon}(\sim K) \leq e^{-s/\epsilon^2} \quad (32)$$

as in the proof of Proposition 5.

Recall now our proof for the lower bound (B.2). We chose a function φ on a finite time interval and with initial point $\mathbf{0}$ which approximated the infimum in (5), or more exactly, approximated the minimizer in (27). We now wish to do a similar thing for the upper bound. Let us now define for each $\Delta > 0$ the $C(S^1)$ -compact set

$$K(\Delta) := \{ \xi \in C(S^1) : \|\xi\|_\kappa \leq \Delta \}.$$

Then we have the result

Lemma 4. *There is a $\Delta > 0$ and a $T' > 0$ such that for all $T \geq T'$,*

$$\{ \varphi[T] : \varphi \in \Phi_{K(\Delta)}^{0,T}(s) \} \subset \{ \xi : \rho_0(\xi, \Phi(s)) < \delta/2 \}.$$

Proof. Suppose not. Then there is a sequence $\{\Delta_n\}_n$ tending to zero, an unbounded increasing sequence $\{T_n\}_n$, and for each n , a φ_n in $\Phi_{K(\Delta_n)}^{0,T_n}(s)$ such that $\rho_0(\varphi_n[T_n], \Phi(s)) \geq \delta/2$. Recall now our arguments and notation of Section 6. Define $\tilde{\varphi}_n$ in $C([-T_n, 0] \times S^1)$ for each n by $\tilde{\varphi}_n[t] := \varphi_n[t+T_n]$ for $-T_n \leq t \leq 0$. Then by a diagonal argument similar to that used in the proof of Proposition 6, there is a subsequence $\{\tilde{\varphi}_{n'}\}_{n'}$ of $\{\tilde{\varphi}_n\}_n$ and a φ in $C((-\infty, 0] \times S^1)$ such that $S^{-\infty,0}(\varphi) \leq s$, $\rho_0(\varphi[0], \Phi(s)) \geq \delta/2$, and for each $t \leq 0$, $\lim_{n'} \rho^{t,0}(\tilde{\varphi}_{n'}, \varphi) = 0$. It is not hard to deduce, in a way similar to the proof of Lemma 1, that

$$\|\varphi[t]\|_{C(S^1)} \leq \Delta_0 + A_\kappa \bar{F} + L_2 M \sqrt{2s}$$

for all $t \leq 0$. As in Lemma 2, this and the fact that $S^{-\infty,0}(\varphi)$ is finite are enough to imply that $\lim_{t \rightarrow -\infty} \rho_0(\varphi[t], \mathbf{0}) = 0$ so that φ is in $\Phi^{-\infty,0}(s)$, implying that $V(\varphi[0]) \leq s$ by Proposition 7. But $\rho_0(\varphi[0], \Phi(s)) \geq \delta/2$; thus a contradiction results, proving our claim. \blacksquare

This lemma allows us to transfer from the set $\Phi^{-\infty,0}(s)$ on an infinite interval and zero ‘initial condition’, to the set $\Phi_{K(\Delta)}^{0,T}(s)$, with $T \geq T'$, which is a collection of functions on a finite time interval with initial point close to zero. The specific calculation we want is that for any $T \geq T'$ and any ζ in $K(\Delta)$,

$$\begin{aligned} P \{ \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta \} &\leq P \{ \rho^{0,T}(v_\zeta^\epsilon, \Phi_{K(\Delta)}^{0,T}(s)) \geq \delta/2 \} \\ &\leq P \{ \rho^{0,T}(v_\zeta^\epsilon, \Phi_\zeta^{0,T}(s)) \geq \delta/2 \}. \end{aligned} \quad (33)$$

The first of these inequalities is not directly obvious. Take ω in Ω such that $\rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta$. Take any φ in $\Phi_{K(\Delta)}^{0,T}(s)$. Then by Lemma 4, $\rho_0(\varphi[T], \Phi(s)) < \delta/2$, so by the triangle inequality, $\rho_0(v_\zeta^\epsilon[T], \varphi[T]) \geq \delta/2$. Hence $\rho^{0,T}(v_\zeta^\epsilon, \varphi) \geq \delta/2$, so since φ was arbitrary in $\Phi_{K(\Delta)}^{0,T}(s)$, we see that $\rho^{0,T}(v_\zeta^\epsilon, \Phi_{K(\Delta)}^{0,T}(s)) \geq \delta/2$. The second inequality naturally follows from the first as we have assumed that ζ is in $K(\Delta)$. From (33), we then have from **(B.3)** that for each $T \geq T'$,

$$\sup_{\zeta \in K(\Delta)} P \{ \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta \} < e^{-(s-\gamma/2)/\epsilon^2} \quad (34)$$

for all $0 < \epsilon < \epsilon_1(T)$, where we note the dependence of ϵ_1 on the value of T , which is as yet unspecified.

We would now like to manipulate (31) to resemble the left-hand side of (34), that is, to restrict in (31) the initial point to be in $K(\Delta)$. Let us show that with large enough probability, v^ϵ with initial point distributed according to $\nu^{0,\epsilon}$, as in (31), reaches $K(\Delta)$ during some finite collection of integer times $\{n : n = 1, 2, \dots, N_0\}$, with N_0 a positive integer yet to be chosen. We first use the estimate (32) to bound (31) by

$$\nu^{0,\epsilon} \{ \xi \in C(S^1) : \rho_0(\xi, \Phi(s)) \geq \delta \} \leq e^{-s/\epsilon^2} + \int_{\zeta \in K} P \{ \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta \} \nu^{0,\epsilon}(d\zeta) \quad (35)$$

which holds for all $T > 0$ and all $0 < \epsilon < \epsilon_2$, where here ϵ_2 will not depend upon the choice of T . By this, we can assume that the Markov process v_ζ^ϵ begins in the compact set K . We may immediately see from the estimates (16) that for some large enough integer n , $v_\zeta^0[n] \in K(\Delta)$ for all ζ in K , that is, for some large enough time, all the nonrandom trajectories v_ζ^0 of (2) with initial point in K will be in $K(\Delta)$. A similar result is probabilistically true for the trajectories of v_ζ^ϵ starting with ζ in K . We should define for each positive integer N the set H_N in $C([0, N] \times S^1)$ by

$$H_N := \{ \varphi \in C([0, N] \times S^1) : \varphi[0] \in K \text{ but } \varphi[j] \notin K(\Delta) \text{ for all } j = 0, 1, \dots, N \}.$$

We have

Lemma 5. *For N_0 large enough,*

$$\inf \{ S^{0,N_0}(\varphi) : \varphi \in H_{N_0} \} > s. \quad (36)$$

Proof. Suppose not. Then

$$\limsup_n (\inf \{ S^{0,n}(\varphi) : \varphi \in H_n \}) \leq s.$$

For each n , select any φ_n in H_n such that

$$S^{0,n}(\varphi) \leq 2 \inf \{ S^{0,n}(\varphi) : \varphi \in H_n \};$$

then for n large enough, $S^{0,n}(\varphi_n) \leq 2s$. By arguments similar to those of Lemma 1, we have that then

$$\|\varphi[t]\|_\kappa \leq d + A_\kappa \bar{F} + (L_{1,\kappa} + L_2)M\sqrt{4s}$$

for all $0 \leq t \leq n$ and all n large enough. But now by arguments similar to those of Lemma 2,

$$\beta := \inf \{ S^{0,k}(\varphi) : \|\varphi[0]\|_\kappa \leq d + A_\kappa \bar{F} + (L_{1,\kappa} + L_2)M\sqrt{4s}, \|\varphi[k]\|_\kappa \geq \Delta \}$$

must be positive for some k large enough. Thus for all n large enough,

$$2s \geq S^{0, nk}(\varphi_{nk}) \geq n\beta$$

which is clearly impossible. ■

Standard arguments using (36) give us that

$$\sup_{\zeta \in K} P \{v_\zeta^\epsilon \in H_{N_0}\} < e^{-(s-\gamma/2)/\epsilon^2} \quad (37)$$

for $0 < \epsilon < \epsilon_3$, where ϵ_3 will not depend upon our choice of T . We now can reduce (31) to the proper form of (34) by using (35) and (37). We have that for $\epsilon > 0$ small enough,

$$\begin{aligned} & \nu^{0, \epsilon} \{ \xi \in C(S^1) : \rho_0(\xi, \Phi(s)) \geq \delta \} \\ & \leq e^{-s/\epsilon^2} + \sup_{\zeta \in K} P \{v_\zeta^\epsilon \in H_{N_0}\} \\ & \quad + \int_{\zeta \in K} P \{v_\zeta^\epsilon \notin H_{N_0}, \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta\} \nu^{0, \epsilon}(d\zeta) \\ & \leq e^{-s/\epsilon^2} + e^{-(s-\gamma/2)/\epsilon^2} \\ & \quad + \int_{\zeta \in K} P \{v_\zeta^\epsilon[n] \in K(\Delta) \text{ for some } n = 0, 1, \dots, N_0, \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta\} \nu^{0, \epsilon}(d\zeta). \end{aligned} \quad (38)$$

This is true for all $T > 0$ and all $0 < \epsilon < \min\{\epsilon_2, \epsilon_3\}$. Let us now finally select our T to be $T := N_0 + T'$. In the last term of the last line of (38), we know that $v_\zeta^\epsilon[n]$ will be in $K(\Delta)$ for some $n = 1, 2, \dots, N_0$. Then at time T , the trajectory will be outside a δ -neighborhood of $\Phi(s)$, so we can use (34) on the portion of the process from n to T . We condition upon the first time at which v_ζ^ϵ is in $K(\Delta)$ and use the Markov property of v_ζ^ϵ to find

$$\begin{aligned} & \int_{\zeta \in K} P \{v_\zeta^\epsilon[n] \in K(\Delta) \text{ for some } n = 0, 1, \dots, N_0, \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta\} \nu^{0, \epsilon}(d\zeta) \\ & \leq \sum_{n=1}^{N_0} \sup_{\zeta \in K(\Delta)} P \{\rho_0(v_\zeta^\epsilon[T-n], \Phi(s)) \geq \delta\}. \end{aligned}$$

Inequality (34) then gives us

$$\int_{\zeta \in K} P \{v_\zeta^\epsilon[n] \in K(\Delta) \text{ for some } n = 0, 1, \dots, N_0, \rho_0(v_\zeta^\epsilon[T], \Phi(s)) \geq \delta\} \nu^{0, \epsilon}(d\zeta) < N_0 e^{-(s-\gamma)/\epsilon^2}$$

for $0 < \epsilon < \min\{\epsilon_1(T-n), \epsilon_2, \epsilon_3; n = 1, 2, \dots, N_0\}$, so (38) is continued as

$$\nu^{0, \epsilon} \{ \xi \in C(S^1) : \rho_0(\xi, \Phi(s)) \geq \delta \} < e^{-s/\epsilon^2} + (N_0 + 1)e^{-(s-\gamma/2)/\epsilon^2}.$$

This completes the proof of **(B.3)**. ■

9. The full result: the cases $0 \leq \kappa < 1/2$.

At this point we have the result

Theorem 2. *The family $\{\nu^{0, \epsilon}\}$ of probability measures on $C(S^1)$ satisfies a large deviations principle with action functional V given by (5).*

We now show that this is in fact sufficient to prove the assertions **(B.1)**–**(B.3)** for any $0 \leq \kappa < 1/2$, using Proposition 5. Our main tool for transferring from the $C(S^1)$ topology to the $C^\kappa(S^1)$ topology, for any $0 \leq \kappa < 1/2$, is the following result found in [8], Lemma 1.5:

Proposition 8. *Let \mathcal{X} and \mathcal{X}' be two Polish spaces. Let $\{P_\epsilon; \epsilon > 0\}$ be an exponentially tight family of probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Let $F : \mathcal{X} \rightarrow \mathcal{X}'$ be a continuous injection, and define the probability measures*

$$P'_\epsilon(\Gamma') := P_\epsilon\{x \in \mathcal{X} : F(x) \in \Gamma'\} \quad \Gamma' \in \mathcal{B}(\mathcal{X}')$$

on $(\mathcal{X}', \mathcal{B}(\mathcal{X}'))$ for each $\epsilon > 0$. Suppose that $\{P'_\epsilon; \epsilon > 0\}$ obeys a large deviations principle with action functional I' . Then the family $\{P_\epsilon; \epsilon > 0\}$ also obeys a large deviations principle, with action functional I given by

$$I(x) := I'(F(x)). \quad x \in \mathcal{X}$$

Let us fix now a $0 \leq \kappa < 1/2$. In our case here of course $C^\kappa(S^1)$ plays the role of \mathcal{X} and $C(S^1)$ plays the role of \mathcal{X}' . We let $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$ be the family $\{P_\epsilon; \epsilon > 0\}$ of probability measures; recall that in Proposition 5 we proved the exponential tightness of $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$. We of course let F be the injection mapping of $C^\kappa(S^1)$ into $C(S^1)$. Then the family $\{P'_\epsilon; \epsilon > 0\}$ is defined as

$$P'_\epsilon(\Gamma') := \nu^{\kappa, \epsilon}\{\xi \in C^\kappa(S^1) : \xi \in \Gamma'\}. \quad \Gamma' \in \mathcal{B}(C(S^1))$$

By the remarks following Theorem 1 concerning the uniqueness of the invariant measure, we then have that for $\epsilon > 0$ small enough,

$$\nu^{0, \epsilon}(\Gamma') = \nu^{0, \epsilon}(C^\kappa(S^1) \cap \Gamma') = \nu^{\kappa, \epsilon}(C^\kappa(S^1) \cap \Gamma') = P'_\epsilon(\Gamma')$$

for all Γ' in $\mathcal{B}(C(S^1))$, using the fact that $\nu^{0, \epsilon}(C^\kappa(S^1)) = 1$. Having proved in the preceding sections that $\{\nu^{0, \epsilon}; \epsilon > 0\}$ obeys a large deviations principle with action functional V given by (5), Proposition 8 immediately gives us that $\{\nu^{\kappa, \epsilon}; \epsilon > 0\}$ also obeys a large deviations principle with action functional

$$V^\kappa(\xi) = V(F(\xi)) = V(\xi). \quad \xi \in C^\kappa(S^1)$$

Thus we have transferred our results of Theorem 2 to the full result:

Theorem 3. *For each $0 \leq \kappa < 1/2$ the family $\{\nu^{\kappa, \epsilon}\}$ of probability measures on $C^\kappa(S^1)$ satisfies a large deviations principle with action functional V given by (5).*

10. Existence and uniqueness of the stationary distribution.

We now return to the proof of Theorem 1; that for every $\epsilon > 0$ small enough, the process v^ϵ is a Markov process in $C^\kappa(S^1)$ with unique stationary distribution $\nu^{\kappa, \epsilon}$. Of course we are primarily interested in the existence and uniqueness of a stationary distribution; by using (12), one can easily show that the process v^ϵ is Markovian, and by Proposition 5 clearly v^ϵ_ζ is $C^\kappa(S^1)$ -valued if ζ is in $C^\kappa(S^1)$.

To prove the existence and uniqueness of a stationary distribution $\nu^{\kappa, \epsilon}$ for a specific $0 \leq \kappa < 1/2$ for v^ϵ , we shall consider the family of distributions $\{\nu^{\kappa, \epsilon, T}; T > 0\}$ of (4). We have from Proposition 5 that the measures $\{\nu^{\kappa, \epsilon, T}; T > 0\}$ are tight; from standard arguments this will then imply the existence of at least one stationary distribution. We then show uniqueness by considering another stationary distribution. We show that asymptotically in time the effect of the initial condition of v^ϵ dies out, so that the second stationary distribution must coincide with the one generated by $\{\nu^{\kappa, \epsilon, T}; T > 0\}$.

Fix now a $0 \leq \kappa < 1/2$ and consider the question of existence. In Proposition 5 we proved the tightness of the family of measures $\{\nu^{\kappa, \epsilon, T}; T > 0\}$ on $C^\kappa(S^1)$. From [3], Theorem 9.3, this is then enough to imply the existence of at least one stationary distribution $\nu^{\kappa, \epsilon}$ of v^ϵ on $C^\kappa(S^1)$, this stationary distribution being the weak limit of some subsequence of $\{\nu^{\kappa, \epsilon, T}; T > 0\}$. Furthermore, from known results (see [15], Theorem VII.5.1), this is also sufficient to imply that for all bounded g in $C(C^\kappa(S^1))$,

$$\lim_T \int_{C^\kappa(S^1)} g(\zeta) \nu^{\kappa, \epsilon, T}(d\zeta) = \int_{C^\kappa(S^1)} g(\zeta) \nu^{\kappa, \epsilon}(d\zeta), \quad (39)$$

i.e., all subsequences of $\{\nu^{\kappa, \epsilon, T}; T > 0\}$ converge to the same measure $\nu^{\kappa, \epsilon}$ on $C^\kappa(S^1)$.

Consider now the question of uniqueness. We shall need the following statement that $v_\zeta^\epsilon[t] - v_{\mathbf{0}}^\epsilon[t]$ converges in probability to $\mathbf{0}$ in $C^\kappa(S^1)$ as t tends to infinity. The technical nature of the proof causes us to delay its proof until the end of the section.

Lemma 6. *For each $0 \leq \kappa < 1/2$ there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and all ζ in $C^\kappa(S^1)$,*

$$P\{\|v_\zeta^\epsilon[t] - v_{\mathbf{0}}^\epsilon[t]\|_\kappa \geq \delta\} = 0$$

for all $\delta > 0$.

To show uniqueness of the stationary distribution for a specific $0 \leq \kappa < 1/2$, we should now take another stationary distribution $\tilde{\nu}^{\kappa,\epsilon}$ and compare it with $\nu^{\kappa,\epsilon}$. Take any bounded function g in $C(C^\kappa(S^1))$. For any $T > 0$ it is easy to check the following inequalities;

$$\begin{aligned} & \left| \int_{C^\kappa(S^1)} g(\zeta) \nu^{\kappa,\epsilon,T}(d\zeta) - \int_{C^\kappa(S^1)} g(\zeta) \tilde{\nu}^{\kappa,\epsilon}(d\zeta) \right| \\ &= \left| \frac{1}{T} \int_0^T E[g(v_{\mathbf{0}}^\epsilon[t])] dt - \frac{1}{T} \int_0^T \int_{C^\kappa(S^1)} E[g(v_\zeta^\epsilon[t])] \tilde{\nu}^{\kappa,\epsilon}(d\zeta) dt \right| \\ &\leq \int_{C^\kappa(S^1)} \frac{1}{T} \int_0^T E[|g(v_{\mathbf{0}}^\epsilon[t]) - g(v_\zeta^\epsilon[t])|] dt \tilde{\nu}^{\kappa,\epsilon}(d\zeta). \end{aligned}$$

From Lemma 6 it is obvious that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T E[|g(v_{\mathbf{0}}^\epsilon[t]) - g(v_\zeta^\epsilon[t])|] dt = 0$$

for all $0 < \epsilon < \epsilon_0$ and all ζ in $C^\kappa(S^1)$. In light of the boundedness of g , we can thus apply dominated convergence to see that

$$\begin{aligned} & \left| \int_{C^\kappa(S^1)} g(\zeta) \nu^{\kappa,\epsilon}(d\zeta) - \int_{C^\kappa(S^1)} g(\zeta) \tilde{\nu}^{\kappa,\epsilon}(d\zeta) \right| \\ &\leq \limsup_T \left| \int_{C^\kappa(S^1)} g(\zeta) \nu^{\kappa,\epsilon,T}(d\zeta) - \int_{C^\kappa(S^1)} g(\zeta) \tilde{\nu}^{\kappa,\epsilon}(d\zeta) \right| \\ &= 0, \end{aligned}$$

recalling (39). Since g in $C(C^\kappa(S^1))$ was arbitrary, $\tilde{\nu}^{\kappa,\epsilon}$ and $\nu^{\kappa,\epsilon}$ coincide, so the stationary distribution must be unique. \blacksquare

Now we furnish the proof of Lemma 6. The proof is rather technical.

Proof of Lemma 6. In fact the essence of the proof is the result for $\kappa = 0$; we can use Proposition 5 and some standard interpolation results for Hölder norms to transfer the result from $\kappa = 0$ to $0 \leq \kappa < 1/2$.

We are interested in the difference $\Delta_\zeta^\epsilon := v_\zeta^\epsilon - v_{\mathbf{0}}^\epsilon$. Using the decomposition (23) and the bounds of (14), it is not hard to see that for all $\epsilon > 0$, all ζ in $C(S^1)$, and all $t \geq 0$

$$\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)} \leq e^{-\alpha t} \|\zeta\|_{C(S^1)} + \bar{f} \int_0^t e^{-\alpha(t-s)} \|\Delta_\zeta^\epsilon[s]\|_{C(S^1)} ds + \epsilon \|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}$$

where $\psi_{\zeta,\Delta}^\epsilon := \psi_\zeta^\epsilon - \psi_{\mathbf{0}}^\epsilon$. By Gronwall's inequality, then

$$\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)} \leq e^{-(\alpha - \bar{f})t} \|\zeta\|_{C(S^1)} + \epsilon \bar{f} \int_0^t e^{-(\alpha - \bar{f})(t-s)} \|\psi_{\zeta,\Delta}^\epsilon[s]\|_{C(S^1)} ds + \epsilon \|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)} \quad (40)$$

for all $\epsilon > 0$, ζ in $C(S^1)$, and $t > 0$.

The next step would naturally be to take the expectations of both sides of (40). We then would hope to find that $E\|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}$ is bounded in some sense by $E\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}$; clearly as Δ_ζ^ϵ becomes small we would expect that $\psi_{\zeta,\Delta}^\epsilon$ also be small. We then would have another Gronwall-type integral inequality which we could use to bound $E\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}$ in terms of $\|\zeta\|_{C(S^1)}$. This approach to continuing (40) is in fact not sufficient; the only relevant tool we have at our disposal is Proposition 4, and from this we cannot easily get an appropriate bound for $E\|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}$ in terms of $E\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}$. What we can do, however, is to use this approach *after* raising both sides of (40) to a large enough power. To be specific, let us take any even integer $N > 4$. Denote $\bar{\alpha} := \alpha - \bar{f}$, and let α^* be a number such that $0 < \alpha^* < \bar{\alpha}$. Hereafter we shall let K be a constant which may change from line to line, but which depends only on our choice of N and α^* . Raising both sides of (40) to the N -th power, we have that for all $\epsilon > 0$, ζ in $C(S^1)$, and all $t \geq 0$,

$$\begin{aligned} & \|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}^N \\ & \leq K e^{-\alpha^* N t} \|\zeta\|_{C(S^1)}^N + K \epsilon^N \left(\int_0^t e^{-\bar{\alpha}(t-s)} \|\psi_{\zeta,\Delta}^\epsilon[s]\|_{C(S^1)} ds \right)^N + K \epsilon^N \|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}^N \\ & \leq K e^{-\alpha^* N t} \|\zeta\|_{C(S^1)}^N + K \epsilon^N \left(\int_0^\infty e^{-(\bar{\alpha}-\alpha^*)(N/(N-1))u} du \right)^{N-1} \cdot \left(\int_0^t e^{-\alpha^* N(t-s)} \|\psi_{\zeta,\Delta}^\epsilon[s]\|_{C(S^1)}^N ds \right) \\ & \quad + K \epsilon^N \|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}^N \end{aligned}$$

so that

$$e^{\alpha^* N t} \|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}^N \leq K \|\zeta\|_{C(S^1)}^N + K \epsilon^N \int_0^t e^{\alpha^* N s} \|\psi_{\zeta,\Delta}^\epsilon[s]\|_{C(S^1)}^N ds + K \epsilon^N e^{\alpha^* N t} \|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}^N.$$

Now we take expectations of both sides to get that

$$\begin{aligned} e^{\alpha^* N t} E \left[\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}^N \right] & \leq K \|\zeta\|_{C(S^1)}^N + K \epsilon^N \int_0^t e^{\alpha^* N s} E \left[\|\psi_{\zeta,\Delta}^\epsilon[s]\|_{C(S^1)}^N \right] ds \\ & \quad + K \epsilon^N e^{\alpha^* N t} \|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}^N. \end{aligned} \tag{41}$$

To bound $E \left[\|\psi_{\zeta,\Delta}^\epsilon[s]\|_{C(S^1)}^N \right]$ by $E \left[\|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^N \right]$ in the hopes of achieving a manageable integral inequality, we shall use the following result:

Lemma 7. *There is a K depending only on N and α^* such that for all $\epsilon > 0$, ζ in $C(S^1)$, and $t \geq 0$,*

$$e^{\alpha^* N t} E \left[\|\psi_{\zeta,\Delta}^\epsilon[t]\|_{C(S^1)}^N \right] \leq K \bar{\sigma}^N \sup_{0 \leq s \leq t} e^{\alpha^* N s} E \left[\|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^N \right].$$

The proof of this Lemma 7 involves estimates similar to the proof of Proposition 4, so we delay the proof until the end of the proof of Lemma 6.

Inserting the estimate of the lemma into (41), we get that

$$e^{\alpha^* N t} E \left[\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}^N \right] e^{-\alpha^* N t} \leq K \|\zeta\|_{C(S^1)}^N + \epsilon^N \bar{\sigma}^N K \int_0^t m_\zeta^\epsilon(s) ds + K \epsilon^N \bar{\sigma}^N m_\zeta^\epsilon(t) \tag{42}$$

where

$$m_\zeta^\epsilon(t) := \sup_{0 \leq s \leq t} e^{\alpha^* N s} E \left[\|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^N \right] \tag{43}$$

for all ϵ , ζ , and t . But then from (42) and the fact that m_ζ^ϵ is increasing, we have that when $\epsilon^N \bar{\sigma}^N K < 1$

$$m_\zeta^\epsilon(t) \leq \frac{K}{1 - \epsilon^N \bar{\sigma}^N K} \|\zeta\|_{C(S^1)}^N + \frac{\epsilon^N \bar{\sigma}^N K}{1 - \epsilon^N \bar{\sigma}^N K} \int_0^t m_\zeta^\epsilon(s) ds.$$

By a final simple argument using Gronwall's inequality we get that

$$E \left[\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)}^N \right] \leq \frac{K}{1 - \epsilon^N \bar{\sigma}^N K} \exp \left[- \left(\alpha^* - \frac{K}{1 - \epsilon^N \bar{\sigma}^N K} \epsilon^N \bar{\sigma}^N \right) Nt \right] \|\zeta\|_{C(S^1)}.$$

Of course this gives us the conclusion of Lemma 6 when $\kappa = 0$.

Consider now the case $0 \leq \kappa < 1/2$. Let us choose a κ' with $\kappa < \kappa' < 1/2$, and use the fact that for any φ in $C^{\kappa'}(S^1)$,

$$\|\varphi\|_\kappa \leq 3 \|\varphi\|_{C(S^1)}^{1-\kappa/\kappa'} \|\varphi\|_{\kappa'}^{\kappa/\kappa'}. \quad (44)$$

We have by Proposition 5, that for any $d > 0$, $\epsilon > 0$, and ζ in $C^\kappa(S^1)$ such that

$$d > \epsilon M K_{\kappa'}^0 + A_{\kappa'} \bar{F} + e^{-\alpha t} \|\zeta\|_{C(S^1)} (1 + \pi^{1/2-\kappa'} (Dt)^{-1/2}), \quad (45)$$

the bounds

$$P\{\|v_0^\epsilon[t]\|_{\kappa'} \geq d\} < \exp[-(d - A_{\kappa'} \bar{F})^2 / (\epsilon^2 M^2) K_{\kappa'}^1] \quad (46)$$

and

$$P\{\|v_\zeta^\epsilon[t]\|_{\kappa'} \geq d\} < \exp \left[- \left(\frac{d - A_{\kappa'} \bar{F} - e^{-\alpha t} \|\zeta\|_{C(S^1)} (1 + \pi^{1/2-\kappa'} (Dt)^{-1/2})}{\epsilon M} \right)^2 K_{\kappa'}^1 \right] \quad (47)$$

hold. Here we have used the calculation, available from (8), that

$$\|\mathcal{T}_\zeta[t]\|_{\kappa'} \leq e^{-\alpha t} (1 + \pi^{1-\kappa'} (\pi Dt)^{-1/2}) \|\zeta\|_{C(S^1)}$$

for all $t > 0$ and all ζ in $C(S^1)$. Thus for all $d > 0$, $\epsilon > 0$, $t > 0$ and ζ in $C^\kappa(S^1)$ satisfying (45), we may use (44) to get the bound

$$\begin{aligned} & P\{\|v_\zeta^\epsilon[t] - v_0^\epsilon[t]\|_\kappa \geq \delta\} \\ & \leq P\{\|v_0^\epsilon[t]\|_{\kappa'} \geq d\} + P\{\|v_\zeta^\epsilon[t]\|_{\kappa'} \geq d\} \\ & \quad + P \left\{ \|v_\zeta^\epsilon[t] - v_0^\epsilon[t]\|_{C(S^1)} \geq \left(\frac{\delta}{3(2d)^{\kappa/\kappa'}} \right)^{\kappa'/(\kappa'-\kappa)} \right\} \\ & \leq \exp[-(d - B_{\kappa'} \bar{F})^2 / (\epsilon^2 M^2) K_{\kappa'}^1] \\ & \quad + \exp \left[- \left(\frac{d - B_{\kappa'} \bar{F} - e^{-\alpha t} \|\zeta\|_{C(S^1)} (1 + \pi^{1/2-\kappa'} (Dt)^{-1/2})}{\epsilon M} \right)^2 K_{\kappa'}^1 \right] \\ & \quad + P \left\{ \|v_\zeta^\epsilon[t] - v_0^\epsilon[t]\|_{C(S^1)} \geq \left(\frac{\delta}{3(2d)^{\kappa/\kappa'}} \right)^{\kappa'/(\kappa'-\kappa)} \right\} \end{aligned}$$

which is valid for each $\delta > 0$. We then argue Lemma 6 for $0 \leq \kappa < 1/2$ by taking first t to infinity and using the result for $\kappa = 0$, and then taking d to infinity using (46) and (47). This is enough to prove the conclusion of Lemma 6 for the general case. \blacksquare

To finally complete the proof of Lemma 6 we now must prove Lemma 7.

Proof of Lemma 7. Let us fix a $0 < \kappa < 1/2 - 2/N$, an $\epsilon > 0$, a ζ in $C(S^1)$, and a $t > 0$. We shall again let K be a constant which may change from line to line but which depends only on our choice of N and κ . We bound the expectation $E \left[\|\Delta_\zeta^\epsilon[t]\|_{C(S^1)} \right]$ in two steps. If we choose any x^* in S^1 , we may write that

$$\|\psi_{\zeta, \Delta}^\epsilon[t]\|_{C(S^1)}^N \leq 2^N |\psi_{\zeta, \Delta}^\epsilon(t, x^*)|^N + 2^N \pi^{\kappa N} [\psi_{\zeta, \Delta}^\epsilon[t]]_\kappa^N \quad (48)$$

and find the expectation of each term separately. We can calculate the expectation of the first term by straightforward methods using representation (18). We can use Proposition 1 to bound the expectation of the second term. Our calculations are similar to those used in proving Proposition 4.

Consider the first term in (48). By the Burkholder-Davis-Gundy inequality ([10], Theorem 3.2.8),

$$\begin{aligned} E [|\psi_{\zeta, \Delta}^\epsilon(t, x^*)|^N] &= E \left[\left(\int_0^t \int_{S^1} G_{t-s}(x^*, y) (\sigma(y, v_\zeta^\epsilon(s, y)) - \sigma(y, v_0^\epsilon(s, y))) W(ds, dy) \right)^N \right] \\ &\leq KE \left[\left(\int_0^t \int_{S^1} G_{t-s}^2(x^*, y) \bar{\sigma}^2 \|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^2 ds dy \right)^{N/2} \right]. \end{aligned}$$

Now if we follow through the proof of Proposition 4, we can see that

$$\begin{aligned} \int_0^t \int_{S^1} G_{t-s}^2(x^*, y) \|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^2 ds dy &\leq \frac{1}{\pi} \int_0^t \sum_{k=0}^{\infty} e^{-2\lambda_k(t-s)} \|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^2 ds \\ &\leq \int_0^t \beta_1(t-s) e^{-2\alpha^*(t-s)} \|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^2 ds. \end{aligned} \quad (49)$$

where for all $u \geq 0$,

$$\beta_1(u) := \frac{1}{\pi} \sum_{k=0}^{\infty} e^{-2(\lambda_k - \alpha^*)u},$$

which is an integrable function. By Hölder's inequality we may continue, as

$$\begin{aligned} E [|\psi_{\zeta, \Delta}^\epsilon(t, x^*)|^N] &\leq K \bar{\sigma}^N \left(\int_0^\infty \beta_1(u) du \right)^{(N-2)/2} \cdot \int_0^t \beta_1(t-s) e^{-\alpha^* N(t-s)} E \left[\|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^N \right] ds \\ &\leq K \bar{\sigma}^N e^{-\alpha^* N t} m_\zeta^\epsilon(t) \left(\int_0^\infty \beta_1(u) du \right)^{N/2}. \end{aligned}$$

where m_ζ^ϵ is defined as in (43); thus

$$E [|\psi_{\zeta, \Delta}^\epsilon(t, x^*)|^N] \leq K \bar{\sigma}^N e^{-\alpha^* N t} m_\zeta^\epsilon(t) \quad (50)$$

for all $\epsilon > 0$, all ζ in $C(S^1)$, and $t \geq 0$. This takes care of the first term of (48).

To bound the expectation of the second term of (48), we may again use Proposition 1 to estimate $[\psi_{\zeta, \Delta}^\epsilon[t]]_\kappa$. Define $\Psi(x) := x^N$ for all x in \mathbb{R} and $p(u) := u^{\kappa+2/N}$ for all $u \geq 0$. Set

$$B_t := \int_{S^1} \int_{S^1} \Psi \left(\frac{\psi_{\zeta, \Delta}^\epsilon(t, x) - \psi_{\zeta, \Delta}^\epsilon(t, y)}{p(r(x, y))} \right) dx dy.$$

Then by Proposition 1, we have similarly to our calculations in the proof of Proposition 4, that

$$[\psi_{\zeta, \Delta}^\epsilon[t]]_\kappa \leq 8 \frac{\kappa + 2/N}{\kappa} B_t^{1/N}$$

so that

$$E [([\psi_{\zeta, \Delta}^\epsilon[t]]_\kappa)^N] \leq 8^N \left(\frac{\kappa + 2/N}{\kappa} \right)^N E[B_t].$$

As in the proof of Proposition 4, we now must calculate $E[B_t]$, making sure that it is finite. For any x and y in S^1 , again the Burkholder-Davis-Gundy inequality gives us that

$$\begin{aligned} E \left[\Psi \left(\frac{\psi_{\zeta, \Delta}^\epsilon(t, x) - \psi_{\zeta, \Delta}^\epsilon(t, y)}{p(r(x, y))} \right) \right] \\ \leq KE \left[\left(\int_0^t \int_{S^1} \frac{(G_{t-s}(x, z) - G_{t-s}(y, z))^2}{r(x, y)^{2\kappa+4/N}} \bar{\sigma}^2 \|\Delta_\zeta^\epsilon[s]\|_{C(S^1)}^2 dz ds \right)^{N/2} \right]. \end{aligned}$$

Proceeding as in (49), we compute

$$\begin{aligned} E \left[\Psi \left(\frac{\psi_{\zeta, \Delta}^{\epsilon}(t, x) - \psi_{\zeta, \Delta}^{\epsilon}(t, y)}{p(r(x, y))} \right) \right] \\ \leq K \bar{\sigma}^N E \left[\left(\sum_{k=1}^{\infty} e^{-2\lambda_k(t-s)} \left(\frac{2}{\pi^{1/2}} \right)^{2-2\kappa-4/N} \left(\frac{\lambda_k}{\pi D} \right)^{\kappa+2/N} \|\Delta_{\zeta}^{\epsilon}[s]\|_{C(S^1)}^2 ds \right)^{N/2} \right] \\ \leq K \bar{\sigma}^N E \left[\left(\int_0^t \beta_2(t-s) e^{-2\alpha^*(t-s)} \|\Delta_{\zeta}^{\epsilon}[s]\|_{C(S^1)}^2 ds \right)^{N/2} \right] \end{aligned}$$

where the integrable function β_2 is given by

$$\beta_2(u) := \left(\frac{2^{2-2\kappa-4/N}}{\pi D^{\kappa+2/N}} \right) \sum_{k=1}^{\infty} e^{-2(\lambda_k - \alpha^*)u} \lambda_k^{\kappa+2/N}$$

for all $u \geq 0$. Hence by Hölder's inequality,

$$E \left[\Psi \left(\frac{\psi_{\zeta, \Delta}^{\epsilon}(t, x) - \psi_{\zeta, \Delta}^{\epsilon}(t, y)}{p(r(x, y))} \right) \right] \leq K \bar{\sigma}^N e^{-\alpha^* N t} m_{\zeta}^{\epsilon}(t) \left(\int_0^{\infty} \beta_2(u) \right)^{N/2}$$

analogously to (50). Hence

$$E[B_t] \leq K \bar{\sigma}^N e^{-\alpha^* N t} m_{\zeta}^{\epsilon}(t)$$

and thus

$$E \left[[\psi_{\zeta, \Delta}^{\epsilon}[t]]_{\kappa}^N \right] \leq K \bar{\sigma}^N e^{-\alpha^* N t} m_{\zeta}^{\epsilon}(t).$$

Combining this, (48), and (50), we get the result of Lemma 7. ■

11. Conclusion.

In this paper we proved two main results concerning the long-term behavior of the Markov process defined by the stochastic PDE (1). We showed that v_{ζ}^{ϵ} has a unique stationary distribution ν^{ϵ} for $\epsilon > 0$ small enough, and we showed that this stationary distribution obeys a large deviations principle with action functional (5). If we compare our proofs with the corresponding standard proofs concerning the stationary distribution of an ordinary stochastic differential equation (see [5], Theorem 4.4.3), we find that our proofs are significantly more technically involved, this primarily being due to the fact that the state space $C^{\kappa}(S^1)$ of the Markov process v_{ζ}^{ϵ} is inherently infinite-dimensional. Nevertheless, our proof here stemmed more directly from the large deviations principle **(B.1)**-**(B.3)** for v_{ζ}^{ϵ} , than do the corresponding proofs for ordinary stochastic differential equations. It is possible that the spirit of our proofs, with only minor modifications, can be adapted to more general Markov processes.

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REFERENCES

- [1] Adler, R.J. (1981). *The Geometry of Random Fields*. Wiley, New York.
- [2] Deuschel, J.-D. and Stroock, D.W. (1989). *Large Deviations*. Vol. 137, Pure and Applied Mathematics, Springer, Academic Press, Inc.
- [3] Ethier, S. N., and Kurtz, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.

- [4] Freidlin, M.I. (1988). Random perturbations of reaction-diffusion equations: the quasi-deterministic approach. *Trans. Amer. Math. Soc.* **305**, 665–697.
- [5] Freidlin, M.I. and Wentzell, A.D. (1984). Translated by Joseph Szücs. *Random Perturbations of Dynamical Systems*. Springer, New York.
- [6] Garsia, A. (1972). Continuity properties of Gaussian processes with multidimensional time parameter. *Proc. Sixth Berkeley Symp. Math. Stat. Prob.*, **2**, 369–374, Univ. of California Press, Berkeley.
- [7] Has'minskiĭ, R. (1980). Translated by D. Louvish. *Stochastic Stability of Differential Equations*. Sijthoff and Noordhoff, Rockville.
- [8] Ioffe, D. On some applicable versions of abstract large deviations. To appear in *Annals of Probability*.
- [9] Kallianpur, G. (1980). *Stochastic Filtering Theory*. Springer, New York.
- [10] Karatzas, J. and Shreve, S. (1987). *Brownian Motion and Stochastic Calculus*. Springer, New York.
- [11] Marcus, R. (1974). Parabolic Itô Equations. *Trans. Amer. Math. Soc.*, **198**, 177–190.
- [12] Mueller, C. Coupling and invariant measures for the heat equation with noise. Preprint.
- [13] Mueller, C. Long-time existence for the heat equation with a noise term. Preprint.
- [14] Royden, H.L. (1968). *Real Analysis*. MacMillan, New York.
- [15] Saperstone, Stephen H. (1981). *Semidynamical Systems in Infinite Dimensional Spaces*. Springer Verlag, New York.
- [16] Sowers, R.B. (1992). Large deviations for a reaction-diffusion equation with non-Gaussian perturbations. To appear in *Annals of Probability*.
- [17] Ventsel', A.D. and Freidlin, M.I. (1970). On small random perturbations of dynamical systems. *Russian Math Surveys*, **25**, No. 1, 1–55.
- [18] Walsh, J.B. (1984). An introduction to stochastic partial differential equations. *École d'Été de Probabilités de Saint-Flour, XIV, Lect. Notes Math. 1180*. Springer, New York.
- [19] Zabczyk, J. (1988). On large deviations for stochastic evolution equations. *Proceedings of the 6th IFIP Working Conference on Stochastic Systems and Optimization*, Warsaw.