INTERMITTENCY-TYPE ESTIMATES FOR SOME NONDEGENERATE SPDE'S

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Abstract. In this paper we prove some intermittency-type estimates for the stochastic partial differential equation $du = \mathcal{L}u dt + \mathcal{M}_l u \circ dW^l_t$, where $\mathcal{L}$ is a strongly elliptic second order partial differential operator and the $\mathcal{M}_l$'s are first-order partial differential operators. Here the W^l's are standard Wiener processes and $\circ$ denotes Stratonovich integration. We assume for simplicity that $u(0,\cdot) \equiv 1$. Our interest here is the behavior of $\mathbb{E}[|u(t,x)|^p]$ for large time and large p—more specifically, our interest is the growth of $(p^2t)^{-1}\log\mathbb{E}[|u(t,x)|^p]$ as t, then p, become large.

0. Introduction

In this paper we prove some intermittency-type estimates for the the stochastic PDE (SPDE)

$$\begin{aligned} du &= \mathcal{L}u dt + \sum_{l=1}^n \mathcal{M}_l u \circ dW^l_t \\ u(0,\cdot) &\equiv 1 \end{aligned} \tag{0.1}$$

where $\mathcal{L}$ is a strongly elliptic second-order partial differential operator and the $\mathcal{M}_l$'s are first-order partial differential operators. The W^l's are standard Wiener processes. For simplicity we let the spatial variable take values in an d-dimensional manifold without boundary. Our interest here is the behavior of $\mathbb{E}[|u(t,x)|^p]$ for large time and large p. More specifically, we are interested in the behavior of the moment Lyapunov exponent

$$\lim_{t\to\infty} \frac{1}{t}\log\mathbb{E}[|u(t,x)|^p] \tag{0.2}$$

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for large p (of course we do not a priori know that this limit exists, nor that it does not depend on x). We are interested in finding when this limit grows quadratically in p, i.e., when

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}[|u(t,x)|^p] = O(|p|^2) \tag{0.3}$$

as $|p|$ becomes large. This has implications about the "patchiness" or "intermittency" of the random field u [15].

Our technique in this paper is to convert the study of (0.2) into some calculations like [3] (see also [8], [12], and [25]). We get some conditions on $\mathcal{L}$ and the $\mathcal{M}_l$'s under which (0.3) is true. We must admit that our aims here are very modest. Many aspects of the calculations of [3] rely on the simple structure of finite-dimensional Euclidean spaces. Here, we will eventually set up a similar problem in an infinite-dimensional space. Hence, to try to extend in full the calculations of [3] would lead to a host of technical calculations. Either we could attempt to derive a complete, but long, analogue of [3], or use simpler, but less complete, estimates, which more quickly lead to some explicit, computable, and geometric conditions under which (0.3) does or does not occur. We chose the shorter and simpler approach. A noticable penalty for adopting these simpler estimates is that we do not get conditions which are both necessary *and* sufficient for (0.3) to occur. Nor, in fact, do we attempt to show that the limit (0.2) exists. We hope, however, that the connection between our problem and that of [3] may in itself be of use.

There recently has been much interest in intermittency in SPDE's. It describes some extreme irregularity in certain magnetohydrodynamic phenomena. One should also note that since (0.1) is the SPDE for the unnormalized conditional density for a certain filtering problem, this problem is connected with other questions about the asymptotics of filters [18], [19], [21 and the references therein]. Intermittency has been found in discrete versions of this problem (see [1], [7], and [15]) and in a strictly parabolic version of the SPDE (0.1) (see [2]). The significance of our results is that we consider here a fully superparabolic SPDE with continuous time and space variables. This entails a conceptually more difficult problem than in [2], since the superparabolicity condition requires that the characteristics of (0.1) contain an auxiliary noise; in the language of [2], we are considering an incompressible and conducting fluid with *finite* magnetic Reynolds number. This necessitates a detailed study of how this additional noise is smoothed out in representing the solution of (0.1), and in seeing how this smoothing operation is affected by higher moments. (see also [14] for different aspects of this sort of difficulty).

This paper is divided into four sections. The next section contains a review of some standard results about (0.1). It also contains some slight reformulations of our problem; these will somewhat simplify our calculations. Section 2 contains a review of some of the calculations of [3]; these will form the model for our analysis of (0.2). In Section 3, we introduce a measure-valued Markov process which will allow us to rewrite our analysis of (0.2) in a form similar to the studies of [3]. Finally, Section 4 contains the final results about (0.2). We give some necessary and some sufficient conditions under which (0.3) is true.

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1. Some Well-Known Results.

We now recall some salient parts of the theory of the SPDE (0.1). Specifically, in this section we shall give more explicit definition to the SPDE (0.1) and secondly we shall state a standard Feynmann-Kac representation formula which will be fundamental to our arguments.

We assume in this paper that M is a connected, compact, C^∞, and d-dimensional manifold with a Riemannian metric tensor $(\cdot,\cdot)$. We let ∇ and Δ respectively be the gradient and Laplace-Beltrami operators associated with $(\cdot,\cdot)$. We then let $\{\sigma_l : l=0,1,\dots,n\}$ be a collection of C^∞ vector fields on M and $\{h_l : l=0,1,\dots,n\}$ be a collection of C^∞ real-valued functions on M. As a convention, we shall always consider tangent vectors on M to be *derivations*, i.e., local linear operators which obey Leibniz' rule [6, Chap. 4.1]. We then assume that the second-order operator $\mathcal{L}$ of (0.1) has the form

$$\mathcal{L}\varphi \overset{\triangle}{=} \frac{1}{2}\Delta\varphi + (\sigma_0, \nabla\varphi) + h_0\varphi \qquad \varphi \in C^\infty(M)$$

and we assume that the first-order operators $\{\mathcal{M}_l : l=1,2,\dots,n\}$ are of the form

$$\mathcal{M}_l\varphi \overset{\triangle}{=} (\sigma_l, \nabla\varphi) + h_l\varphi. \qquad \varphi \in C^\infty(M),\ l=1,2,\dots,n$$

To complete the background of (0.1), let $(\Omega, \mathcal{F}, \mathbb{P})$ be an underlying probability triple on which a n-dimensional standard Wiener process $\{W^l : l=1,2,\dots,n\}$ is defined. Then we know that the solution of (0.1) exists and is unique for all time (see [24, Chap 3]).

Next we wish to write the solution u of (0.1) by means of a Feynman-Kac function-space integral. This will in fact be crucial to our analysis. In order to make our future arguments easier, however, let us first change our problem in several simple ways.

To begin, let us more clearly state the goal of this paper. We wish to find the behavior of

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \tag{1.1}$$

and

$$\underline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right] \tag{1.2}$$

for $|p|$ large, for any specific x in M. We simply note in passing that the initial condition $u(0,\cdot) \equiv 1$ is sufficient to imply that $u(t,x)$ is $\mathbb{P}$-a.s. nonnegative for each $t \geq 0$ and each x in M, so the absolute value signs in (1.1) and (1.2) are in fact extraneous. Instead of directly studying (1.1) and (1.2), we shall find it more convenient in this paper to consider a slightly more general problem. Let $\mathcal{P}(M)$ denote the collection of probability measures on $(M, \mathcal{B}(M))$, where $\mathcal{B}(M)$ is the Borel σ-field of subsets of M. Instead of (1.1) and (1.2), we shall be interested in

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}\left[\left|\int_M u(t,x)\mu(dx)\right|^p\right] \tag{1.3}$$

and

$$\varlimsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}\left[\left|\int_M u(t,x)\mu(dx)\right|^p\right] \tag{1.4}$$

for $|p|$ large, for any fixed μ in $\mathcal{P}(M)$. Clearly by taking μ to be a point mass at x, for any chosen x in M, we can study (1.1) and (1.2). Note that again, the absolute value signs are extraneous due to the nonnegativity of our initial condition.

Our second transformation of the problem is a bit more subtle. In the same way that deterministic parabolic PDE's admit a Feynman-Kac representation in terms of stochastic characteristics, the solution u of (0.1) admits a representation in terms of stochastic characteristics (see [22], [24, Theorem 5.1.1]). These characteristics, however, evolve *backward*; for any $t \geq 0$ and any x in M, we must solve a *backward* Ito equation. While this is not in itself very difficult, it requires us to study the *forward* evolution of u by studying the *backward* evolution of the characteristics. As most of the machinery in our arguments relating to the characteristics is traditionally written in terms of forward evolution, it will make our efforts easier to study the *forward*-evolving characteristics of a corresponding *backward* SPDE. Let us consider then the *terminal*-value problem

$$\begin{aligned} -dv^T &= \mathcal{L}v^T dt + \sum_{l=1}^n \mathcal{M}_l v^T \overleftarrow{\circ} dW_t^l \\ v^T(T,\cdot) &\equiv 1 \end{aligned} \tag{1.5}$$

where $\overleftarrow{\circ}$ denotes backward Stratonovich integration (see [24, Definitions 1.4.13 and 1.4.14]). The theory of (1.5) is exactly analogous to that of (0.1), and we refer the reader to [24, Chap 3] for a complete treatment. Since the solution to (1.5) is unique, the statistics of $v^T(0,\cdot)$ are the same as the statistics of $u(T,\cdot)$ for any $T \geq 0$; more specifically,

$$\mathbb{E}\left[\left|\int_M u(T,x)\mu(dx)\right|^p\right] = \mathbb{E}\left[\left|\int_M v^T(0,x)\mu(dx)\right|^p\right] \tag{1.6}$$

for any μ in $\mathcal{P}(M)$, any $T \geq 0$, and any p.

Now we recall the well-known Feynman-Kac representation formula for v^T. Expand $(\Omega, \mathcal{F}, \mathbb{P})$ as necessary to support an d-dimensional standard Wiener process $\{\tilde{W}^l : l = 1, 2, \dots, d\}$. Let $\{\tilde{\sigma}_l : l = 1, 2 \dots d\}$ be C^∞ vector fields on M such that in Hörmander form

$$\frac{1}{2}\Delta + \sigma_0 = \frac{1}{2}\sum_{l=1}^d \tilde{\sigma}_l^2 + \tilde{\sigma}_0.$$

Then we let $\{\vartheta_t : t \geq 0\}$ be the stochastic flow of diffeomorphisms of M defined by

$$\begin{aligned} d\vartheta_t(x) &= \sum_{l=1}^d \tilde{\sigma}_l(\vartheta_t(x)) \circ d\tilde{W}_t^l + \tilde{\sigma}_0(\vartheta_t(x))dt \\ &\quad + \sum_{l=1}^n \sigma_l(\vartheta_t(x)) \circ dW_t^l \qquad x \in M \\ \vartheta_0(x) &= x. \end{aligned} \tag{1.7}$$

(see [10], [11], and [20] for a treatment of stochastic differential equations on manifolds). Next define the sub sigma-fields of $\mathcal{F}$

$$\mathcal{W}_s^t \triangleq \sigma\{W_r^l - W_s^l : s \le r \le t,\ l = 1, 2, \dots, n\} \tag{1.8}$$

for each $0 \le s \le t$, and set $\mathcal{W}_t^\infty \triangleq \vee_{T \ge t} \mathcal{W}_t^T$ for each $t \ge 0$. Then (see [22] and [24, Theorem 5.1.1]) $\mathbb{P}$-a.s

$$v^T(0,x) = \mathbb{E}\left[\exp\left[\int_0^T h_0(\vartheta_s(x))ds + \sum_{l=1}^n \int_0^T h_l(\vartheta_s(x)) \circ dW_s^l\right] \middle| \mathcal{W}_0^T\right] \tag{1.9}$$

for each x in M. For future ease of notation, let us define, analogously to (1.8),

$$\tilde{\mathcal{W}}_s^t \triangleq \sigma\{\tilde{W}_r^l - \tilde{W}_s^l : s \le r \le t,\ l = 1, 2, \dots, d\}$$

for all $0 \le s \le t$ with $\tilde{\mathcal{W}}_t^\infty \triangleq \vee_{T \ge t} \tilde{\mathcal{W}}_t^T$ for each $t \ge 0$. For any μ in $\mathcal{P}(M)$, we may further expand $(\Omega, \mathcal{F}, \mathbb{P})$ to support an M-valued random variable having law μ and which is independent of $\mathcal{W}_0^\infty \vee \tilde{\mathcal{W}}_0^\infty$. Then it is easy to see, from (1.9), that $\mathbb{P}$-a.s.

$$\begin{aligned}
&\int_M v^T(0,x)\mu(dx) \\
&\quad = \mathbb{E}\left[\exp\left[\int_0^T h_0(\vartheta_s(x))ds + \sum_{l=1}^n \int_0^T h_l(\vartheta_s(x)) \circ dW_s^l\right] \middle| \mathcal{W}_0^T\right] \mu(dx) \\
&\quad = \mathbb{E}\left[\int_M \exp\left[\int_0^T h_0(\vartheta_s(x))ds + \sum_{l=1}^n \int_0^T h_l(\vartheta_s(x)) \circ dW_s^l\right] \mu(dx) \middle| \mathcal{W}_0^T\right] \\
&\quad = \mathbb{E}\left[\exp\left[\int_0^T h_0(\vartheta_s(\xi))ds + \sum_{l=1}^n \int_0^T h_l(\vartheta_s(\xi)) \circ dW_s^l\right] \middle| \mathcal{W}_0^T\right].
\end{aligned} \tag{1.10}$$

Before closing this preliminary section, we define a certain collection of test functions on $\mathcal{P}(M)$. We say that a *smooth* function Φ on $\mathcal{P}(M)$ is one which may be represented as

$$\Phi(\pi) = \varphi(\langle\psi_1, \pi\rangle, \langle\psi_2, \pi\rangle, \dots, \langle\psi_m, \pi\rangle) \qquad \pi \in \mathcal{P}(M) \tag{1.11}$$

for some m, where where $\{\psi_i : i = 1, 2, \dots, m\}$ are some functions in $C^\infty(M)$ and φ is some bounded and twice-differentiable function from $\mathbb{R}^m$ to $\mathbb{R}$ whose first and second derivatives are bounded. Here we have used the standard notation that if ψ is in $C(M)$ and π is in $\mathcal{P}(M)$, then

$$\langle\psi, \pi\rangle \triangleq \int_M \psi(x)\pi(dx).$$

Let $\mathfrak{S}$ denote the collection of such smooth functions. By Stone-Weierstrass, (see [23, Sec. 9.7]), $\mathfrak{S}$ is dense in $C(\mathcal{P}(M))$.

2. Some Exemplary Finite-Dimensional Calculations

We now briefly review the results of [3], which will serve as a guide to our studies of (1.3) and (1.4). The calculations of [3] are directly relevant when we replace $\mathcal{L}$ by a first-order operator (this is done in [2]). We shall also see that the calculations of [3], when applied to a certain auxiliary process, give us some insight into the behavior of (1.3) and (1.4).

Let $\underline{M}$ be a compact, connected, $\underline{d}$-dimensional manifold. Let $\{\underline{\sigma}_0, \underline{\sigma}_1, \dots, \underline{\sigma}_n\}$ be C^∞ vector fields on $\underline{M}$ and let $\{\underline{h}_0, \underline{h}_1, \dots, \underline{h}_n\}$ be in $C^\infty(\underline{M})$. Let $\{\underline{\vartheta}_t : t \ge 0\}$ be the stochastic flow of diffeomorphisms of $\underline{M}$ which are defined by

$$\begin{aligned} d\underline{\vartheta}_t(\underline{x}) &= \sum_{l=1}^n \underline{\sigma}_l(\underline{\vartheta}_t(\underline{x})) \circ dW^l_t + \underline{\sigma}_0(\underline{\vartheta}_t(\underline{x}))dt \qquad \underline{x} \in \underline{M} \\ \underline{\vartheta}_0(\underline{x}) &= \underline{x}. \end{aligned} \tag{2.1}$$

We can then study the asyptotics of the quantity

$$\underline{I}(t,p;\underline{x}) \triangleq \mathbb{E}\left[\exp\left[p\int_0^t \underline{h}_0(\underline{\vartheta}_s(\underline{x}))ds + p\sum_{l=1}^n \int_0^t \underline{h}_l(\underline{\vartheta}_s(\underline{x})) \circ dW^l_s\right]\right] \tag{2.2}$$

for $t > 0$, $p \in \mathbb{R}$, and $\underline{x} \in \underline{M}$. This is somewhat related to our studies of (0.1) in that the solution of the first-order (Hörmander-form) SPDE

$$\begin{aligned} d\underline{u} &= (-\underline{\sigma}_0\underline{u} + \underline{h}_0\underline{u})\,dt + \sum_{l=1}^n (-\underline{\sigma}_l\underline{u} + \underline{h}_l\underline{u}) \circ dW_t \\ \underline{u}(0,\cdot) &\equiv 1 \end{aligned}$$

satisfies

$$\underline{u}(t,\underline{\vartheta}_t(\underline{x})) = \exp\left[\int_0^t \underline{h}_0(\underline{\vartheta}_s(\underline{x}))ds + \sum_{l=1}^n \int_0^t \underline{h}_l(\underline{\vartheta}_s(\underline{x})) \circ dW^l_s\right] \tag{2.3}$$

for all $t \ge 0$ and $\underline{x} \in \underline{M}$; thus

$$\underline{I}(t,p;\underline{x}) = \mathbb{E}\left[|\underline{u}(t,\underline{\vartheta}_t(\underline{x}))|^p\right]. \qquad t \ge 0,\ p \in \mathbb{R},\ \underline{x} \in \underline{M}$$

This is explored in more detail in [2]. The philosophical difference between (1.7) and (2.1) is that the manifold $\underline{M}$ on which $\{\underline{\vartheta}_t : t \ge 0\}$ is defined is different than M; in particular, it need not be Riemannian and it may be "larger" than M.

The results of [3] state that under some nondegeneracy conditions,

$$\lim_{|p|\to\infty} \lim_{t\to\infty} \frac{1}{p^2 t} \log \underline{I}(t,p;\underline{x}) = \frac{1}{2} \inf_{\phi \in C^\infty(\underline{M})} \sup_{\underline{x}\in\underline{M}} \sum_{l=1}^n \left(\underline{\sigma}_l\phi - \underline{h}_l\right)^2(\underline{x}). \tag{2.4}$$

These nondegeneracy conditions are essentially two-fold:

$$\begin{aligned} \text{Lie}\{\underline{\sigma}_0, \underline{\sigma}_1, \dots, \underline{\sigma}_n\}(\underline{x}) &= T_{\underline{x}}\underline{M} \\ \underline{\sigma}_0(x) &\in \text{Span}\{\underline{\sigma}_0(\underline{x}), \underline{\sigma}_1(\underline{x}), \dots, \underline{\sigma}_n(\underline{x})\}. \end{aligned} \qquad \underline{x} \in \underline{M} \tag{2.5}$$

Unfortunately, neither of these conditions will be reasonable in our study of (1.3)–(1.4); it will thus be useful to restate some of the calculations of [3] so that we will be able to derive some weaker results than (2.4).

The ideas of [3] can be understood as follows. Fix $\phi \in C^\infty(\underline{M})$. Then

$$\begin{aligned} \underline{I}(t,p;\underline{x}) = \mathbb{E}\,[&\exp\left[p\left\{\phi(\underline{\vartheta}_t(\underline{x})) - \phi(\underline{x})\right\}\right] \\ &\times \exp\left[p\left\{\int_0^t \underline{h}_0(\underline{\vartheta}_s(\underline{x}))ds + \sum_{l=1}^n \int_0^t \underline{h}_l(\underline{\vartheta}_s(\underline{x})) \circ dW_s^l \right.\right. \\ &\left.\left. - (\phi(\underline{\vartheta}_t(\underline{x})) - \phi(\underline{x}))\right\}\right]]\,. \end{aligned} \tag{2.6}$$

By Ito's rule,

$$\begin{aligned} &\int_0^t \underline{h}_0(\underline{\vartheta}_s(\underline{x}))ds + \sum_{l=1}^n \int_0^t \underline{h}_l(\underline{\vartheta}_s(\underline{x})) \circ dW_s^l - (\phi(\underline{\vartheta}_t(\underline{x})) - \phi(\underline{x})) \\ &\qquad = \int_0^t (\underline{h}_0 - \underline{\sigma}_0\phi)\,(\underline{\vartheta}_s(\underline{x}))ds + \sum_{l=1}^n \int_0^t (\underline{h}_l - \underline{\sigma}_l\phi)\,(\underline{\vartheta}_s(\underline{x})) \circ dW_s^l. \end{aligned}$$

Inserting this into the exponent in (2.6), we can form a Girsanov exponent and write that

$$\begin{aligned} \underline{I}(t,p;\underline{x}) = \mathbb{E}\Big[&\exp\left[p\left\{\phi(\underline{\vartheta}_t^{p,\phi}(\underline{x})) - \phi(\underline{x})\right\}\right] \\ &\times \exp\left[\frac{p^2}{2}\int_0^t \sum_{l=1}^n (\underline{h}_l - \underline{\sigma}_l\phi)^2\,(\underline{\vartheta}_s^{p,\phi}(\underline{x}))ds\right] \\ &\times \exp\left[p\int_0^t (\underline{h}_0 - \underline{\sigma}_0\phi)\,(\underline{\vartheta}_s^{p,\phi}(\underline{x}))ds \right. \\ &\left. + \frac{p}{2}\int_0^t \sum_{l=1}^n \left(\underline{\sigma}_l\underline{h}_l - \underline{\sigma}_l^2\phi\right)(\underline{\vartheta}_s^{p,\phi}(\underline{x}))ds\right]\Big] \end{aligned} \tag{2.7}$$

where $\{\underline{\vartheta}_t^{p,\phi};\, t \ge 0\}$ is the stochastic flow of diffeomorphisms of $\underline{M}$ defined as

$$\begin{aligned} d\underline{\vartheta}_t^{p,\phi}(\underline{x}) &= \sum_{l=1}^n \underline{\sigma}_l(\underline{\vartheta}_t^{p,\phi}(\underline{x})) \circ dW_t^l + \underline{\sigma}_0(\underline{\vartheta}_t^{p,\phi}(\underline{x}))dt \\ &\quad + p\sum_{l=1}^n (\underline{h}_l - \underline{\sigma}_l\phi)\,(\underline{\vartheta}_t^{p,\phi}(\underline{x}))\underline{\sigma}_l(\underline{\vartheta}_t^{p,\phi}(\underline{x}))dt \qquad \underline{x} \in \underline{M} \\ \underline{\vartheta}_0^{p,\phi}(\underline{x}) &= \underline{x}. \end{aligned} \tag{2.8}$$

Next we can rescale, writing $s = r/p$ in the integrands in (2.7). We get that

$$\begin{aligned}
\underline{I}(t,p;\underline{x}) = \mathbb{E}\Big[&\exp\Big[p\Big\{\phi(\underline{\tilde{\vartheta}}^{p,\phi}_{tp}(\underline{x})) - \phi(\underline{x})\Big\}\Big] \\
&\times \exp\left[\frac{p}{2}\int_0^{tp}\sum_{l=1}^{n}(\underline{h}_l - \underline{\sigma}_l\phi)^2\,(\underline{\tilde{\vartheta}}^{p,\phi}_r(\underline{x}))dr\right] \\
&\times \exp\Bigg[\int_0^{tp}(\underline{h}_0 - \underline{\sigma}_0\phi)\,(\underline{\tilde{\vartheta}}^{p,\phi}_r(\underline{x}))dr \\
&\qquad + \frac{1}{2}\int_0^{tp}\sum_{l=1}^{n}\left(\underline{\sigma}_l\underline{h}_l - \underline{\sigma}_l^2\phi\right)(\underline{\tilde{\vartheta}}^{p,\phi}_r(\underline{x}))dr\Bigg]\Bigg]
\end{aligned} \tag{2.9}$$

where now $\{\underline{\tilde{\vartheta}}^{p,\phi}_t;\ t \geq 0\}$ is the stochastic flow of diffeomorphisms of $\underline{M}$ defined as

$$\begin{aligned}
d\underline{\tilde{\vartheta}}^{p,\phi}_t(\underline{x}) &= \frac{1}{\sqrt{p}}\sum_{l=1}^{n}\underline{\sigma}_l(\underline{\tilde{\vartheta}}^{p,\phi}_t(\underline{x}))\circ dW^l_t + \frac{1}{p}\underline{\sigma}_0(\underline{\tilde{\vartheta}}^{p,\phi}_t(\underline{x}))dt \\
&\quad + \sum_{l=1}^{n}(\underline{h}_l - \underline{\sigma}_l\phi)\,(\underline{\tilde{\vartheta}}^{p,\phi}_t(\underline{x}))\underline{\sigma}_l(\underline{\tilde{\vartheta}}^{p,\phi}_t(\underline{x}))dt \qquad \underline{x} \in \underline{M} \\
\underline{\tilde{\vartheta}}^{p,\phi}_0(\underline{x}) &= \underline{x}.
\end{aligned} \tag{2.10}$$

From (2.9), it is very easy to see that

$$\overline{\lim_{|p|\to\infty}}\ \overline{\lim_{t\to\infty}}\ \frac{1}{p^2 t}\log\underline{I}(t,p;\underline{x}) \leq \frac{1}{2}\inf_{\phi\in C^\infty(\underline{M})}\sup_{\underline{x}\in\underline{M}}\sum_{l=1}^{n}\left(\underline{\sigma}_l\phi - \underline{h}_l\right)^2(\underline{x}).$$

The other direction is more difficult. Here is where the nondegeneracy requirements are needed.

3. A Measure-Valued Markov Process.

In this section we will transform the representation (1.10) into something like the right side of (2.3). This will allow us to use calculations like those outlined in Section 2. As we shall see, however, the cost of this is that in place of $\underline{\vartheta}$ of (2.1), we will need to consider a stochastic process which is *measure-valued*; more precisely, we will replace $\underline{M}$ of Section 2 by $\mathcal{P}(M)$.

The basic difference between (1.10) and the right side of (2.3) is the *conditional expectation.* Let us first *pass the conditioning to the exponent.* Define the $C^\infty(M;\mathbb{R})$-valued process $\{q_t : t \ge 0\}$ as

$$q_t(x) \triangleq \exp\left[\int_0^t h_0(\vartheta_s(x))ds + \sum_{l=1}^n \int_0^t h_l(\vartheta_s(x)) \circ dW_s^l\right]. \qquad t \ge 0,\ x \in M$$

It is easy to see that $\{q_t : t \ge 0\}$ is indeed $\mathbb{P}$-a.s. in $C^\infty(M;\mathbb{R})$ for all $t \ge 0$; one may use the techniques of [10], [11], and [20] to construct a stochastic flow of diffeomorphisms of $M \times \mathbb{R}$ which coincides with the mapping $(x,y) \mapsto (\vartheta_t(x), \log q_t(x) + y)$ for each $t \ge 0$. Let us now fix a $\mu \in \mathcal{P}(M)$ and assume, as before, that ξ is an M-valued random variable which is independent of $\mathcal{W}_0^\infty \vee \tilde{\mathcal{W}}_0^\infty$ and which has law μ. Then we may rewrite (1.10) as

$$\langle v^T(0,\cdot), \mu\rangle = \mathbb{E}[q_T(\xi)|\mathcal{W}_0^T]. \qquad T > 0$$

Writing down the stochastic differential equation for $\{q_t(\xi) : t \ge 0\}$ and then taking the conditional expectation, we find that

$$\begin{aligned}\mathbb{E}[q_t(\xi)|\mathcal{W}_0^t] &= 1 + \int_0^t \mathbb{E}[h_0(\vartheta_s(\xi))q_s(\xi)|\mathcal{W}_0^s]ds \\ &\quad + \sum_{l=1}^n \int_0^t \mathbb{E}[h_l(\vartheta_s(\xi))q_s(\xi)|\mathcal{W}_0^s] \circ dW_s^l \\ &= 1 + \int_0^t \frac{\mathbb{E}[h_0(\vartheta_s(\xi))q_s(\xi)|\mathcal{W}_0^s]}{\mathbb{E}[q_s(\xi)|\mathcal{W}_0^s]}\mathbb{E}[q_s(\xi)|\mathcal{W}_0^s]ds \qquad t \ge 0 \\ &\quad + \sum_{l=1}^n \int_0^t \frac{\mathbb{E}[h_l(\vartheta_s(\xi))q_s(\xi)|\mathcal{W}_0^s]}{\mathbb{E}[q_s(\xi)|\mathcal{W}_0^s]}\mathbb{E}[q_s(\xi)|\mathcal{W}_0^s] \circ dW_s^l.\end{aligned} \tag{3.1}$$

The proof of (3.1) requires, of course, an intermediate representation in Ito form, and also requires some calculations similar to those used in the proof of the Zakai equation of nonlinear filtering. For each $t \ge 0$ and $\mu \in \mathcal{P}(M)$, define a random element $\pi_t(\mu)$ of $\mathcal{P}(M)$ by

$$\pi_t(\mu)(A) := \frac{\mathbb{E}\left[\int_M \chi_A(\vartheta_t(x))q_t(x)\mu(dx)\middle|\mathcal{W}_0^t\right]}{\mathbb{E}\left[\int_M q_t(x)\mu(dx)\middle|\mathcal{W}_0^t\right]} \qquad A \in \mathcal{B}(M),\ t \ge 0$$

(One may assume that our underlying space of events is Polish since everything can be defined in terms of Wiener processes and M-valued random variables. Thus we can indeed assume that $\pi_t(\mu) \in \mathcal{P}(M)$ for every $t \geq 0$ and $\mu \in \mathcal{P}(M)$—see [26, Thm. 5.1.15]. We will have more to say about the technicalities of $\{\pi_t(\mu) : t \geq 0,\ \mu \in \mathcal{P}(M)\}$ in a moment). Then (3.1) can easily be rewritten as

$$\begin{aligned}\mathbb{E}[q_t(\xi)|\mathcal{W}_0^t] = 1 &+ \int_0^t \langle h_0, \pi_s(\mu)\rangle\, \mathbb{E}[q_s(\xi)|\mathcal{W}_0^s]ds \\ &+ \sum_{l=1}^n \int_0^t \langle h_0, \pi_s(\mu)\rangle\, \mathbb{E}[q_s(\xi)|\mathcal{W}_0^s] \circ dW_s^l.\end{aligned} \qquad t \geq 0$$

This directly leads to

Proposition 3.1. *For any $\mu \in \mathcal{P}(M)$,*

$$\left\langle v^t(0,\cdot), \mu\right\rangle = \exp\left[\int_0^t \langle h_0, \pi_s(\mu)\rangle\, ds + \sum_{l=1}^n \int_0^t \langle h_0, \pi_s(\mu)\rangle \circ dW_s^l\right]. \qquad t \geq 0$$

Using (1.6) to return to our original problem, this implies that

Corollary 3.2. *For any $\mu \in \mathcal{P}(M)$ and any $p \in \mathbb{R}$,*

$$\mathbb{E}[|\langle u(t,\cdot), \mu\rangle|^p] = \mathbb{E}\left[\exp\left[p\int_0^t \langle h_0, \pi_s(\mu)\rangle\, ds + p\sum_{l=1}^n \int_0^t \langle h_0, \pi_s(\mu)\rangle \circ dW_s^l\right]\right]. \qquad t \geq 0 \quad (3.2)$$

We will treat this as an analogue of (2.2). In place of $\{\underline{\vartheta}_t : t \geq 0,\ \underline{x} \in \underline{M}\}$, we use $\{\pi_t(\mu) : t \geq 0,\ \mu \in \mathcal{P}(M)\}$, and in place of the $\underline{h}_l$'s, we use the mappings $\{\mathcal{H}_l : l = 0, 1, \dots, n\}$ from $\mathcal{P}(M)$ to $\mathbb{R}$ which are defined as

$$\mathcal{H}_l(\pi) \triangleq \langle h_l, \pi\rangle. \qquad \pi \in \mathcal{P}(M),\ l = 0, 1, \dots, n$$

To close this section, we will give some basic properties of $\{\pi_t(\mu) : t \geq 0,\ \mu \in \mathcal{P}(M)\}$; clearly $\{\pi_t(\mu) : t \geq 0,\ \mu \in \mathcal{P}(M)\}$ will play a central role in our studies of (3.2). It is not hard to see that for any $\psi \in C^2(M)$ and any $\mu \in \mathcal{P}(M)$, $\{\langle \psi, \pi_t(\mu)\rangle : t \geq 0\}$ satisfies

$$\begin{aligned} d\langle \psi, \pi_t(\mu)\rangle &= \mathcal{L}^+(\psi, \pi_t(\mu))dt + \sum_{l=1}^n \mathcal{M}_l^+(\psi, \pi_t(\mu)) \circ dW_t^l \\ \langle \psi, \pi_0(\mu)\rangle &= \langle \psi, \mu\rangle \end{aligned} \qquad t \geq 0 \qquad (3.3)$$

where $\mathcal{L}^+$ and $\{\mathcal{M}_l^+ : l = 1, 2, \dots, n\}$ are the mappings from $C^2(M) \times \mathcal{P}(M)$ to $\mathbb{R}$ defined as

$$\begin{aligned} \mathcal{L}^+(\psi, \pi) &\triangleq \langle \mathcal{L}\psi, \pi\rangle - \langle \psi, \pi\rangle \langle \mathcal{L}\mathbf{1}, \pi\rangle \\ \mathcal{M}_l^+(\psi, \pi) &\triangleq \langle \mathcal{M}_l\psi, \pi\rangle - \langle \psi, \pi\rangle \langle \mathcal{M}_l\mathbf{1}, \pi\rangle. \qquad l = 1, 2, \dots, n \\ &\psi \in C^2(M),\ \pi \in \mathcal{P}(M) \end{aligned}$$

Here $\mathbf{1}$ is the mapping from $\mathcal{P}(M)$ to $\mathbb{R}$ defined as $\mathbf{1} : \mathcal{P}(M) \mapsto 1$. Of course $\mathcal{L}\mathbf{1}$ and the $\mathcal{M}_l\mathbf{1}$'s have a much simpler representation, namely

$$\begin{aligned} (\mathcal{L}\mathbf{1})(x) &= h_0(x) \\ (\mathcal{M}_l\mathbf{1})(x) &= h_l(x). \qquad l = 1, 2, \dots, n \end{aligned} \qquad x \in M$$

By standard results in partial Malliavin calculus [4], [5], [24, Chap. 7.2] $\mathbb{P}$-a.s. $\pi_t(\mu)$ possesses a smooth density with respect to the Riemannian volume measure $\mathfrak{V}$ on $(M, \mathcal{B}(M))$ for every $t > 0$ and $\mu \in \mathcal{P}(M)$. If we then define

$$p_t(\mu)(x) \triangleq \frac{d\pi_t(\mu)}{d\mathfrak{V}}(x), \qquad x \in M,\ t > 0,\ \mu \in \mathcal{P}(M)$$

the equations (3.3) imply that $\{p_t(\mu) : t > 0\}$ satisfies the SPDE

$$\begin{aligned} dp_t(\mu) &= \mathcal{L}^* p_t(\mu)dt - \left\{\int_{z\in M} h_0(z)p_t(\mu)(z)\mathfrak{V}(dz)\right\} p_t(\mu)dt \\ &\quad + \sum_{l=1}^n \mathcal{M}_l^* p_t(\mu) \circ dW_t^l \\ &\quad\quad - \sum_{l=1}^n \left\{\int_{z\in M} h_l(z)p_t(\mu)(z)\mathfrak{V}(dz)\right\} p_t(\mu) \circ dW_t^l \\ p_0(\mu) &= \mu. \end{aligned} \qquad t > 0 \quad (3.4)$$

Here $\mathcal{L}^*$ and the $\mathcal{M}_l^*$'s are respectively the adjoint operators of $\mathcal{L}$ and the $\mathcal{M}_l$'s with respect to the measure $\mathfrak{V}$. The reader acquainted with nonlinear filtering theory will recognize (3.3) or alternately (3.4) as the Kushner equations for a certain filtering problem [17, Sec. 11.2], [20, Sec. 6.3]. We can also show that for each $\mu \in \mathcal{P}(M)$, $\{\pi_t(\mu) : t > 0\}$ is a $\mathcal{P}(M)$-valued Markov process. For any $\Phi \in \mathfrak{S}$ which has representation (1.11), define

$$\begin{aligned} \mathbf{B}_0\Phi(\pi) &\triangleq \sum_{l=1}^m \frac{\partial\varphi}{\partial x_i}(\langle\psi_1, \pi\rangle, \langle\psi_2, \pi\rangle, \dots, \langle\psi_m, \pi\rangle)\mathcal{L}^+(\psi_l, \pi) \\ \mathbf{B}_l\Phi(\pi) &\triangleq \sum_{l=1}^m \frac{\partial\varphi}{\partial x_i}(\langle\psi_1, \pi\rangle, \langle\psi_2, \pi\rangle, \dots, \langle\psi_m, \pi\rangle)\mathcal{M}_l^+(\psi_l, \pi). \end{aligned} \qquad \pi \in \mathcal{P}(M)$$

Then we have from (3.3) that for any $\mu \in \mathcal{P}(M)$ and any $\Phi \in \mathfrak{S}$,

$$\begin{aligned} d\Phi(\pi_t(\mu)) &= (\mathbf{B}_0\Phi)(\pi_t(\mu))dt + \sum_{l=1}^n (\mathbf{B}_l\Phi)(\pi_t(\mu)) \circ dW_t^l \\ &= \left(\mathbf{B}_0\Phi + \frac{1}{2}\sum_{l=1}^n \mathbf{B}_l^2\Phi\right)(\pi_t(\mu))dt + \sum_{l=1}^n (\mathbf{B}_l\Phi)(\pi_t(\mu))dW_t^l \end{aligned} \qquad t > 0 \qquad (3.5)$$

Summing up all of this, we have

Proposition 3.3. *For each $\mu \in \mathcal{P}(M)$, $\{\pi_t(\mu) : t \geq 0\}$ is a Markov process with respect to the filtration $\{\mathcal{W}_0^t : t \geq 0\}$ and its associated semigroup of transition operators is Feller. For any $\Phi \in \mathfrak{S}$ and $\mu \in \mathcal{P}(M)$, $\{\Phi(\pi_t(\mu)) : t \geq 0\}$ evolves according to (3.5).*

Proof. The fact that for each $\mu \in \mathcal{P}(M)$, $\{\pi_t(\mu) : t \geq 0\}$ is Markovian with respect to $\{\mathcal{W}_0^t : t \geq 0\}$ and its associated semigroup is Feller comes from calculations like those of [18, Thm. 2.3]. $\square$

4. Asymptotics.

The next step is to use calculations like those of Section 3 to get, as much as possible, the analogue of (2.4). Of course, as we pointed out earlier, we probably can't get a complete analogue of (2.4) since in general it would be too restrictive to impose the counterpart of the non-degeneracy assumptions like those of (2.5).

As a first step, let's transform (3.2) in ways analogous to the transformations of (2.8) and (2.10). We give the results in the language of semigroups. Let $B(\mathcal{P}(M);\mathbb{R})$ be the Banach space of bounded and measurable mappings from $\mathcal{P}(M)$ to $\mathbb{R}$. For each $p \in \mathbb{R}$ and $\Phi \in \mathfrak{S}$, define a collection $\{T_t^{p,\Phi} : t \geq 0\}$ of mappings from $B(\mathcal{P}(M);\mathbb{R})$ to itself by

$$\left(T_t^{p,\Phi}F\right)(\mu) \stackrel{\triangle}{=} \mathbb{E}\left[F(\pi_{t/p}(\mu)) \exp\left[p\sum_{l=1}^n \int_0^{t/p} (\mathcal{H}_l - \mathbf{B}_l\Phi)(\pi_s(\mu))dW_s^l \right.\right.$$
$$\left.\left. -\frac{p^2}{2}\sum_{l=1}^n \int_0^{t/p} (\mathcal{H}_l - \mathbf{B}_l\Phi)^2 (\pi_s(\mu))ds\right]\right].$$
$$t \geq 0,\ \mu \in \mathcal{P}(M),\ F \in B(\mathcal{P}(M);\mathbb{R}) \qquad (4.1)$$

Then we have

Lemma 4.1. *For any $p \in \mathbb{R}$ and $\Phi \in \mathfrak{S}$, $\{T_t^{p,\Phi} : t \geq 0\}$ is a Feller semigroup on $B(\mathcal{P}(M);\mathbb{R})$ whose generator, when restricted to $\mathfrak{S}$, is*

$$\mathbf{A}^{p,\Phi} \stackrel{\triangle}{=} \sum_{l=1}^d (\mathcal{H}_l - \mathbf{B}_l\Phi)\mathbf{B}_l + \frac{1}{p}\mathbf{B}_0 + \frac{1}{2p}\sum_{l=1}^n \mathbf{B}_l^2.$$

Proof. That $\{T_t^{p,\Phi} : t \geq 0\}$ is a semigroup on $B(\mathcal{P}(M);\mathbb{R})$ follows from the fact that for each $\mu \in \mathcal{P}(M)$, $\{\pi_t(\mu) : t \geq 0\}$ is a Markov process with respect to the filtration $\{\mathcal{W}_0^t : t \geq 0\}$. The Feller property comes from the analogous statement in Proposition 3.3 and the fact that the mapping $\varpi \mapsto (\mathcal{H}_l - \mathbf{B}_l\Phi)(\varpi)$ from $\mathcal{P}(M)$ to $\mathbb{R}$ is continuous for each $l = 1, 2, \ldots, n$. Finally, the formula for the generator is a simple consequence of (3.5) and the definition (4.1). □

Now set $\Omega' \stackrel{\triangle}{=} C(\mathbb{R}_+;\mathcal{P}(M))$ and let $\mathcal{F}'$ be the canonical Borel sigma-field on Ω' (defined using the Prohorov topology on $\mathcal{P}(M)$ [13, Chap. 3.1 and Problem 25 in Chap 3]). Let $\{\mathbf{X}_t : t \geq 0\}$ be the collection of $\mathcal{P}(M)$-valued random variables on $(\Omega', \mathcal{F}')$ defined by

$$\mathbf{X}_t(\omega') \stackrel{\triangle}{=} \omega'(t). \qquad t \geq 0,\ \omega' \in \Omega'$$

Finally, for each $p \in \mathbb{R}$, $\Phi \in \mathfrak{S}$, and $\mu \in \mathcal{P}(M)$, let $\bar{\mathbb{P}}_\mu^{p,\Phi}$ be the unique probability measure on $(\Omega', \mathcal{F}')$ such that

$$\bar{\mathbb{P}}_\mu^{p,\Phi}\left\{\mathbf{X}_T \in A \middle| \sigma\{\mathbf{X}_s : 0 \leq s \leq t\}\right\} = \left(T_{T-t}^{p,\Phi}\chi_A\right)(\mathbf{X}_t), \qquad \mathbb{P}_\mu^{p,\Phi} - a.s.$$
$$0 \leq t \leq T,\ A \in \mathcal{B}(\mathcal{P}(M))$$

where $\mathcal{B}(\mathcal{P}(M))$ is the natural Borel sigma-field of subsets of $\mathcal{P}(M)$; let $\bar{\mathbb{E}}_\mu^{p,\Phi}$ be the associated expectation operator.

We can now write down the analogue of (2.9).

Proposition 4.2. *For each $\Phi \in \mathfrak{S}$, every $p \in \mathbb{R}$, every $\mu \in \mathcal{P}(M)$, and every $t \geq 0$,*

$$\begin{aligned}\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p] = \bar{\mathbb{E}}^{p,\Phi}_{\mu}&\left[\exp\left[p\left\{\Phi(\mathbf{X}_t)-\Phi(\mu)\right\}\right]\right.\\ &\times\exp\left[\frac{p}{2}\int_0^{tp}\sum_{l=1}^{n}\left(\mathcal{H}_l-\mathbf{B}_l\Phi\right)^2(\mathbf{X}_s)ds\right]\\ &\times\exp\left[\int_0^{tp}\left(\mathcal{H}_0-\mathbf{B}_0\Phi\right)(\mathbf{X}_s)ds\right.\\ &\left.\left.+\frac{1}{2}\int_0^{tp}\sum_{l=1}^{n}\left(\mathbf{B}_l\mathcal{H}_l-\mathbf{B}_l^2\Phi\right)(\mathbf{X}_s)ds\right]\right].\end{aligned}$$

From this, we immediately get that

Proposition 4.3. *We have that*

$$\varlimsup_{|p|\to\infty}\varlimsup_{t\to\infty}\frac{1}{p^2t}\log\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p] \leq \frac{1}{2}\inf_{\Phi\in\mathfrak{S}}\sup_{\varpi\in\mathcal{P}(M)}\sum_{l=1}^{n}\left(\mathbf{B}_l\Phi-\mathcal{H}_l\right)^2(\varpi) \qquad (4.2)$$

for all $\mu \in \mathcal{P}(M)$.

This of course immediately leads to

Corollary 4.4. *We have that*

$$\varlimsup_{|p|\to\infty}\varlimsup_{t\to\infty}\frac{1}{p^2t}\log\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p] \leq \frac{1}{2}\sup_{\varpi\in\mathcal{P}(M)}\sum_{l=1}^{n}\mathcal{H}_l^2(\varpi)$$

for all $\mu \in \mathcal{P}(M)$.

Proof. Simply take $\Phi \equiv 0$ in (4.2).

Now comes the lower bound. Here is where we must be satisfied with weaker results than the full analogue of (2.4). Of course we have

Proposition 4.5. *We have that*

$$\varliminf_{|p|\to\infty}\varliminf_{t\to\infty}\frac{1}{p^2t}\log\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p] \geq \frac{1}{2}\sup_{\Phi\in\mathfrak{S}}\inf_{\varpi\in\mathcal{P}(M)}\sum_{l=1}^{n}\left(\mathbf{B}_l\Phi-\mathcal{H}_l\right)^2(\varpi)$$

for all $\mu \in \mathcal{P}(M)$.

and

Corollary 4.6. *We have that*

$$\varliminf_{|p|\to\infty}\varliminf_{t\to\infty}\frac{1}{p^2t}\log\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p] \geq \frac{1}{2}\inf_{\varpi\in\mathcal{P}(M)}\sum_{l=1}^{n}\mathcal{H}_l^2(\varpi)$$

for all $\mu \in \mathcal{P}(M)$.

Our final trick in bounding (1.4) from below is to use Jensen's inequality to see that for any $p \in \mathbb{R}$ and $\mu \in \mathcal{P}(M)$,

$$\begin{aligned}
&\varliminf_{t\to\infty} \frac{1}{p^2 t} \log \mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p] \\
&\quad \geq \varliminf_{t\to\infty} \frac{1}{p^2 t} \bar{\mathbb{E}}^{p,\Phi}_{\mu} \left[\frac{p}{2} \int_0^{tp} \sum_{l=1}^n (\mathcal{H}_l - \mathbf{B}_l\Phi)^2 (\mathbf{X}_s) ds \right. \\
&\qquad \left. + \int_0^{tp} \left\{ (\mathcal{H}_0 - \mathbf{B}_0\Phi) + \frac{1}{2}\sum_{l=1}^n \left(\mathbf{B}_l\mathcal{H}_l - \mathbf{B}_l^2\Phi\right) \right\} (\mathbf{X}_s) ds \right] \\
&\quad = \varliminf_{t\to\infty} \frac{1}{t} \int_0^t \bar{\mathbb{E}}^{p,\Phi} \left[\frac{1}{2}\sum_{l=1}^n (\mathcal{H}_l - \mathbf{B}_l\Phi)^2 (\mathbf{X}_s) \right] ds \\
&\qquad + \frac{1}{p} \varliminf_{t\to\infty} \frac{1}{t} \int_0^t \bar{\mathbb{E}}^{p,\Phi}_{\mu} \left[\left\{ (\mathcal{H}_0 - \mathbf{B}_0\Phi) + \frac{1}{2}\sum_{l=1}^n \left(\mathbf{B}_l\mathcal{H}_l - \mathbf{B}_l^2\Phi\right) \right\} (\mathbf{X}_s) \right] ds.
\end{aligned} \tag{4.3}$$

From this, we easily get

Proposition 4.7. *We have that*

$$\begin{aligned}
&\varliminf_{|p|\to\infty} \varliminf_{t\to\infty} \frac{1}{p^2 t} \log \bar{\mathbb{E}}^{p,\Phi}_{\mu} [|\langle u(t,\cdot),\mu\rangle|^p] \\
&\qquad \geq \sup_{\Phi\in\mathfrak{S}} \varliminf_{|p|\to\infty} \varliminf_{t\to\infty} \frac{1}{t} \int_0^t \bar{\mathbb{E}}^{p,\Phi}_{\mu} \left[\frac{1}{2}\sum_{l=1}^n \left(\mathcal{H}_l^2 - \mathbf{B}_l\Phi\right)(\mathbf{X}_s) \right] ds
\end{aligned} \tag{4.4}$$

for all $\mu \in \mathcal{P}(M)$.

The goal now is to see when the right side of (4.4) is positive. To do so, let's fix $\Phi \in \mathfrak{S}$. Define $V^\Phi \in \mathfrak{S}$ as

$$V^\Phi(\varpi) \triangleq \frac{1}{2} \inf_{\varpi\in\mathcal{P}(M)} \sum_{k=1}^n (\mathcal{H}_k - \mathbf{B}_k\Phi)^2 (\varpi). \qquad \varpi \in \mathcal{P}(M)$$

We want to show that, as p and t tend to infinity, $\{\mathbf{X}_t : t \geq 0\}$ stays away from the set

$$\daleth^\Phi \triangleq \left\{\varpi \in \mathcal{P}(M) : V^\Phi(\varpi) = 0\right\}.$$

Note that

$$\begin{aligned}\mathbf{A}^{p,\Phi}V^\Phi(\mathbf{X}_t) &= \sum_{1\le k,l\le n}\{(\mathcal{H}_k-\mathbf{B}_k\Phi)(\mathcal{H}_l-\mathbf{B}_l\Phi)(\mathbf{B}_l\mathcal{H}_k-\mathbf{B}_l\mathbf{B}_k\Phi)\}(\mathbf{X}_t)dt\\ &\quad+\frac{1}{p}\sum_{k=1}^n\{(\mathcal{H}_k-\mathbf{B}_k\Phi)(\mathbf{B}_0\mathcal{H}_k-\mathbf{B}_0\mathbf{B}_k\Phi)\}(\mathbf{X}_t)dt\\ &\quad+\frac{1}{2p}\sum_{1\le k,l\le n}\left\{(\mathcal{H}_k-\mathbf{B}_k\Phi)(\mathbf{B}_l^2\mathcal{H}_k-\mathbf{B}_l^2\Phi)\right\}(\mathbf{X}_t)dt\\ &\quad+\frac{1}{2p}\sum_{1\le k,l\le n}(\mathbf{B}_l\mathcal{H}_k-\mathbf{B}_l\mathbf{B}_k\Phi)^2(\mathbf{X}_t)dt\\ &\hspace{20em} t\ge 0 \quad (4.5)\end{aligned}$$

and

$$d\langle V^\Phi(\mathbf{X})\rangle_t=\frac{1}{p}\sum_{l=1}^n(\mathbf{B}_lV^\Phi)(\mathbf{X}_t)^2dt. \qquad t\ge 0 \tag{4.6}$$

Define now

$$\lambda^\Phi \triangleq \inf\left\{\sum_{1\le k,l\le n}\xi_l(\mathbf{B}_l\mathcal{H}_k-\mathbf{B}_l\mathbf{B}_k\Phi)(\varpi)\xi_k : \varpi\in\beth^\Phi,\ \sum_{k=1}^n\xi_k^2=1\right\}. \tag{4.7}$$

If $\lambda^\Phi>0$, then the evolution of $\{V^\Phi(\mathbf{X}_t) : t\ge 0\}$ looks like a perturbation of the unstable ODE

$$\dot V=\lambda^\Phi V. \tag{4.8}$$

This turns out to be correct. We have

Proposition 4.8. *If $\lambda^\Phi>0$ for some $\Phi\in\mathfrak{S}$, then*

$$\underline{\lim}_{|p|\to\infty}\ \underline{\lim}_{t\to\infty}\frac{1}{p^2t}\log\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p]>0 \tag{4.9}$$

for all $\mu\in\mathcal{P}(M)$.

Proof. Given in the Appendix. □

A simple example of this is when $\Phi=\mathbf{0}$, where $\mathbf{0}\in\mathfrak{S}$ is the identically zero mapping; i.e., $\mathbf{0}:\mathcal{P}(M)\mapsto 0$.

Corollary 4.9. *If $\lambda^{\mathbf{0}}>0$, then*

$$\underline{\lim}_{|p|\to\infty}\ \underline{\lim}_{t\to\infty}\frac{1}{p^2t}\log\mathbb{E}[|\langle u(t,\cdot),\mu\rangle|^p]>0$$

for all $\mu\in\mathcal{P}(M)$.

Let's investigate this last result a bit more closely. Rewriting (4.7) with $\Phi=\mathbf{0}$, we have that

$$\lambda^{\mathbf{0}}=\inf\left\{\sum_{1\le k,l\le n}\xi_l(\mathbf{B}_l\mathcal{H}_k)(\varpi)\xi_k : \varpi\in\beth^{\mathbf{0}},\ \sum_{k=1}^n\xi_k^2=1\right\}. \tag{4.10}$$

Fix now $\varpi \in \beth^\Phi$ and $(\xi_1, \xi_2, \dots \xi_n)$ such that $\sum_{k=1}^n \xi_k^2 = 1$. Then

$$\begin{aligned}
\sum_{1\le k,l\le n} \xi_l (\mathbf{B}_l \mathcal{H}_k)(\varpi)\xi_k &= \sum_{1\le k,l\le n} \xi_l \left\{ \langle \mathcal{M}_l h_k, \varpi\rangle - \langle h_k, \varpi\rangle \langle h_l, \varpi\rangle \right\} \xi_k \\
&= \sum_{1\le k,l\le n} \xi_l \left\{ \langle \sigma_l h_k, \varpi\rangle + \langle h_l h_k, \varpi\rangle \right\} \xi_k \\
&= \left\langle \sum_{1\le k,l\le n} \xi_l (\sigma_l, \nabla h_k)\xi_k + \sum_{1\le k,l\le n} (\xi_l h_l h_k \xi_k), \varpi \right\rangle .
\end{aligned} \tag{4.11}$$

The second equality uses the fact that $\varpi \in \beth^\mathbf{0}$. Define

$$\bar\lambda \triangleq \inf\left\{ \sum_{1\le k,l\le n} \xi_l(\sigma_l, \nabla h_k)(x)\xi_k + \left(\sum_{k=1}^n \xi_l h_l(x)\right)^2 : x \in M, \sum_{k=1}^n \xi_k^2 = 1 \right\}.$$

Then by (4.11),

$$\lambda^\mathbf{0} \ge \bar\lambda, \tag{4.12}$$

so we have

Corollary 4.10. *If $\bar\lambda > 0$, then*

$$\varliminf_{|p|\to\infty} \varliminf_{t\to\infty} \frac{1}{p^2 t} \log \mathbb{E}[\,|\langle u(t,\cdot), \mu\rangle|^p] > 0$$

for all $\mu \in \mathcal{P}(M)$.

Note that in getting the bound (4.12), we have not fully used the fact that we are concerned only with $\varpi \in \beth^\mathbf{0}$ in (4.10). In particular, we have not used this fact in bounding the last line of (4.11) from below. We will, however, be content with Corollary 4.10, since a more detailed investigation of (4.11) would take us rather far afield.

In the interest of seeing what Corollary 4.10 means, let's consider an example.

Example 1. Let $\sigma_l = \nabla h_l$ for all $l = 1, 2, \dots, n$. Then

$$\bar\lambda = \inf\left\{ \left\| \sum_{l=1}^n \xi_l \nabla h_l(x) \right\|^2 + \left(\sum_{l=1}^n \xi_l h_l(x)\right)^2 : x \in M, \sum_{k=1}^n \xi_k^2 = 1 \right\}$$

where $\|\cdot\|$ is the Riemannian metric on M. Thus $\bar\lambda > 0$ if

$$\left\{ (\xi_1, \xi_2, \dots \xi_n) \in \mathbb{R}^n : \sum_{l=1}^n \xi_l \nabla h_l(x) = 0_x \right\} \bigcap \left\{ (\xi_1, \xi_2, \dots \xi_n) \in \mathbb{R}^n : \sum_{l=1}^n \xi_l h_l(x) = 0 \right\} = (0, 0, \dots 0) \tag{4.13}$$

for each $x \in M$, where 0_x is the zero element of $T_x M$ for each $x \in M$, and $(0, 0, \dots 0)$ is the zero element of $\mathbb{R}^n$.

An even simpler example is given by further specifying that $n = 1$;

Example 2. Let $n = 1$ and $\sigma_1 = \nabla h_1$. Then

$$\bar{\lambda} = \inf \left\{ \|\nabla h_1(x)\|^2 + |h_1(x)|^2 : x \in M \right\},$$

so $\bar{\lambda} > 0$ if $\nabla h_1(x) \neq 0_x$ whenever $h_1(x) = 0$. The condition (4.13) is simply a multidimensional generalization of this.

Recall that we embarked upon the analysis of (4.3) in order to see what happens when

$$\inf_{\varpi \in \mathcal{P}(M)} \sum_{l=1}^{n} \mathcal{H}_l^2(\varpi) = 0; \tag{4.14}$$

i.e., when Corollary 4.6 is not refined enough to give us (0.3). If (4.14) is true, then the zero sets of the h_l's must each be nonempty; i.e., $\{x \in M : h_l(x) = 0\} \neq \emptyset$ for each $l = 1, 2, \dots, n$. The above examples illustrate that we still have (4.9) if the gradient of the h_l's are nonzero in enough directions; one might somehow understand this in the sense that the flow of the σ_l's push away from the zero sets of the h_l's. We shall not attempt to make this idea rigorous here. Also, we shall leave it to the reader to devise other, perhaps more elaborate, examples of when $\lambda^{\mathbf{0}} > 0$. Many such examples are obvious; all of them share with the above examples the idea of imposing some additional structure on the σ_l's and h_l's which allow a more explicit lower bound on $\bar{\lambda}$.

Appendix

This appendix is devoted to the proof of Proposition 4.8. The expression on the right of (4.4) reflects some aspect of the invariant measures of $\{T_t^{p,\Phi} : t \geq 0\}$. An ergodic decomposition allows us to consider only invariant measures for $\{T_t^{p,\Phi} : t \geq 0\}$ which are also ergodic. For such ergodic probability measures, a Has'minskiĭ-type representation [16, Chap. 4.4], followed by some standard stability analyses, lead to Proposition 4.8. Let us fix now for the rest of this appendix a $p \in \mathbb{R}$ and a $\Phi \in \mathfrak{S}$. Recall that we are assuming that $\lambda^\Phi > 0$.

We begin with the fact that for any $\mu \in \mathcal{P}(M)$,

$$\varliminf_{t \to \infty} \frac{1}{t} \int_0^t \bar{\mathbb{E}}_\mu^{p,\Phi} \left[V^\Phi(\mathbf{X}_s) \right] ds = \int_{\mathcal{P}(M)} V^\Phi(\varpi) Q(d\varpi)$$

where $Q \in \mathcal{P}(\mathcal{P}(M))$ is an invariant probability measure for $\{T_t^{p,\Phi} : t \geq 0\}$. For convenience, let $\mathfrak{I}^{p,\Phi}$ be the collection of elements of $\mathcal{P}(\mathcal{P}(M))$ which are invariant for $\{T_t^{p,\Phi} : t \geq 0\}$; then (4.9) is true if

$$\varliminf_{|p| \to \infty} \inf \left\{ \int_{\mathcal{P}(M)} V^\Phi(\varpi) Q(d\varpi) : Q \in \mathfrak{I}^{p,\Phi} \right\} > 0. \tag{A.1}$$

By virtue of the ergodic decomposition theorem [9, Chap. 5.2], [26, Chap. 7.4], this is true if

$$\gamma \triangleq \varliminf_{|p| \to \infty} \inf \left\{ \int_{\mathcal{P}(M)} V^\Phi(\varpi) Q(d\varpi) : Q \in \mathfrak{E}\mathfrak{I}^{p,\Phi} \right\} \tag{A.2}$$

is positive, where $\mathfrak{E}\mathfrak{I}^{p,\Phi}$ is the collection of elements of $\mathfrak{I}^{p,\Phi}$ which are ergodic.

We want to consider the infimum in (A.2) via a Has'minskiĭ-type representation. To do so, define

$$L^\Phi(\eta) \triangleq \left\{ \varpi \in \mathcal{P}(M) : V^\Phi(\varpi) \leq \eta \right\}, \qquad \eta \geq 0$$

and fix $\eta' > 0$ such that

$$\inf \left\{ \sum_{1 \leq k,l \leq n} \xi_l (\mathbf{B}_l \mathcal{H}_k - \mathbf{B}_l \mathbf{B}_k \Phi)(\varpi) \xi_k : \varpi \in L^\Phi(\eta'), \sum_{k=1}^n \xi_k^2 = 1 \right\} \geq \lambda^\Phi / 2.$$

Note that

$$\gamma_0 \triangleq \inf \left\{ \int_{\mathcal{P}(M)} V^\Phi(\varpi) Q(d\varpi) : Q \in \mathcal{P}(\mathcal{P}(M)), Q(L(\eta')) = 0 \right\}$$

is positive; indeed, $\gamma_0 \geq \eta'$. Thus, we need to show that

$$\gamma_1 \triangleq \inf \left\{ \int_{\mathcal{P}(M)} V^\Phi(\varpi) Q(d\varpi) : Q \in \mathfrak{E}\mathfrak{I}^{p,\Phi}, Q(L(\eta')) > 0 \right\} \tag{A.3}$$

is positive; $\gamma \geq \min\{\gamma_0, \gamma_1\}$. To do this, fix $0 < \eta_1 < \eta_2 < \eta'$; we will write a Has'minskiĭ representation using the Markov process induced by $\{\mathbf{X}_t : t \geq 0\}$ on $\partial L(\eta_1)$ and $\partial L(\eta_2)$. As a preliminary, of course, we need to ensure that $\{\mathbf{X}_t : t \geq 0\}$ hits $\partial L(\eta_1)$ and $\partial L(\eta_2)$ infinitely often. Note that for any $Q \in \mathfrak{EI}^{p,\Phi}$ such that $Q(L(\eta')) > 0$,

$$\begin{aligned} 1 &= \int_{\mathcal{P}(M)} \bar{\mathbb{P}}^{p,\Phi}_\mu \left\{ \lim_{t\to\infty} \frac{1}{t}\int_0^t \chi_{L(\eta')}(\mathbf{X}_s)ds = Q(L(\eta'))\right\} Q(d\mu) \\ &\leq \int_{\mathcal{P}(M)} \bar{\mathbb{P}}^{p,\Phi}_\mu \left\{ \overline{\lim_{t\to\infty}}\, \chi_{L(\eta')}(\mathbf{X}_s) = 1\right\} Q(d\mu) \end{aligned}$$

so at least we know that $L(\eta')$ is recurrent for $\{\mathbf{X}_t : t \geq 0\}$ under $\int_{\mathcal{P}(M)} \bar{\mathbb{P}}^{p,\Phi}_\mu Q(d\mu)$ To prove furthermore that $\partial L(\eta_1)$ and $\partial L(\eta_2)$ are recurrent under $\int_{\mathcal{P}(M)} \bar{\mathbb{P}}^{p,\Phi}_\mu Q(d\mu)$ for any $Q \in \mathfrak{EI}^{p,\Phi}$ such that $Q(L(\eta')) > 0$, let's look more closely at the evolution of $\{V^\Phi(\mathbf{X}_t) : t \geq 0\}$ as described by (4.5) and (4.6). Define the two constants

$$\begin{aligned} \zeta_0 &\triangleq \inf\left\{ \frac{1}{2}\sum_{1\leq k,l\leq n} (\mathbf{B}_l\mathcal{H}_k - \mathbf{B}_l\mathbf{B}_k\Phi)^2(\varpi) : \varpi \in L(\eta')\right\} \\ \zeta_1 &\triangleq \sup\left\{ \frac{1}{2}\sum_{1\leq k\leq n} (\mathbf{B}_0\mathcal{H}_k - \mathbf{B}_0\mathbf{B}_k\Phi)^2(\varpi) \right. \\ &\qquad \left. + \frac{1}{4}\sum_{1\leq k,l\leq n} (\mathbf{B}_l^2\mathcal{H}_k - \mathbf{B}_l^2\Phi)^2(\varpi) : \varpi \in L(\eta')\right\}. \end{aligned}$$

Note that under our assumption that $\lambda^\Phi > 0$ and our choice of η', $\zeta_0 > 0$. Then by an easy application of Young's inequality,

$$(\mathbf{A}^{p,\Phi}V^\Phi)(\varpi) \geq \left(\frac{\lambda^\Phi}{2} - \frac{1}{4\varepsilon p} - \frac{n}{8\varepsilon p}\right) V^\Phi(\varpi) + \frac{1}{2p}(\zeta_0 - \varepsilon\zeta_1) \tag{A.4}$$

for all $\varpi \in L(\eta')$, all $p \in \mathbb{R}$, and all $\varepsilon > 0$. Thus, if we take $\varepsilon > 0$ sufficiently small and $|p|$ sufficiently large,

$$(\mathbf{A}^{p,\Phi}V^\Phi)(\varpi) \geq \frac{\lambda^\Phi}{4} V^\Phi(\varpi) + \frac{\zeta_0}{4p} \geq \frac{\zeta_0}{4p} \tag{A.5}$$

for all $\varpi \in L(\eta')$. Turning now to (4.6), we also see that since $\lambda^\Phi > 0$ and by our choice of η', there is a constant $\kappa > 0$ such that

$$\kappa V^\Phi(\varpi) \leq \sum_{l=1}^n (\mathbf{B}_l V^\Phi)^2(\varpi) \leq \kappa^{-1} V^\Phi(\varpi) \tag{A.6}$$

for all $\varpi \in L(\eta')$. Consider now any $Q \in \mathfrak{EI}^{p,\Phi}$ such that $Q(L(\eta')) > 0$. We already know that $L(\eta')$ is recurrent under $\int_{\mathcal{P}(M)} \bar{\mathbb{P}}^{p,\Phi}_\mu Q(d\mu)$, and combining this fact with

(A.5) and (A.6), we can also conclude that $\partial L(\eta_1)$ and $\partial L(\eta_2)$ are recurrent. Indeed, for sufficiently small $\eta'' > 0$, we can compare $\{V^\Phi(\mathbf{X}_t) : t \geq 0\}$ to the deterministic process $\{\frac{\zeta_0}{4p}t : t \geq 0\}$ on $L(\eta'')$, and on $L(\eta') \sim L(\eta'')$, we can compare $\{V^\Phi(\mathbf{X}_t) : t \geq 0\}$ to a process $\{\frac{\zeta_0}{4p}t + \frac{1}{\sqrt{p}}M_t : t \geq 0\}$, where $\{M_t : t \geq 0\}$ is a continuous martingale whose bracket grows linearly (thus $\{M_t : t \geq 0\}$ is a time-changed Brownian motion whose clock grows linearly). We leave the details of the argument to the reader. We are now assured that a Has'minskiĭ-type representation will be valid. Define the mappings τ_1 and τ_2 from Ω' to $\mathbb{R}_+$ by

$$\begin{aligned}\tau_1 &\triangleq \inf\{t \geq 0 : V^\Phi(\mathbf{X}_t) = \eta_1\}\\ \tau_2 &\triangleq \inf\{t \geq \tau_1 : V^\Phi(\mathbf{X}_t) = \eta_2\}.\end{aligned}$$

Clearly τ_1 and τ_2 are stopping times with respect to the filtration of $\mathcal{F}'$ generated by $\{\mathbf{X}_t : t \geq 0\}$. Then for any $Q \in \mathfrak{E}\mathfrak{I}^{p,\Phi}$ such that $Q(L(\eta')) > 0$,

$$\int_{\mathcal{P}(M)} V^\Phi(\varpi)Q(d\varpi) = \frac{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu \left[\int_0^{\tau_2} V^\Phi(\mathbf{X}_s)ds\right] \hat{Q}(d\mu)}{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2]\hat{Q}(d\mu)}$$

where $\hat{Q}$ is a probability measure on $(\partial L(\eta_2), \mathcal{B}(\partial L(\eta_2)))$ which is the distribution of $\mathbf{X}_{\tau_2}$ under $\int_{\mathcal{P}(M)} \bar{\mathbb{P}}^{p,\Phi}_\mu Q(d\mu)$.

The way is now clear for some simple stability calculations. For any measure $\hat{Q}$ on $(\partial L(\eta_2), \mathcal{B}(\partial L(\eta_2)))$,

$$\begin{aligned}&\frac{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu \left[\int_0^{\tau_2} V^\Phi(\mathbf{X}_s)ds\right] \hat{Q}(d\mu)}{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2]\hat{Q}(d\mu)}\\ &\quad\geq \frac{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu \left[\int_0^{\tau_1} V^\Phi(\mathbf{X}_s)ds\right] \hat{Q}(d\mu)}{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_1]\hat{Q}(d\mu) + \sup_{\mu \in L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2 - \tau_1]}\\ &\quad\geq \frac{\eta_1 \int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_1]\hat{Q}(d\mu)}{\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_1]\hat{Q}(d\mu) + \sup_{\mu \in L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2 - \tau_1]}.\end{aligned} \tag{A.7}$$

We want to show that the right side of (A.7) is bounded away from zero uniformly in p and $\hat{Q}$. This entails showing that $\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_1]\hat{Q}(d\mu)$ is bounded from below and $\sup_{\mu \in L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2 - \tau_1]$ is bounded from above. Using (4.5) and (4.6) and the bound (A.4) to consider $\{V^\Phi(\mathbf{X}_t) : t \geq 0\}$ as a perturbation of (4.8), we find that both of these are not unreasonable. Let's first bound $\bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2 - \tau_1]$ from above. Returning to (A.5), we see that when $|p|$ is sufficiently large,

$$\begin{aligned}\eta_2 - \eta_1 &= \bar{\mathbb{E}}^{p,\Phi}_\mu \left[\int_{\tau_1}^{\tau_2} (\mathbf{A}^{p,\Phi}V^\Phi)(\varpi)ds\right] \qquad \mu \in \partial L(\eta_2)\\ &\geq \frac{\zeta_0}{4p}\bar{\mathbb{E}}^{p,\Phi}_\mu[\tau_2 - \tau_1],\end{aligned}$$

so

$$\bar{\mathbb{E}}^{p,\Phi}_{\mu}[\tau_2 - \tau_1] \le \frac{4p(\eta_2 - \eta_1)}{\zeta_0}. \qquad \mu \in \partial L(\eta_2).$$

Secondly, we bound $\int_{\partial L(\eta_2)} \bar{\mathbb{E}}^{p,\Phi}_{\mu}[\tau_1]\hat{Q}(d\mu)$ from below. Taking $\varepsilon = 1$ and $|p|$ sufficiently large in (A.4), we get that

$$(\mathbf{A}^{p,\Phi}V^{\Phi})(\varpi) \ge \frac{\lambda^{\Phi}}{4}V^{\Phi}(\varpi) - \frac{\zeta_1}{4p} \ge -\frac{\zeta_1}{4p}$$

for all $\varpi \in L(\eta_2)$. Thus for all $\mu \in \partial L(\eta_2)$,

$$\begin{aligned} \eta_1 - \eta_2 &= \bar{\mathbb{E}}^{p,\Phi}_{\mu}\left[\int_0^{\tau_1} (\mathbf{A}^{p,\Phi}V^{\Phi})(\varpi)ds\right] \\ &\ge -\frac{\zeta_1}{2p}\bar{\mathbb{E}}^{p,\Phi}_{\mu}[\tau_1], \end{aligned}$$

so

$$\bar{\mathbb{E}}^{p,\Phi}_{\mu}[\tau_1] \ge \frac{2p(\eta_2 - \eta_1)}{\zeta_1}. \qquad \mu \in \partial L(\eta_2).$$

Using simple properties of the mapping $(x, y) \mapsto (\eta_1 x)/(x + y)$, we may thus conclude that for any $Q \in \mathfrak{E}\mathfrak{I}^{p,\Phi}$ such that $Q(L(\eta')) > 0$,

$$\begin{aligned} \int_{\mathcal{P}(M)} V^{\Phi}(\varpi)Q(d\varpi) &\ge \frac{\eta_1 \frac{2p(\eta_2-\eta_1)}{\zeta_1}}{\frac{2p(\eta_2-\eta_1)}{\zeta_1} + \frac{4p(\eta_2-\eta_1)}{\zeta_0}} \\ &= \frac{\eta_1}{1 + \frac{2\zeta_1}{\zeta_0}}. \end{aligned}$$

Thus, returning to (A.3), we have that

$$\gamma_1 \ge \frac{\eta_1}{1 + \frac{2\zeta_1}{\zeta_1}}.$$

Thus γ in (A.2) is positive, so (A.1) is true, and Proposition 4.8 is consequently verified.

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