

A COMPARISON OF HOMOGENIZATION AND LARGE DEVIATIONS, WITH APPLICATIONS TO WAVEFRONT PROPAGATION

MARK I. FREIDLIN AND RICHARD B. SOWERS

University of Maryland, College Park and
University of Illinois at Urbana-Champaign

January 18, 1999

ABSTRACT. We consider the combined effects of homogenization and large deviations in a stochastic differential equation. We show that there are three regimes, depending on the relative rates at which the small viscosity parameter and the homogenization parameter tend to zero. We prove some large-deviations type estimates, and then apply these results to study wavefronts in both a single reaction-diffusion equation and in a system of reaction-diffusion equations.

§0. INTRODUCTION

In this paper we will consider the stochastic differential equation (SDE) on \mathbb{R}^d (with $d \geq 1$)

$$dX_t^{x,\varepsilon,\delta} = \sqrt{\varepsilon} \sum_{l=1}^d A_l \left(\frac{X_t^{x,\varepsilon,\delta}}{\delta} \right) dW_t^l + B^{\varepsilon,\delta_\varepsilon} \left(\frac{X_t^{x,\varepsilon,\delta}}{\delta} \right) dt \quad t \geq 0 \quad (0.1)$$
$$X_0^{x,\varepsilon,\delta} = x.$$

Here the A_l 's and $B^{\varepsilon,\delta}$ are smooth mappings from \mathbb{R}^d to itself which are periodic of period 1 in each variable (we will later have a bit more to say about the effect of $B^{\varepsilon,\delta}$; assume for now that $B^{\varepsilon,\delta}$ does not depend on ε or δ). As ε and δ tend to zero, two well-known effects come into play. If we fix $\delta > 0$ and let ε tend to zero, the theory of *large deviations* tells us how quickly $X^{x,\varepsilon,\delta}$ tends to the deterministic ODE given by actually setting ε to zero. On the other hand, if we fix $\varepsilon > 0$ and

1991 *Mathematics Subject Classification*. Primary 35B27, 35K57 60F10, Secondary 35K40, 35B40.

Key words and phrases. Homogenization, Huygen's principle, KPP equations, large deviations, Minkowski geometry, reaction-diffusion equations, wavefront propagation.

The work of the first author was supported by ARO grant DAA 103-92-G-0219 and by NSF grant DMS 95-04177. The work of the second author was supported by an NSF Postdoctoral Fellowship at the University of Maryland at College Park and Northwestern University. During revision, the work of the second author was supported by NSF grants DMS 96-26398 and DMS 96-15877.

let δ tend to zero, *homogenization* occurs; instead of coefficients which oscillate rapidly, we see effective constant coefficients. The goal of this paper is to study the *combination* of large deviations and homogenization through (0.1).

Of course an understanding of the asymptotics of the SDE (0.1) can immediately be transferred to the theory of PDE's, and through standard arguments be used to consider a PDE of Kolmogorov-Petrovskii-Piskunov type [18]. Namely, if we fix a nonlinear function f of KPP type (see [10] and [11]), then we can consider the PDE

$$\begin{aligned} \frac{\partial u^{\varepsilon, \delta}}{\partial t} &= \frac{\varepsilon}{2} \langle a(x/\delta), D_x^2 u^{\varepsilon, \delta}(t, x) \rangle_{\mathcal{M}_{d \times d}} + \langle B^{\varepsilon, \delta}(x/\delta), \nabla_x u^{\varepsilon, \delta}(t, x) \rangle + \frac{1}{\varepsilon} f(u^{\varepsilon, \delta}) \\ u^{\varepsilon, \delta}(0, \cdot) &= g(x). \end{aligned} \tag{0.2}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^d , D^2 is the Hessian and ∇ is the standard gradient operator (above, we affix the subscripts x to D^2 and ∇ to emphasize that they act on the spatial variables), and $\langle A, B \rangle_{\mathcal{M}_{d \times d}} \stackrel{\text{def}}{=} \text{trace}(AB^T)$ is the standard inner product on $\mathcal{M}_{d \times d}$, the space of $d \times d$ matrices, with \cdot^T denoting matrix transpose. The matrix-valued function a is given by

$$a(x) = \sum_{l=1}^d A_l(x) A_l^T(x), \quad (\text{for all } 1 \leq i, j \leq d)$$

We assume that $a(x)$ is uniformly nondegenerate for all $x \in \mathbb{R}^d$ (see (1.1)). In (0.2), the initial condition $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is a nonnegative and continuous function of compact support G_\circ with nonempty interior. Large-deviations-type behavior of (0.1) then leads to a proof of wavefront propagation in (0.2). Questions of wavefront propagation for PDE's such as (0.2) have already been studied in [22]. We will discuss this a bit more later.

The limiting behavior of the SDE (0.1) or alternately of the PDE (0.2) clearly should depend on the relation between ε and δ . Thus, if ε tends to zero sufficiently quickly (compared to δ), we should first treat δ as fixed and carry out these calculations for slowly-varying coefficients as in [10]; we should then let δ tend to zero in the resulting formulæ. On the other hand, if ε tends to zero sufficiently slowly (compared to δ), then we expect that the equation (0.1) should be first homogenized—the coefficients should be replaced by their corresponding effective coefficients; then we should consider large deviations for the homogenized equation. Finally, if δ and ε tend to zero at the same rate, we have the case considered in [9], [10, Ch. 7], and [15]. We can organize all of this by considering $X^{x, \varepsilon, \delta_\varepsilon}$, where

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = 0.$$

We shall see that the limiting behavior of $X^{x, \varepsilon, \delta_\varepsilon}$ can be taxonomized by the limits of the ratio $\delta_\varepsilon/\varepsilon$. Specifically, we have exactly three *regimes*;

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon} = \begin{cases} 0 & : \text{Regime 1} \\ c \in (0, \infty) & : \text{Regime 2} \\ \infty & : \text{Regime 3.} \end{cases} \tag{0.3}$$

The basic set of calculations of this paper involves showing that in all three regimes, $X_1^{x,\varepsilon,\delta_\varepsilon}$ has a large deviations principle (LDP) as ε tends to zero, and in identifying the rate function \mathcal{J} . By scaling time and using the Markov property, the entire path $\{X_t^{x,\varepsilon,\delta_\varepsilon}; 0 \leq t \leq T\}$ should then have a large deviations principle in $C([0, T]; \mathbb{R}^d)$ (for any fixed $T > 0$) with rate function $S_{0,T}(\varphi) = \int_0^T \mathcal{J}(\dot{\varphi}(s)) ds$ (if φ is absolutely continuous and $S_{0,T}(\varphi) = \infty$ if not). The wavefront of the PDE (0.2) can then be analyzed (this will be done in §6) using the function

$$V(t, x) \stackrel{\text{def}}{=} f'(0)t - \inf_{\substack{\varphi \in C^1([0,t]; \mathbb{R}^d) \\ \varphi(0)=x \\ \varphi(t) \in G_0}} S_{0,t}(\varphi) = f'(0)t - t \inf_{y \in G_0} \mathcal{J}\left(\frac{y-x}{t}\right). \quad (0.4)$$

It turns out that \mathcal{J} is convex (see Lemma A.1), which implies the last expression since the extremal paths in the middle expression will be linear (see §6). The position of the wavefront for (0.2) at time t is then defined by the equation $V(t, x) = 0$; more exactly,

$$\lim_{\varepsilon \rightarrow 0} u^{\varepsilon, \delta_\varepsilon}(t, x) = \begin{cases} 1 & \text{if } V(t, x) > 0 \\ 0 & \text{if } V(t, x) < 0. \end{cases} \quad (0.5)$$

The main technique for showing that $X_1^{x,\varepsilon,\delta_\varepsilon}$ has a large deviations principle is the following standard result. For each $T > 0$ and $x \in \mathbb{R}^d$, define

$$g_{T,x}^\varepsilon(\theta) \stackrel{\text{def}}{=} \varepsilon \log \mathbb{E} \left[\exp \left[\frac{1}{\varepsilon} \langle \theta, X_T^{x,\varepsilon,\delta_\varepsilon} \rangle \right] \right]. \quad \varepsilon > 0, \theta \in \mathbb{R}^d$$

By standard results (see (1.6)), $|g_{T,x}^\varepsilon(\theta)| < \infty$ for all $\varepsilon > 0$ and $\theta \in \mathbb{R}^d$. Now define

$$g_{T,x}(\theta) \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} g_{T,x}^\varepsilon(\theta). \quad \theta \in \mathbb{R}^d$$

when this limit exists. We then have ([4, Chap. 2.3], [7], [13, Chap. 5.1], [14]):

Theorem 0.1. *Fix $T > 0$ and $x \in \mathbb{R}^d$. Assume that*

- i) For each $\theta \in \mathbb{R}^d$, $g_{T,x}(\theta)$ is well-defined in $[-\infty, \infty]$.*
- ii) The origin is in the interior of the set $\{\theta \in \mathbb{R}^d : g_{T,x}(\theta) < \infty\}$.*
- iii) The set $A \stackrel{\text{def}}{=} \{\theta \in \mathbb{R}^d : |g_{T,x}(\theta)| < \infty\}$ has a nonempty interior A° , $\nabla g_{T,x}(\theta)$ is well-defined for all $\theta \in A^\circ$, and $\lim_{\theta \rightarrow \partial A, \theta \in A^\circ} \|\nabla g_{T,x}(\theta)\| = \infty$.*

Then the random variables $\{X_T^{x,\varepsilon,\delta_\varepsilon} : \varepsilon > 0\}$ have a large deviations principle with rate function $I_{T,x}$ defined by

$$I_{T,x}(z) \stackrel{\text{def}}{=} \sup_{\theta' \in \mathbb{R}^d} \{\langle \theta', z \rangle - g_{T,x}(\theta')\}. \quad z \in \mathbb{R}^d$$

As we pointed out earlier, the PDE (0.2) has been studied in [22]. Essentially, they consider Regimes 2 and 3 by setting $\delta_\varepsilon = \varepsilon^\kappa$, where $0 < \kappa \leq 1$; the case $\kappa = 1$ implies Regime 2, and $\kappa \in (0, 1)$ corresponds to Regime 3. They directly describe $g_{T,x}(\theta)$ as the unique constant such that the *cell problem* can be solved. Part of our interest is to show what is going on *probabilistically*. We hope that our analysis helps clarify how Regimes 2 and 3 differ. In particular, Regime 3 can be understood as a *limit* of Regime 2 as the constant c (of (0.3)) tends to infinity, and this involves

some computations in [2] and [5] concerning the limits of Donsker-Varadhan-type large deviations for occupation measures. On the other hand, the rate function of Regime 3 can also be understood in a *pathwise* sense by very carefully using the heat kernel asymptotics of Norris and Stroock [25]. In other words, Regime 3 is sort of a transition regime between large deviations of occupation measures and large deviations of paths.

The organization of this paper is as follows. In the next section, we set up some notation and make precise our assumptions. In §2–4 we study large deviations for $X_T^{x,\varepsilon,\delta_\varepsilon}$ for any fixed $T > 0$, as ε tends to zero. In §5, we use these results to calculate the functional large deviations principle for $\{X_t^{x,\varepsilon,\delta_\varepsilon} : 0 \leq t \leq T\}$ as ε tends to zero; there are some complications in Regime 1 to be dealt with (see Remark 5.3). In §6, we return to the wavefront propagation problem. We close in §7 with some generalizations; in particular, some systems of RDE’s are considered there.

Finally, we should say that questions related to this paper, besides those of the references mentioned earlier, can be found in [1], [8], [19], [23], [24], [26], and [29].

§1. ASSUMPTIONS, NOTATION, AND PROBLEM FORMULATION

Let’s now be a bit more precise about things. First, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability triple on which a d -dimensional Brownian motion $(W^1, W^2 \dots W^d)$ is defined. Let \mathbb{E} be the corresponding expectation operator. We have already defined $\langle \cdot, \cdot \rangle$ as the standard Euclidean inner product on \mathbb{R}^d ; now let $\|\cdot\|$ be the associated norm. We have also defined ∇ , D^2 , $\mathcal{M}_{d \times d}$, $\langle \cdot, \cdot \rangle_{\mathcal{M}_{d \times d}}$, and \cdot^T . Let div be the standard divergence operator (i.e., the negative of the adjoint of ∇ with respect to Lebesgue measure). Let $C_p(\mathbb{R}^d; \mathbb{R}^d)$ be the collection of continuous mappings from \mathbb{R}^d into \mathbb{R}^d which are periodic of period 1 in each coordinate of the argument and let $\|\cdot\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)}$ be the associated supremum norm. For each $T > 0$, let $C([0, T]; \mathbb{R}^d)$ be the standard Banach space of continuous functions from $[0, T]$ into \mathbb{R}^d ; let $\|\cdot\|_{C([0, T]; \mathbb{R}^d)}$ be the supremum norm on this space and let $\rho_{C([0, T]; \mathbb{R}^d)}$ be the corresponding metric. Also, for $T > 0$ and $\alpha \in (0, 1)$, let $C^\alpha([0, T]; \mathbb{R}^d)$ be the corresponding space of Hölder-continuous functions of exponent α and let $\|\cdot\|_{C^\alpha([0, T]; \mathbb{R}^d)}$ be the associated norm. Let \mathfrak{T} be the d -dimensional torus of size 1, and let $\pi : \mathbb{R}^d \rightarrow \mathfrak{T}$ be any fixed covering map. Let $\|\cdot\|_{C(\mathfrak{T}; \mathbb{R}^d)}$ be the standard supremum norm on $C(\mathfrak{T}; \mathbb{R}^d)$, the space of continuous maps from \mathfrak{T} to \mathbb{R}^d .

The A_l ’s of (0.1) are assumed to be in $C_p(\mathbb{R}^d; \mathbb{R}^d)$, and we also assume that

$$\kappa \stackrel{\text{def}}{=} \inf \left\{ \sum_{l=1}^d \langle \theta, A_l(x) \rangle^2 : x \in \mathbb{R}^d, \theta \in \mathbb{R}^d, \|\theta\| = 1 \right\} > 0. \quad (1.1)$$

We assume that $B^{\varepsilon, \delta}$ is of the form

$$B^{\varepsilon, \delta} = \frac{\varepsilon}{\delta} B_0 + B_1 + B_2^{\varepsilon, \delta}, \quad (1.2)$$

where B_0 , B_1 , and (for every $\varepsilon > 0$ and $\delta > 0$) $B_2^{\varepsilon, \delta}$ are in $C_p(\mathbb{R}^d; \mathbb{R}^d)$. We furthermore assume that $\lim_{\varepsilon, \delta \rightarrow 0} \|B_2^{\varepsilon, \delta}\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)} = 0$.

Next let’s move the SDE (0.1) to \mathfrak{T} and make some rescalings. We will find an SDE for $\pi(\delta^{-1} X_{(\delta/\sqrt{\varepsilon})^2 t}^{x, \varepsilon, \delta})$. First, let’s write down the generator of $\{\delta^{-1} X_{(\delta/\sqrt{\varepsilon})^2 t}^{x, \varepsilon, \delta} : t \geq$

0} and its toroidal counterpart; define

$$\begin{aligned} (\mathcal{L}^{\varepsilon,\delta}\phi)(x) &\stackrel{\text{def}}{=} \frac{1}{2}\langle a(x), D^2\phi(x) \rangle_{\mathcal{M}_{d \times d}} + \frac{\delta}{\varepsilon}\langle B^{\varepsilon,\delta}(x), \nabla\phi(x) \rangle \\ (\hat{\mathcal{L}}^{\varepsilon,\delta}\phi')(\pi(x)) &= (\mathcal{L}^{\varepsilon,\delta}(\phi' \circ \pi))(x) \end{aligned} \quad (1.3)$$

for all $x \in \mathbb{R}^d$, $\phi \in C^\infty(\mathbb{R}^d)$, and $\phi' \in C^\infty(\mathfrak{X})$. To move to the torus, define vector fields $\hat{\sigma}_1, \hat{\sigma}_2 \dots \hat{\sigma}_d$ and $\hat{\sigma}_0^{\varepsilon,\delta}$ (for $\varepsilon > 0$ and $\delta > 0$) on \mathfrak{X} as

$$\begin{aligned} (\hat{\sigma}_l\phi)(\pi(x)) &\stackrel{\text{def}}{=} \langle A_l, \nabla(\phi \circ \pi) \rangle(x) \\ \hat{\sigma}_0^{\varepsilon,\delta}\phi &\stackrel{\text{def}}{=} \hat{\mathcal{L}}^{\varepsilon,\delta}\phi - \frac{1}{2} \sum_{l=1}^d \hat{\sigma}_l^2\phi. \end{aligned} \quad x \in \mathbb{R}^d, \phi \in C^\infty(\mathfrak{X})$$

Defining $\hat{X}^{x,\varepsilon,\delta}$ as the solution of the \mathfrak{X} -valued SDE

$$\begin{aligned} d\hat{X}_t^{x,\varepsilon,\delta} &= \sum_{l=1}^d \hat{\sigma}_l(\hat{X}_t^{x,\varepsilon,\delta}) \circ dW_t^l + \hat{\sigma}_0^{\varepsilon,\delta}(\hat{X}_t^{x,\varepsilon,\delta})dt \quad t \geq 0 \\ \hat{X}_0^{x,\varepsilon,\delta} &= \pi(x/\delta), \end{aligned}$$

we have that $\hat{X}_t^{x,\varepsilon,\delta} \stackrel{(\mathcal{L})}{=} \pi(\delta^{-1}X_{(\delta/\sqrt{\varepsilon})^2t}^{x,\varepsilon,\delta})$ for all $t \geq 0$. We use this in rewriting $g_{T,x}^\varepsilon$. We rewrite $\langle \theta, X_T^{x,\varepsilon,\delta} \rangle$ in integral form, rescale, and then use the periodicity of the coefficients of (0.1). Define the following elements of $C^\infty(\mathfrak{X}; \mathbb{R}^d)$:

$$\begin{aligned} \hat{A}_l(\pi(x)) &= A_l(x) \quad (\text{for all } 1 \leq l \leq d) \\ \hat{B}^{\varepsilon,\delta}(\pi(x)) &= B^{\varepsilon,\delta}(x) \quad (\text{for all } \varepsilon > 0 \text{ and } \delta > 0); \\ \hat{B}_0(\pi(x)) &= B_0(x) \quad \text{and} \quad \hat{B}_1(\pi(x)) = B_1(x) \end{aligned} \quad (1.4)$$

for all $x \in \mathbb{R}^d$. Then we have that

$$\begin{aligned} g_{T,x}^\varepsilon(\theta) &= \langle \theta, x \rangle + \varepsilon \log \mathbb{E} \left[\exp \left[\frac{\delta_\varepsilon}{\varepsilon} \sum_{l=1}^d \int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T} \langle \hat{A}_l(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}), \theta \rangle dW_s^l \right. \right. \\ &\quad \left. \left. + \left(\frac{\delta_\varepsilon}{\varepsilon} \right)^2 \int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T} \langle \hat{B}^{\varepsilon,\delta_\varepsilon}(\hat{X}_s^{x,\varepsilon,\delta_\varepsilon}), \theta \rangle ds \right] \right] \quad \varepsilon > 0, \theta \in \mathbb{R}^d \end{aligned} \quad (1.5)$$

for all $T > 0$ and $x \in \mathbb{R}^d$. Note that a simple Girsanov transformation implies that

$$\begin{aligned} \langle \theta, x \rangle + \left(\frac{1}{2}\kappa\|\theta\|^2 - \|\hat{B}^{\varepsilon,\delta_\varepsilon}\|_{C(\mathfrak{X};\mathbb{R}^d)}\|\theta\| \right) T &\leq g_{T,x}^\varepsilon(\theta) \\ &\leq \langle \theta, x \rangle + \left(\frac{1}{2}\kappa^{-1}\|\theta\|^2 + \|\hat{B}^{\varepsilon,\delta_\varepsilon}(z)\|_{C(\mathfrak{X};\mathbb{R}^d)}\|\theta\| \right) T \end{aligned} \quad (1.6)$$

for all $T > 0$, $x \in \mathbb{R}^d$, $\varepsilon > 0$, and $\theta \in \mathbb{R}^d$.

Finally, let's define some other operators on the torus. Define

$$\begin{aligned} (\hat{\mathcal{L}}_0\phi)(\pi(x)) &\stackrel{\text{def}}{=} \frac{1}{2}\langle a(x), D^2(\phi \circ \pi)(x) \rangle_{\mathcal{M}_{d \times d}} + \langle B_0(x), \nabla(\phi \circ \pi)(x) \rangle \\ (\hat{\beta}_1\phi)(\pi(x)) &\stackrel{\text{def}}{=} \langle B_1(x), \nabla(\phi \circ \pi)(x) \rangle \end{aligned} \quad (1.7)$$

for all $x \in \mathbb{R}^d$ and $\phi \in C^\infty(\mathfrak{X})$. Then $\hat{\mathcal{L}}_0$ is the part of $\mathcal{L}^{\varepsilon,\delta}$ (of (1.3)), properly translated to \mathfrak{X} , which does not depend on ε and δ , and $\hat{\beta}_1$ is the corresponding effect of B_1 .

§2. REGIME 1

We here assume that $\lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon / \varepsilon) = 0$. In this case, δ_ε tends to zero much more quickly than ε , and homogenization dominates; the large deviations of $X^{x, \varepsilon, \delta_\varepsilon}$ are the same as those of a constant-coefficient system. Essentially, this case was covered by Makhno [21]; we include the calculations here for completeness. Here, the role of B_0 is important; this is the case of “diffusion with unbounded drift” in [3, Chapter 3.4.2]. The case where $B_0 \equiv 0$ was considered in [1], using slightly different techniques. The term B_0 can change the effective homogenized diffusion coefficient; it changes the action functional for $\{X_t^{x, \varepsilon, \delta_\varepsilon}; t \geq 0\}$ in the corresponding way. Note that in this case, the generator of $\hat{X}^{x, \varepsilon, \delta_\varepsilon}$ tends to $\hat{\mathcal{L}}_0$ as ε tends to zero. Let $\hat{\mu}_0 \in \mathcal{P}(\mathfrak{X})$ be the unique invariant measure of $\hat{\mathcal{L}}_0$ [16, Prop. 4.5 of Chap. 5]. We now enforce a standard homogenization assumption

Assumption 2.1. We assume that $\int_{\mathfrak{X}} \hat{B}_0(z) \hat{\mu}_0(dz) = 0$.

Here \hat{B}_0 is as in (1.4). This assumption means that the $O(\varepsilon/\delta_\varepsilon)$ term in (1.2) does not cause $\hat{X}^{x, \varepsilon, \delta_\varepsilon}$ to “blow up” (see [3]). Under this assumption, the Fredholm alternative implies that there must be a unique $\psi \in C^\infty(\mathfrak{X}; \mathbb{R}^d)$ such that $\hat{\mathcal{L}}_0 \psi = \hat{B}_0$ and $\int_{\mathfrak{X}} \psi(z) \hat{\mu}_0(dz) = 0$ (see [3, Theorem 3.4 of Chap. 3]; $-\psi$ is in some cases called the “corrector”). Define now

$$\begin{aligned} \bar{\mathcal{J}}^{(1)}(\theta) &\stackrel{\text{def}}{=} \int_{\mathfrak{X}} \frac{1}{2} \sum_{l=1}^d \left\langle \left(\hat{A}_l - \hat{\sigma}_l \psi \right) (z), \theta \right\rangle^2 + \left\langle \left(\hat{B}_1 - \hat{\beta}_1 \psi \right) (z), \theta \right\rangle \hat{\mu}_0(dz). \\ &= \frac{1}{2} \langle \bar{A} \theta, \theta \rangle + \langle \bar{B}, \theta \rangle \end{aligned}$$

for all $\theta \in \mathbb{R}^d$, where the vector field $\hat{\beta}_1$ on \mathfrak{X} is defined as in (1.7), the \hat{A}_l 's and \hat{B}_1 are as in (1.4), and the $n \times n$ matrix \bar{A} and the d -vector \bar{B} are given by

$$\begin{aligned} \bar{A} &\stackrel{\text{def}}{=} \frac{1}{2} \sum_{l=1}^d \int_{\mathfrak{X}} \left(\hat{A}_l - \hat{\sigma}_l \psi \right) (z) \left(\hat{A}_l - \hat{\sigma}_l \psi \right)^T (z) \hat{\mu}_0(dz) \\ \bar{B}_i &\stackrel{\text{def}}{=} \int_{\mathfrak{X}} \left(\hat{B}_1 - \hat{\beta}_1 \psi \right) (z) \hat{\mu}_0(dz). \end{aligned}$$

Some calculations as in [24] show that \bar{A} must be strictly positive-definite; letting \bar{A}^{-1} be the matrix inverse of \bar{A} , we define

$$\mathcal{J}^{(1)}(\theta) \stackrel{\text{def}}{=} \sup_{\theta' \in \mathbb{R}^d} \left\{ \langle \theta, \theta' \rangle - \bar{\mathcal{J}}^{(1)}(\theta') \right\} = \frac{1}{2} \langle \bar{A}^{-1}(\theta - \bar{B}), \theta - \bar{B} \rangle. \quad \theta \in \mathbb{R}^d \quad (2.1)$$

Clearly, $\mathcal{J}^{(1)}$ is convex. Then

Proposition 2.2. Fix $T_0 > 0$ and assume that Assumption 2.1 is true. For every $x \in \mathbb{R}^d$ and $0 < T \leq T_0$, the family $\{X_T^{x, \varepsilon, \delta_\varepsilon} : \varepsilon > 0\}$ of \mathbb{R}^d -valued random variables has a large deviations principle with rate function $I_{T, x}^{(1)}(z) \stackrel{\text{def}}{=} T \mathcal{J}^{(1)}\left(\frac{z-x}{T}\right)$ (all $z \in \mathbb{R}^d$). Furthermore, this LDP is uniform for all $0 < T \leq T_0$ and $x \in \mathbb{R}^d$.

Proof. The proof is in three steps.

Step 1—Rewrite things. Fix $\theta \in \mathbb{R}^d$. Use Ito's formula on $\frac{\delta_\varepsilon}{\varepsilon}\psi(\hat{X}^{x,\varepsilon,\delta_\varepsilon})$, put this formula into the exponent in (1.5), and use Girsanov's formula, writing the result in terms of occupation measures (which tend to invariant measures). For each $\varepsilon > 0$, let $\{\hat{Y}_t^{x,\varepsilon} : t \geq 0\}$ be the solution of the \mathfrak{X} -valued SDE

$$\begin{aligned} d\hat{Y}_t^{x,\varepsilon} &= \sum_{l=1}^d \hat{\sigma}_l(\hat{Y}_t^{x,\varepsilon}) \circ dW_t^l + \hat{\sigma}_0^{\varepsilon,\delta_\varepsilon}(\hat{Y}_t^{x,\varepsilon}) dt \\ &\quad + \frac{\delta_\varepsilon}{\varepsilon} \sum_{l=1}^d \left\langle (\hat{A}_l - \hat{\sigma}_l \psi)(\hat{Y}_t^{x,\varepsilon}), \theta \right\rangle \hat{\sigma}_l(\hat{Y}_t^{x,\varepsilon}) dt \quad t \geq 0 \\ \hat{Y}_0^\varepsilon &= \pi(x/\delta_\varepsilon). \end{aligned}$$

Let $\{L_t^{x,\varepsilon} : t > 0\}$ be the occupation measure of $\hat{Y}^{x,\varepsilon}$; $L_t^{x,\varepsilon} \stackrel{\text{def}}{=} \frac{1}{t} \int_0^t \delta_{\hat{Y}_s^{x,\varepsilon}} ds$ for all $t > 0$. Let $\Phi^\varepsilon \in C^\infty(\mathfrak{X})$ be defined as

$$\Phi^\varepsilon(z) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{l=1}^d \left\langle (\hat{A}_l - \sigma_l \psi)(z), \theta \right\rangle^2 + \left\langle \left(\hat{B}^{\varepsilon,\delta_\varepsilon} - \frac{\varepsilon}{\delta_\varepsilon} \hat{\mathcal{L}}^{\varepsilon,\delta_\varepsilon} \psi \right)(z), \theta \right\rangle. \quad z \in \mathfrak{X}$$

Then for each $\varepsilon > 0$,

$$\begin{aligned} g_{T,x}^\varepsilon(\theta) &= \langle \theta, x \rangle + \varepsilon \log \mathbb{E} \left[\exp \left[\frac{T}{\varepsilon} \int_{\mathfrak{X}} \Phi^\varepsilon(z) L_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T}^{x,\varepsilon}(dz) \right] \right. \\ &\quad \left. \times \exp \left[\frac{\delta_\varepsilon}{\varepsilon} \left\{ \psi(\hat{Y}_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T}^{x,\varepsilon}) - \psi(\hat{Y}_0^{x,\varepsilon}) \right\} \right] \right]. \end{aligned}$$

Step 2—Asymptotics. Let's now expand things around the fact that $L_t^{x,\varepsilon}$ tends to the invariant measure of $\hat{Y}^{x,\varepsilon}$ as t tends to infinity. For each $\varepsilon > 0$, define the second-order operator $\hat{\mathcal{L}}_1^\varepsilon \stackrel{\text{def}}{=} \hat{\mathcal{L}}^{\varepsilon,\delta_\varepsilon} + \frac{\delta_\varepsilon}{\varepsilon} \sum_{l=1}^d \left\langle \hat{A}_l - \hat{\sigma}_l \psi, \theta \right\rangle \hat{\sigma}_l$ and let $\hat{\mu}_1^\varepsilon$ be its invariant measure. For each $\varepsilon > 0$, now define Ψ^ε as the unique element of $C^\infty(\mathfrak{X})$ such that $\hat{\mathcal{L}}_1^\varepsilon \Psi^\varepsilon = \Phi^\varepsilon - \int_{\mathfrak{X}} \Phi^\varepsilon(z') \hat{\mu}_1^\varepsilon(dz')$ and $\int_{\mathfrak{X}} \Psi^\varepsilon(z) \hat{\mu}_1^\varepsilon(dz) = 0$. Such a Ψ^ε must exist again by the Fredholm alternative. Using Ito's formula on $\left(\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)^2 \Psi^\varepsilon(\hat{Y}^{x,\varepsilon})$, we have that

$$\begin{aligned} g_{T,x}^\varepsilon(\theta) &= \langle \theta, x \rangle + T \int_{\mathfrak{X}} \Phi^\varepsilon(z) \hat{\mu}_1^\varepsilon(dz) \\ &\quad + \varepsilon \log \mathbb{E} \left[\exp \left[- \left(\frac{\delta_\varepsilon}{\varepsilon} \right)^2 \sum_{l=1}^d \int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T} (\hat{\sigma}_l \Psi^\varepsilon)(\hat{Y}_s^{x,\varepsilon}) dW_s^l \right] \right. \\ &\quad \times \exp \left[\left(\frac{\delta_\varepsilon}{\varepsilon} \right)^2 \left\{ \Psi^\varepsilon(\hat{Y}_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T}^{x,\varepsilon}) - \Psi^\varepsilon(\hat{Y}_0^{x,\varepsilon}) \right\} \right] \\ &\quad \left. \times \exp \left[\frac{\delta_\varepsilon}{\varepsilon} \left\{ \psi(\hat{Y}_{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T}^{x,\varepsilon}) - \psi(\hat{Y}_0^{x,\varepsilon}) \right\} \right] \right] \quad (2.2) \end{aligned}$$

for all $\varepsilon > 0$.

Let's now take a look at the asymptotics of Φ^ε and Ψ^ε as ε tends to zero. From some simple calculations, we see that $\lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon = \Phi^0$, where

$$\Phi^0(z) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{l=1}^d \left\langle (\hat{A}_l - \hat{\sigma}_l \psi)(z), \theta \right\rangle^2 + \left\langle (\hat{B}_1 - \hat{\beta}_1 \psi)(z), \theta \right\rangle. \quad z \in \mathfrak{X}$$

The convergence of Φ^ε to Φ occurs in $C^k(\mathfrak{X})$ for all $k \geq 0$. We see that, as ε tends to zero, $\hat{\mathcal{L}}_1^\varepsilon$ tends to $\hat{\mathcal{L}}_0$, thus $\hat{\mu}_1^\varepsilon$ tends to $\hat{\mu}_0$ in the weak topology (see [17, Chap. VIII, Section 1]), so $\lim_{\varepsilon \rightarrow 0} \Psi^\varepsilon = \Psi^0$, where Ψ^0 is the unique element of $C^\infty(\mathfrak{X})$ such that $\hat{\mathcal{L}}_0 \Psi^0 = \Phi^0 - \int_{\mathfrak{X}} \Phi^0(z') \hat{\mu}_0(dz')$ and $\int_{\mathfrak{X}} \Psi^0(z) \hat{\mu}_0(dz) = 0$. Since Φ^ε tends to Φ in $C^k(\mathfrak{X})$ for every $k \geq 0$, Ψ^ε tends to Ψ in $C^k(\mathfrak{X})$ for every $k \geq 0$ (this is clear from the calculations of [3, Theorem 3.4 of Chap. 3]). We now put everything together into (2.2). Since Φ^ε tends to Φ in $C(\mathfrak{X})$, $\lim_{\varepsilon \rightarrow 0} \int_{\mathfrak{X}} \Phi^\varepsilon(z) \hat{\mu}_1^\varepsilon(dz) = \int_{\mathfrak{X}} \Phi^0(z) \hat{\mu}_0(dz)$. Also, the first and second exponential terms on the right side of (2.2) are negligible since $\lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon/\varepsilon) = 0$. We also have that

$$\begin{aligned} & \left| \varepsilon \log \mathbb{E} \left[\exp \left[- \left(\frac{\delta_\varepsilon}{\varepsilon} \right)^2 \sum_{l=1}^d \int_0^{(\sqrt{\varepsilon}/\delta_\varepsilon)^2 T} (\hat{\sigma}_l \Psi^\varepsilon)(\hat{Y}_s^{x,\varepsilon}) dW_s^l \right] \right] \right| \\ & \leq \left| \varepsilon \left(\frac{\delta_\varepsilon}{\varepsilon} \right)^4 (\sqrt{\varepsilon}/\delta_\varepsilon)^2 \frac{T}{2} \sum_{l=1}^d \|\hat{\sigma}_l \Psi^\varepsilon\|_{C(\mathfrak{X})}^2 \right| \leq \left(\frac{\delta_\varepsilon}{\varepsilon} \right)^2 \frac{T}{2} \sum_{l=1}^d \|\hat{\sigma}_l \Psi^\varepsilon\|_{C(\mathfrak{X})}^2 \end{aligned}$$

for all $\varepsilon > 0$. Since $\lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon/\varepsilon) = 0$, this last expression tends to zero as ε tends to zero. Since Ψ^ε tends to Ψ^0 in $C^1(\mathfrak{X})$, $\overline{\lim}_{\varepsilon \rightarrow 0} \max_{1 \leq l \leq d} \|\hat{\sigma}_l \Psi^\varepsilon\|_{C(\mathfrak{X})} < \infty$, so $\lim_{\varepsilon \rightarrow 0} g_{T,x}^\varepsilon(\theta) = \langle \theta, x \rangle + T \int_{\mathfrak{X}} \Phi^0(z) \hat{\mu}_0(dz)$. It is clear from our calculations that this limit is uniform over all $x \in \mathbb{R}^d$ and all $T > 0$ in a compact subset of \mathbb{R}_+ .

Step 3—Large deviations. Hypotheses *i*) and *ii*) of Theorem 0.1 are clearly true. From the simple form $\mathcal{J}^{(1)}$ and the fact that A is strictly positive-definite, we know that *iii*) is true. Thus, the stated LDP holds for each $T > 0$ and $x \in \mathbb{R}^d$. The uniformity stems from the uniformity of the limit $\lim_{\varepsilon \rightarrow 0} g_{T,x}^\varepsilon(\theta)$. \square

We will need the uniformity of the large deviations principle in our later consideration of the trajectories of $X^{x,\varepsilon,\delta_\varepsilon}$.

Finally, we note that under some additional restrictions on the relative behavior of δ_ε and ε , Proposition 2.2 follows from a direct Laplace-type expansion using some heat kernel estimates of [24].

§3. REGIME 2

We here assume that $c \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon/\varepsilon)$ is well-defined and in $(0, \infty)$. It turns out that in this case we can almost directly apply the calculations of [2], [10, Section 7.2], or [13, Section 7.4.2] to the expression (1.5). In this case, the operator $\hat{\mathcal{L}}^{\varepsilon,\delta_\varepsilon}$ tends to the operator $\hat{\mathcal{L}}_0 + c\hat{\beta}_1$ as ε tends to zero. Define

$$\begin{aligned} \bar{\mathcal{J}}^{(2)}(\theta) \stackrel{\text{def}}{=} & \inf_{\phi \in C^\infty(\mathfrak{X})} \sup_{\mu \in \mathcal{P}(\mathfrak{X})} \int_{\mathfrak{X}} \left\{ \frac{1}{2} \sum_{l=1}^d \left(\left\langle \hat{A}_l(z), \theta \right\rangle - (\hat{\sigma}_l \phi)(z) \right)^2 \right. \\ & + \frac{1}{c} \left(\left\langle \hat{B}_0(z), \theta \right\rangle - (\hat{\mathcal{L}}_0 \phi)(z) \right) \\ & \left. + \left(\left\langle \hat{B}_1(z), \theta \right\rangle - (\hat{\beta}_1 \phi)(z) \right) \right\} \mu(dx) \quad \theta \in \mathbb{R}^d \quad (3.1) \end{aligned}$$

and let $\mathcal{J}^{(2)}$ be its Legendre-Fenchel transform;

$$\mathcal{J}^{(2)}(\theta) \stackrel{\text{def}}{=} \sup_{\theta' \in \mathbb{R}^d} \left\{ \langle \theta, \theta' \rangle - \bar{\mathcal{J}}^{(2)}(\theta') \right\}. \quad \theta \in \mathbb{R}^d$$

Again, $\mathcal{J}^{(2)}$ is clearly convex. Then we have

Theorem 3.1. *Fix $T > 0$ and $x \in \mathbb{R}^d$. The family $\{X_T^{x,\varepsilon,\delta_\varepsilon} : \varepsilon > 0\}$ of \mathbb{R}^d -valued random variables has a large deviations principle with rate function $I_{T,x}^{(2)}(z) \stackrel{\text{def}}{=} T\mathcal{J}^{(2)}\left(\frac{z-x}{T}\right)$ (all $z \in \mathbb{R}^d$).*

Proof. By [10, Section 7.2], $g_{T,x}$ is well-defined and equal to $\langle \theta, x \rangle + T\bar{\mathcal{J}}^{(2)}(\theta)$ for all $T > 0$, $x \in \mathbb{R}^d$, and $\theta \in \mathbb{R}^d$. This and (1.6) imply that in Theorem 0.1, hypotheses *i)*, *ii)*, and the first requirement of hypothesis *iii)* are true. For every $\theta \in \mathbb{R}^d$, $\bar{\mathcal{J}}^{(2)}(\theta)$ is the first eigenvalue of the operator

$$\frac{1}{c^2}\hat{\mathcal{L}}_0 + \frac{1}{c} \left\{ \beta_1 + \sum_{l=1}^d \langle \hat{A}_l, \theta \rangle \hat{\sigma}_l \right\} + \left\{ \frac{1}{2} \sum_{l=1}^d \langle \hat{A}_l, \theta \rangle^2 + \left\langle \frac{1}{c}\hat{B}_0 + \hat{B}_1, \theta \right\rangle \right\}.$$

By standard results [17, Ch. VIII], it follows that $\bar{\mathcal{J}}^{(2)}$ and thus $g_{T,x}$ is differentiable. The $g_{T,x}^\varepsilon$'s are convex in θ (since they are logarithmic moment generating functions), so $g_{T,x}$ is also convex. Thus, for all $\theta \neq 0$,

$$\begin{aligned} \|\nabla g_{T,x}(\theta)\| &= \sup_{\substack{\zeta \in \mathbb{R}^d \\ \|\zeta\|=1}} \langle \nabla g_{T,x}(\theta), \zeta \rangle \geq \left\langle \nabla g_{T,x}(\theta), \frac{\theta}{\|\theta\|} \right\rangle \\ &\geq \left\langle \frac{\partial}{\partial r} \Big|_{\theta}, \nabla g_{T,x} \right\rangle \geq \frac{g_{T,x}(\theta) - g_{T,x}(0)}{\|\theta\|}. \end{aligned}$$

By (1.6), we see that the remaining requirement of hypothesis *iii)* of Theorem 1.1 is also true. \square

We will also need to know that $X^{x,\varepsilon,\delta_\varepsilon}$ is “exponentially tight” in $C([0, T]; \mathbb{R}^d)$ for each $T > 0$. We will use this in §5.

Proposition 3.2. *For any fixed $T > 0$, $x \in \mathbb{R}^d$, and $\alpha \in (0, 1/2)$,*

$$\overline{\lim}_{L \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \|X^{x,\varepsilon,\delta_\varepsilon}\|_{C^\alpha([0,T]; \mathbb{R}^d)} \geq L \right\} = -\infty.$$

Proof. We can write $X_t^{x,\varepsilon,\delta_\varepsilon} = x + \sqrt{\varepsilon}M_t^{x,\varepsilon} + S_t^{x,\varepsilon}$ for all $t \geq 0$, where $M_t^{x,\varepsilon} \stackrel{\text{def}}{=} \sum_{l=1}^d \int_0^t A_l \left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) dW_s^l$ and $S_t^{x,\varepsilon} \stackrel{\text{def}}{=} \int_0^t B^{\varepsilon,\delta_\varepsilon} \left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) ds$ for all $t \geq 0$. By standard calculations,

$$\overline{\lim}_{L \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \sqrt{\varepsilon} \|M^{x,\varepsilon}\|_{C^\alpha([0,T]; \mathbb{R}^d)} \geq L \right\} = -\infty \quad (3.2)$$

Now $\|S^{x,\varepsilon}\|_{C^\alpha([0,T]; \mathbb{R}^d)} \leq \|B^{\varepsilon,\delta_\varepsilon}\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)}(T + T^{1-\alpha})$ for all $\varepsilon > 0$, and

$$\overline{\lim}_{\varepsilon \rightarrow 0} \|B^{\varepsilon,\delta_\varepsilon}\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)} \leq \frac{1}{c} \|B_0\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)} + \|B_1\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)}. \quad (3.3)$$

Clearly, then

$$\begin{aligned} \overline{\lim}_{L \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \|X^{x,\varepsilon,\delta_\varepsilon}\|_{C^\alpha([0,T]; \mathbb{R}^d)} \geq L \right\} \\ \leq \overline{\lim}_{L \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \sqrt{\varepsilon} \|M^{x,\varepsilon}\|_{C^\alpha([0,T]; \mathbb{R}^d)} \geq L - \|x\| - \|B^{\varepsilon,\delta_\varepsilon}\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)}(T + T^{1-\alpha}) \right\} \end{aligned}$$

and the result follows from this and (3.2). \square

§4. REGIME 3

We finally assume that $\lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon / \varepsilon) = \infty$. Here ε tends to zero much more quickly than δ_ε and, in this case, the large deviations principle for (0.1) is effectively given by first finding the large deviations principle for $X^{x, \varepsilon, \delta}$ with δ fixed and then letting δ tend to zero. Here, $\lim_{\varepsilon \rightarrow 0} B^{\varepsilon, \delta_\varepsilon} = B_1$. For each $z \in \mathbb{R}^d$, let $a^{-1}(z)$ be the matrix inverse of $a(z)$; since κ of (1.1) is positive, a^{-1} is well-defined on all of \mathbb{R}^d . For each $z \in \mathbb{R}^d$, define the norm $\|\theta\|_{a^{-1}(z)} \stackrel{\text{def}}{=} \sqrt{\langle \theta, a^{-1}(z)\theta \rangle}$ for all $\theta \in \mathbb{R}^d$. For each $L > 0$, now define a ‘‘quasipotential’’ $V_L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ as

$$V_L(z, z') \stackrel{\text{def}}{=} \inf_{\substack{\varphi \in C^1([0, L]; \mathbb{R}^d) \\ \varphi(0) = z \\ \varphi(L) = z'}} \frac{1}{2} \int_0^L \|\dot{\varphi}(s) - B_1(\varphi(s))\|_{a^{-1}(\varphi(s))}^2 ds \quad z, z' \in \mathbb{R}^d$$

(see [13]). Next define $\mathcal{J}^{(3)} : \mathbb{R}^d \rightarrow [0, \infty)$ as

$$\mathcal{J}^{(3)}(z) \stackrel{\text{def}}{=} \lim_{L \rightarrow \infty} \frac{1}{L} V_L(0, Lz). \quad z \in \mathbb{R}^d$$

Elementary subadditivity considerations reveal that $V(z)$ is indeed well-defined and finite for all $z \in \mathbb{R}^d$. We have

Theorem 4.1. *Fix $T > 0$ and $x \in \mathbb{R}^d$. The family $\{X_T^{x, \varepsilon, \delta_\varepsilon} : \varepsilon > 0\}$ of \mathbb{R}^d -valued random variables has a large deviations principle with rate function $I_{T, x}^{(3)}(z) \stackrel{\text{def}}{=} T \mathcal{J}^{(3)}\left(\frac{z-x}{T}\right)$ (all $z \in \mathbb{R}^d$).*

Note that our definition here is more direct than the definitions of the action functionals in Regimes 1 and 2; we do not define the action functional via a Legendre-Fenchel transform. This is reflected in the fact that our proof of Theorem 4.1 is a direct analysis of the trajectories of $X^{x, \varepsilon, \delta_\varepsilon}$, rather than an analysis of the logarithmic moment generating function $g_{T, x}$. Again, it is sufficient to look first at $T = 1$ and $x \in \mathbb{R}^d$ fixed.

We begin with some deterministic results about $\mathcal{J}^{(3)}$.

Lemma 4.2. *The function $\mathcal{J}^{(3)}$ is continuous, convex, and there are positive α and β such that $\frac{1}{\alpha} \|z\|^2 - \beta \leq \mathcal{J}^{(3)}(z) \leq \alpha \|z\|^2 + \beta$ for all $z \in \mathbb{R}^d$. Hence for each $s \geq 0$, the set $\{z \in \mathbb{R}^d : \mathcal{J}^{(3)}(z) \leq s\}$ is a compact subset of \mathbb{R}^d .*

Proof. The continuity and bounds on $\mathcal{J}^{(3)}$ are left to the reader. In light of the continuity of $\mathcal{J}^{(3)}$, convexity will follow if we can show that

$$\mathcal{J}^{(3)}\left(\frac{k}{m}x + \frac{m-k}{m}y\right) \leq \frac{k}{m} \mathcal{J}^{(3)}(x) + \frac{m-k}{m} \mathcal{J}^{(3)}(y) \quad (4.1)$$

for all k and m in \mathbb{Z} with $0 < k \leq m$ and all x and y in \mathbb{R}^d with rational coordinates (i.e., x and y in \mathbb{Q}^d). Fix such a k , m , x , and y and set $z \stackrel{\text{def}}{=} \frac{k}{m}x + \frac{m-k}{m}y$. Fix $\eta > 0$ and choose $L > 0$ such that $V_L(0, Lx) \leq L(\mathcal{J}^{(3)}(x) + \eta)$ and $V_L(0, Ly) \leq L(\mathcal{J}^{(3)}(y) + \eta)$ and such that Lx and Ly are in \mathbb{Z}^d . Thus, there exists a φ_1 and φ_2 in $C^1([0, L]; \mathbb{R}^d)$ such that $\varphi_1(0) = \varphi_2(0) = 0$, $\varphi_1(L) = Lx$, $\varphi_2(L) = Ly$, and

$$\begin{aligned} \frac{1}{2} \int_0^L \|\dot{\varphi}_1(s) - B_1(\varphi_1(s))\|_{a^{-1}(\varphi_1(s))}^2 ds &\leq L(\mathcal{J}^{(3)}(x) + 2\eta) \\ \frac{1}{2} \int_0^L \|\dot{\varphi}_2(s) - B_1(\varphi_2(s))\|_{a^{-1}(\varphi_2(s))}^2 ds &\leq L(\mathcal{J}^{(3)}(y) + 2\eta). \end{aligned} \quad (4.2)$$

Multiply z by mL ; thus $mLz = kLx + (m - k)Ly$. Fix now any large integer M . Let's construct a curve $\psi \in C([0, MmL])$ which is piecewise C^1 such that $\psi(0) = 0$, $\psi(MmL) = MmLz$, and which consists of Mk copies of φ_1 , laid end to end, followed by $M(m - k)$ copies of φ_2 , laid end to end. In other words,

$$\begin{aligned}\psi(0) &= 0 \\ \psi(s) &= \psi(L(\lceil s/L \rceil - 1)) + \varphi_1(s - L(\lceil s/L \rceil - 1)) \quad 0 < s \leq MkL \\ \psi(s) &= \psi(L(\lceil s/L \rceil - 1)) + \varphi_2(s - L(\lceil s/L \rceil - 1)), \quad MkL < s \leq MmL\end{aligned}$$

where $\lceil \cdot \rceil$ is the integer ceiling function. Using the periodicity of B_1 and a^{-1} (and in particular the fact that the integrals of (4.2) are invariant under translations by \mathbb{Z}^d), we see that

$$\begin{aligned}\frac{1}{2} \int_0^{MmL} \left\| \dot{\psi}(s) - B_1(\psi(s)) \right\|_{a^{-1}(\psi(s))}^2 ds \\ = Mk \frac{1}{2} \int_0^L \left\| \dot{\varphi}_1(s) - B_1(\varphi_1(s)) \right\|_{a^{-1}(\varphi_1(s))}^2 ds \\ + M(m - k) \int_0^L \left\| \dot{\varphi}_2(s) - B_1(\varphi_2(s)) \right\|_{a^{-1}(\varphi_2(s))}^2 ds \\ \leq MkL \left(\mathcal{J}^{(3)}(x) + 2\eta \right) + M(m - k)L \left(\mathcal{J}^{(3)}(y) + 2\eta \right).\end{aligned}$$

Thus,

$$\frac{1}{MmL} V_L(0, MmLz) \leq \frac{k}{m} \left(\mathcal{J}^{(3)}(x) + 2\eta \right) + \frac{m - k}{m} \left(\mathcal{J}^{(3)}(y) + 2\eta \right),$$

and (4.1) follows by letting M tend to infinity and then η tend to zero. \square

Let's next prove the large deviations lower bound; this will turn out to be a simple modification of standard calculations. For each $z \in \mathbb{R}^d$, let $M_z \in \mathbb{L}(\mathbb{R}^d; \mathbb{R}^d)$ be defined as

$$M_z(\theta) \stackrel{\text{def}}{=} \sum_{l=1}^d \theta_l A_l(z). \quad (\theta_1, \theta_2 \dots \theta_d) \in \mathbb{R}^d \quad (4.3)$$

Note that $\|M_z^{-1}(\theta)\| = \|\theta\|_{a^{-1}(z)}$ for all $\theta \in \mathbb{R}^d$. Let's collect together some regularity results. Observe that there are constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, and κ_5 , such that

$$\begin{aligned}\|\theta\|_{a^{-1}(z)}^2 &\leq \|\theta\|_{a^{-1}(z')}^2 (1 + \kappa_1 \|z - z'\|), & \|\theta\|_{a^{-1}(z)}^2 &\leq \kappa_2 \|\theta\|_{a^{-1}(z')}^2 \\ \|\theta\|_{a^{-1}(z)}^2 &\leq \kappa_3 \|\theta\|^2, & \|M_z(\theta)\|^2 &\leq \kappa_4 \|\theta\|^2 \\ \|B_0(z) - B_0(z')\| &\leq \kappa_5 \|z - z'\|\end{aligned} \quad (4.4)$$

for all z, z' , and θ in \mathbb{R}^d .

Proposition 4.3. *For each open subset G of \mathbb{R}^d ,*

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ X_1^{x, \varepsilon, \delta_\varepsilon} \in G \right\} \geq - \inf_{z \in G} \bar{\mathcal{J}}^{(3)}(z - x). \quad (4.5)$$

Proof. Fix $z \in G$. Fix also $L > 0$ and let $\varepsilon' > 0$ be small enough that G contains the set $\{z' \in \mathbb{R}^d : \|z' - z\| \leq L\delta_{\varepsilon'}\}$. Choose any $\varphi \in C^1([0, 1]; \mathbb{R}^d)$ such that $\varphi(0) = x$ and $\varphi(1) = z$. Fix $0 < \varepsilon \leq \varepsilon'$. Then

$$\begin{aligned} \mathbb{P} \left\{ X_1^{x, \varepsilon, \delta_\varepsilon} \in G \right\} &\geq \mathbb{P} \left\{ \|X_1^{x, \varepsilon, \delta_\varepsilon} - z\| \leq L\delta_\varepsilon \right\} \\ &\geq \mathbb{P} \left\{ \|X^{x, \varepsilon, \delta_\varepsilon} - \varphi\|_{C([0, 1]; \mathbb{R}^d)} \leq L\delta_\varepsilon \right\} \geq \mathbb{P} \left\{ \|\check{X}\|_{C([0, 1]; \mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon}) \right\}, \end{aligned}$$

where $\check{X}_t \stackrel{\text{def}}{=} \frac{1}{\sqrt{\varepsilon}}(X_t^{x, \varepsilon, \delta_\varepsilon} - \varphi(t))$ for all $0 \leq t \leq 1$. If we set

$$\begin{aligned} \xi(t) &\stackrel{\text{def}}{=} M_{X_t^{x, \varepsilon, \delta_\varepsilon}/\delta_\varepsilon}^{-1} \left(\dot{\varphi}(t) - B^{\varepsilon, \delta_\varepsilon} \left(\frac{X_t^{x, \varepsilon, \delta_\varepsilon}}{\delta_\varepsilon} \right) \right) \quad 0 \leq t \leq 1 \\ \check{W}_t &\stackrel{\text{def}}{=} W_t - \frac{1}{\sqrt{\varepsilon}} \int_0^t \xi(s) ds, \end{aligned}$$

we have that

$$\check{X}_t = \sum_{l=1}^d \int_0^t A_l \left(\frac{X_s^{x, \varepsilon, \delta_\varepsilon}}{\delta_\varepsilon} \right) d\check{W}_s. \quad 0 \leq t \leq 1 \quad (4.6)$$

Define next the measure $\check{\mathbb{P}}$ on (Ω, \mathcal{F}) as

$$\check{\mathbb{P}}(A) \stackrel{\text{def}}{=} \mathbb{E} \left[\chi_A \exp \left[\frac{1}{\sqrt{\varepsilon}} \sum_{l=1}^d \int_0^1 \xi^l(s) dW_s^l - \frac{1}{2\varepsilon} \int_0^1 \|\xi(s)\|^2 ds \right] \right]$$

for all $A \in \mathcal{F}$ and let $\check{\mathbb{E}}$ be the associated expectation operator. Then

$$\begin{aligned} \mathbb{P} \left\{ \|\check{X}\|_{C([0, 1]; \mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon}) \right\} &= \check{\mathbb{E}} \left[\chi_{\{\|\check{X}\|_{C([0, 1]; \mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \right. \\ &\quad \left. \times \exp \left[-\frac{1}{\sqrt{\varepsilon}} \sum_{l=1}^d \int_0^1 \xi(s) d\check{W}_s^l - \frac{1}{2\varepsilon} \int_0^1 \|\xi(s)\|^2 ds \right] \right]. \end{aligned}$$

Now note that

$$\left\{ \|\check{X}\|_{C([0, 1]; \mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon}) \right\} = \left\{ \sup_{0 \leq t \leq 1} \left\| \frac{X_t^{x, \varepsilon, \delta_\varepsilon}}{\delta_\varepsilon} - \frac{\varphi(t)}{\delta_\varepsilon} \right\| \leq L \right\},$$

and on this set $\frac{1}{2} \int_0^1 \|\xi(s)\|^2 ds \leq \check{V}(\varphi; \varepsilon, L)$, where

$$\check{V}(\varphi; \varepsilon, L) \stackrel{\text{def}}{=} \sup \left\{ \frac{1}{2} \int_0^1 \left\| \dot{\varphi}(t) - B^{\varepsilon, \delta_\varepsilon} \left(\frac{\varphi(t)}{\delta_\varepsilon} + \psi(t) \right) \right\|_{a^{-1}(\varphi(t)/\delta_\varepsilon + \psi(t))}^2 dt : \right. \\ \left. \|\psi\|_{C([0, 1]; \mathbb{R}^d)} \leq L \right\}$$

(recall the statement immediately following (4.3)). For any $\psi \in C([0, 1]; \mathbb{R}^d)$ with $\|\psi\|_{C([0, 1]; \mathbb{R}^d)} \leq L$,

$$\begin{aligned}
& \frac{1}{2} \left\| \dot{\varphi}(t) - B^{\varepsilon, \delta_\varepsilon} \left(\frac{\varphi(t)}{\delta_\varepsilon} + \psi(t) \right) \right\|_{a^{-1}(\varphi(t)/\delta_\varepsilon + \psi(t))}^2 \\
& \leq \frac{1}{2} (1 + \kappa_1 L) \left\| \dot{\varphi}(t) - B_1 \left(\frac{\varphi(t)}{\delta_\varepsilon} \right) \right. \\
& \quad \left. + B_1 \left(\frac{\varphi(t)}{\delta_\varepsilon} \right) - B^{\varepsilon, \delta_\varepsilon} \left(\frac{\varphi(t)}{\delta_\varepsilon} + \psi(t) \right) \right\|_{a^{-1}(\varphi(t)/\delta_\varepsilon)}^2 \\
& \leq \frac{1}{2} (1 + \kappa_1 L) (1 + \tilde{L}) \left\| \dot{\varphi}(t) - B_1 \left(\frac{\varphi(t)}{\delta_\varepsilon} \right) \right\|_{a^{-1}(\varphi(t)/\delta_\varepsilon)}^2 dt + \omega(\varepsilon, L, \tilde{L})
\end{aligned} \tag{4.7}$$

for all $0 \leq t \leq 1$, where

$$\begin{aligned}
\omega(\varepsilon, L, \tilde{L}) & \stackrel{\text{def}}{=} \frac{1}{2} (1 + \kappa_1 L) (1 + 1/\tilde{L}) \kappa_3 \\
& \quad \times \sup_{\substack{y, y' \in \mathbb{R}^d \\ \|y - y'\| \leq L}} \left(\frac{\varepsilon}{\delta_\varepsilon} \|B_0(y)\| + \|B_1(y) - B_1(y')\| + \|B_2^{\varepsilon, \delta_\varepsilon}(y)\| \right)^2.
\end{aligned}$$

The last line of (4.7) uses a simple application of Young's inequality. Thus,

$$\check{V}(\varphi; \varepsilon, L) \leq (1 + \kappa_1 L) (1 + \tilde{L}) \frac{1}{2} \int_0^1 \left\| \dot{\varphi}(t) - B_1 \left(\frac{\varphi(t)}{\delta_\varepsilon} \right) \right\|_{a^{-1}(\varphi(t)/\delta_\varepsilon)}^2 dt + \omega(\varepsilon, L, \tilde{L}). \tag{4.8}$$

Observe that $\omega(\varepsilon, L, \tilde{L})$ does not depend on our choice of x or z in \mathbb{R}^d , or our choice of $\varphi \in C^1([0, 1]; \mathbb{R}^d)$ and that $\lim_{L \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \omega(\varepsilon, L, \tilde{L}) = 0$ for every $\tilde{L} > 0$.

Some calculations as in [28, Proposition 3 of Sec. 4] now show that

$$\begin{aligned}
\mathbb{P}\{X_1^{x, \varepsilon, \delta_\varepsilon} \in G\} & \geq \exp \left[-\frac{\check{V}(\varphi; \varepsilon, L)}{\varepsilon} \right] \\
& \quad \times \exp \left[-\frac{1}{\sqrt{\varepsilon}} \frac{\check{\mathbb{E}} \left[\left| \sum_{l=1}^d \int_0^1 \xi^l(s) dW_s^l \right| \chi_{\{\|\check{X}\|_{1,0} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \right]}{\check{\mathbb{P}}\{\|\check{X}\|_{C([0, 1]; \mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \right] \\
& \quad \times \check{\mathbb{P}}\{\|\check{X}\|_{C([0, 1]; \mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}. \tag{4.9}
\end{aligned}$$

The following result is proved in Appendix B:

Lemma 4.4. *Let $\kappa_6 \geq d\kappa_3\kappa_4$ be an even number. Define*

$$\check{h}(\bar{L}) \stackrel{\text{def}}{=} \left(\frac{2e^{-1/2}}{\sqrt{2\pi}} \min\{1, \bar{L}/\sqrt{\kappa_4\kappa_6}\} \right)^{\kappa_6}. \quad \bar{L} > 0$$

Then for each $\bar{L} > 0$, $\check{\mathbb{P}}\{\|\check{X}\|_{C([0, T]; \mathbb{R}^d)} \leq \bar{L}\} \geq \check{h}(\bar{L})$.

Proof. The proof is in Appendix B. \square

In (4.9), we can use the Burkholder-Davis-Gundy inequality to get that

$$\begin{aligned}
& \frac{\check{\mathbb{E}} \left[\left| \sum_{l=1}^d \int_0^1 \xi^l(s) dW_s^l \right| \chi_{\{\|\check{X}\|_{1,0} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \right]}{\check{\mathbb{P}}\{\|\check{X}\|_{1,0} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \\
& \leq \left(\frac{\check{\mathbb{E}} \left[\left| \sum_{l=1}^d \int_0^1 \xi^l(s) dW_s^l \right|^{\kappa_6} \chi_{\{\|\check{X}\|_{C([0,1];\mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \right]}{\check{\mathbb{P}}\{\|\check{X}\|_{C([0,1];\mathbb{R}^d)} \leq L(\delta_\varepsilon/\sqrt{\varepsilon})\}} \right)^{1/\kappa_6} \\
& \leq \frac{\varsigma_{\kappa_6/2}^{1/\kappa_6} \sqrt{2\check{V}(\varphi; \varepsilon, L)}}{\{\check{h}(L(\delta_\varepsilon/\sqrt{\varepsilon}))\}^{1/\kappa_6}}
\end{aligned}$$

where $\varsigma_{\kappa_6/2}$ is the Burkholder-Davis-Gundy constant for $\kappa_6/2$ [20, Theorem 3.3.28], [27, Ch. 4.4].

Finally, we put everything together, rescale the integral on the right of (4.8), and vary φ (over all $\varphi \in C^1([0,1]; \mathbb{R}^d)$ such that $\varphi(0) = x$ and $\varphi(1) = z$). We get that

$$\begin{aligned}
& \mathbb{P}\{X_1^{x,\varepsilon,\delta_\varepsilon} \in G\} \\
& \geq \exp \left[-\frac{1}{\varepsilon} \left((1 + \kappa_1 L)(1 + \tilde{L})\delta_\varepsilon V_{1/\delta_\varepsilon}(x/\delta_\varepsilon, z/\delta_\varepsilon) + \omega(\varepsilon, L, \tilde{L}) \right) \right. \\
& \quad \left. - \frac{\varsigma_{\kappa_6/2}^{1/\kappa_6} \sqrt{2 \left((1 + \kappa_1 L)(1 + \tilde{L})\delta_\varepsilon V_{1/\delta_\varepsilon}(x/\delta_\varepsilon, z/\delta_\varepsilon) + \omega(\varepsilon, L, \tilde{L}) \right)}}{\sqrt{\varepsilon} \{\check{h}(L(\delta_\varepsilon/\sqrt{\varepsilon}))\}^{1/\kappa_6}} \right] \\
& \quad \times \check{h}(L(\delta_\varepsilon/\sqrt{\varepsilon})). \quad (4.10)
\end{aligned}$$

Let's now let ε tend to zero. Simple calculations show that

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon V_{1/\delta_\varepsilon}(x/\delta_\varepsilon, z/\delta_\varepsilon) = \bar{\mathcal{J}}^{(3)}(z - x) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\sqrt{\varepsilon} \{\check{h}(L(\delta_\varepsilon/\sqrt{\varepsilon}))\}^{1/\kappa_6}} = 0.$$

Thus, letting first ε , then L , and then \tilde{L} tend to zero in (4.10), we get that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{X_1^{x,\varepsilon,\delta_\varepsilon} \in G\} \geq -\bar{\mathcal{J}}^{(3)}(z - x).$$

Vary finally z over G ; we get exactly (4.5). \square

Next let's prove the large deviations upper bound. We shall do this by appealing to the bounds of [25]. For each $\varepsilon > 0$, let $\mathcal{L}^{\varepsilon,\delta_\varepsilon,*}$ be the adjoint of $\mathcal{L}^{\varepsilon,\delta_\varepsilon}$ with respect to Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and let $\{P_t^\varepsilon : t \geq 0\}$ be the semigroup on $B(\mathbb{R}^d)$ defined by $\mathcal{L}^{\varepsilon,\delta_\varepsilon,*}$, where $B(\mathbb{R}^d)$ is the Banach space of bounded and measurable functions on \mathbb{R}^d . Then, by virtue of the periodicity of the A_l 's and the positivity of κ of (1.1), for each $\varepsilon > 0$ there is a $p^\varepsilon \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \sim \{(0, z, z) : z \in \mathbb{R}^d\})$ such that

$$(P_t^\varepsilon \phi)(z) = \int_{\mathbb{R}^d} p^\varepsilon(t, z, y) \phi(y) dy, \quad t \geq 0, z \in \mathbb{R}^d, f \in B(\mathbb{R}^d)$$

and thus

$$\mathbb{P}\{X_t^{x,\varepsilon,\delta_\varepsilon} \in A\} = \int_{A/\delta_\varepsilon} p^\varepsilon((\sqrt{\varepsilon}/\delta_\varepsilon)^2 t, z, x/\delta_\varepsilon) dz. \quad t \geq 0, A \in \mathcal{B}(\mathbb{R}^d) \quad (4.11)$$

Let's begin by reconciling the notation of [25] with ours. We first reverse the direction of time since the semigroups of [25] evolve backwards; for each $\varepsilon > 0$, let $\{\hat{P}^\varepsilon : t \leq 0\}$ be the semigroup on $B(\mathbb{R}^d)$ defined as $\hat{P}_t^\varepsilon \phi \stackrel{\text{def}}{=} P_{-t}^\varepsilon \phi$ for all $t \leq 0$ and $\phi \in B(\mathbb{R}^d)$. Fix $\phi \in B(\mathbb{R}^d)$ and define, for each $\varepsilon > 0$, $\phi_t^\varepsilon \stackrel{\text{def}}{=} \hat{P}_t^\varepsilon \phi$ for all $t \leq 0$. Fix also $\psi \in C_c^2(\mathbb{R}^d)$. Then for each $t \leq 0$,

$$\int_{\mathbb{R}^d} \psi(z) \dot{\phi}_t^\varepsilon(z) dz = - \int_{\mathbb{R}^d} \psi(z) (\mathcal{L}^{\varepsilon,\delta_\varepsilon,*} \phi_t^\varepsilon)(z) dz = - \int_{\mathbb{R}^d} (\mathcal{L}^{\varepsilon,\delta_\varepsilon} \psi)(z) \phi_t^\varepsilon(z) dz$$

and we want to rewrite this as

$$\int_{\mathbb{R}^d} \psi(z) \dot{\phi}_t^\varepsilon(z) dz = \int_{\mathbb{R}^d} \left\{ \langle \nabla \psi, \underline{a}^\varepsilon \nabla \phi_t^\varepsilon \rangle - \psi \langle \underline{b}^\varepsilon, \underline{a}^\varepsilon \nabla \phi_t^\varepsilon \rangle - \phi_t^\varepsilon \langle \hat{\underline{b}}^\varepsilon, \underline{a}^\varepsilon \nabla \psi \rangle - \underline{c}^\varepsilon \psi \phi_t^\varepsilon \right\}(z) dz \quad (4.12)$$

for some appropriate $d \times d$ matrix-valued function $\underline{a}^\varepsilon$ on \mathbb{R}^d , some $\underline{b}^\varepsilon$ and $\hat{\underline{b}}^\varepsilon$ mapping \mathbb{R}^d into itself, and some $\underline{c}^\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}^d$. We will take

$$\begin{aligned} \underline{a}^\varepsilon(z) &= \frac{1}{2} a(z) \underline{b}^\varepsilon(z) \stackrel{\text{def}}{=} - \left(\frac{\delta_\varepsilon}{\varepsilon} \right) a^{-1}(z) B^{\varepsilon,\delta_\varepsilon}(z) \\ \hat{\underline{b}}^\varepsilon(z) &\stackrel{\text{def}}{=} \left(\frac{\delta_\varepsilon}{\varepsilon} \right) a^{-1}(z) B^{\varepsilon,\delta_\varepsilon}(z) - a^{-1}(z) (\text{div } a)^T(z) \quad z \in \mathbb{R}^d \\ \underline{c}^\varepsilon(z) &\stackrel{\text{def}}{=} - \frac{1}{2} \left(\frac{\delta_\varepsilon}{\varepsilon} \right) (\text{div } B^{\varepsilon,\delta_\varepsilon})(z). \end{aligned} \quad (4.13)$$

Here $\text{div } a$ is of course the row vector whose elements are the divergences of the corresponding columns of a . We leave it to the reader to verify that (4.12) is true with these definitions. It is easy to see that $\underline{a}^\varepsilon$ can only be chosen as above; however, $\underline{b}^\varepsilon$ and $\hat{\underline{b}}^\varepsilon$ are *not* uniquely specified by (4.12). This turns out to be important (see (4.16) below).

We now appeal to [25]. For each $\varepsilon > 0$, define

$$\begin{aligned} \mathcal{E}^\varepsilon(t, z, y) &\stackrel{\text{def}}{=} \inf_{\substack{\varphi \in C^1([t,0]; \mathbb{R}^d) \\ \varphi(t)=z \\ \varphi(0)=y}} \frac{1}{4} \int_t^0 \left\langle \dot{\varphi}(s) - \left(\underline{a}^\varepsilon(\underline{b}^\varepsilon - \hat{\underline{b}}^\varepsilon) \right) (\varphi(s)), \right. \\ &\quad \left. \left(\underline{a}^\varepsilon \right)^{-1}(\varphi(s)) \left(\dot{\varphi}(s) - \left(\underline{a}^\varepsilon(\underline{b}^\varepsilon - \hat{\underline{b}}^\varepsilon) \right) (\varphi(s)) \right) \right\rangle ds. \end{aligned} \quad t < 0, z, y \in \mathbb{R}^d \quad (4.14)$$

Then there is a $C > 0$ such that

$$\begin{aligned} &p^\varepsilon(-t, z, y) \\ &\leq C \left(\frac{1 + (-t) \sup_{w \in \mathbb{R}^d} \left\{ \langle \underline{b}^\varepsilon, \underline{a}^\varepsilon \underline{b}^\varepsilon \rangle + \langle \hat{\underline{b}}^\varepsilon, \underline{a}^\varepsilon \hat{\underline{b}}^\varepsilon \rangle + |\underline{c}^\varepsilon| \right\} (w) + \mathcal{E}^\varepsilon(t, z, y)}{-t} \right)^{d/2} \\ &\quad \times \exp \left[-\mathcal{E}^\varepsilon(t, z, y) + \frac{(-t)}{4} \sup_{w \in \mathbb{R}^d} \left\{ \langle \underline{b}^\varepsilon + \hat{\underline{b}}^\varepsilon, \underline{a}^\varepsilon(\underline{b}^\varepsilon + \hat{\underline{b}}^\varepsilon) \rangle + \underline{c}^\varepsilon \right\} (w) \right]. \end{aligned} \quad t < 0, z, y \in \mathbb{R}^d \quad (4.15)$$

Let's now simplify some of this. Note that

$$\underline{a}^\varepsilon(\underline{b}^\varepsilon - \hat{\underline{b}}^\varepsilon) = -\frac{\delta_\varepsilon}{\varepsilon} B^{\varepsilon, \delta_\varepsilon} + \frac{1}{2}(\operatorname{div} a)^T, \quad \text{and} \quad \underline{b}^\varepsilon + \hat{\underline{b}}^\varepsilon = -\frac{1}{2}a^{-1}(\operatorname{div} a)^T. \quad (4.16)$$

The second equality is the crucial one which in fact guided our choice of $\underline{b}^\varepsilon$ and $\hat{\underline{b}}^\varepsilon$; although $\underline{b}^\varepsilon$ and $\hat{\underline{b}}^\varepsilon$ grow like $\frac{\delta_\varepsilon}{\varepsilon}$ as ε tends to zero, $\underline{b}^\varepsilon + \hat{\underline{b}}^\varepsilon$ is bounded as ε tends to zero, and thus the second exponential term in (4.15) grows in a manageable way as ε tends to zero. Using (4.13) and (4.16), we can rewrite (4.14) as

$$\begin{aligned} & \mathcal{E}^\varepsilon(-t, z, y) \\ &= \inf_{\substack{\varphi \in C^1([0, t]; \mathbb{R}^d) \\ \varphi(0) = y \\ \varphi(t) = z}} \frac{1}{2} \int_0^t \left\| \dot{\varphi}(s) - \left(\frac{\delta_\varepsilon}{\varepsilon} \right) B^{\varepsilon, \delta_\varepsilon}(\varphi(s)) + \frac{1}{2} \Gamma(\varphi(s)) \right\|_{a^{-1}(\varphi(s))}^2 ds, \\ & \hspace{20em} t > 0, z, y \in \mathbb{R}^d \end{aligned}$$

and, using (4.16) and the growth rates of $\underline{b}^\varepsilon$, $\hat{\underline{b}}^\varepsilon$, and $\underline{c}^\varepsilon$ as ε tends to zero, we see that there is thus a second constant $C' > 0$ such that

$$\begin{aligned} p^\varepsilon(t, z, y) &\leq C' \left(\frac{1 + \left(\frac{\delta_\varepsilon}{\varepsilon} \right) \left(\frac{\delta_\varepsilon}{\varepsilon} + 1 \right) t + \mathcal{E}^\varepsilon(-t, z, y)}{t} \right)^{d/2} \\ &\quad \times \exp \left[-\mathcal{E}^\varepsilon(-t, z, y) + K \left(\frac{\delta_\varepsilon}{\varepsilon} + 1 \right) t \right]. \\ & \hspace{20em} t > 0, z, y \in \mathbb{R}^d \end{aligned}$$

The upper large deviations bound of Theorem 4.1 now follows from a simple Laplace-type bound;

Proposition 4.5. *For each closed subset F of \mathbb{R}^d ,*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{X_1^{x, \varepsilon, \delta_\varepsilon} \in F\} \leq - \inf_{z \in F} \bar{\mathcal{J}}^{(3)}(z - x). \quad (4.17)$$

Proof. From (4.11),

$$\begin{aligned} \mathbb{P}\{X_1^{x, \varepsilon, \delta_\varepsilon} \in F\} &= \int_{z \in F/\delta_\varepsilon} p^\varepsilon \left(\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon} \right)^2, z, \frac{x}{\delta_\varepsilon} \right) dz \\ &= \delta_\varepsilon^{-d} \int_{z \in F} p^\varepsilon \left(\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon} \right)^2, \frac{z}{\delta_\varepsilon}, \frac{x}{\delta_\varepsilon} \right) dz \\ &\leq C' \delta_\varepsilon^{-d} \int_{z \in F} \left(\frac{1 + \left(\frac{\delta_\varepsilon}{\varepsilon} \right) \left(\frac{\delta_\varepsilon}{\varepsilon} + 1 \right) \left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon} \right)^2 + \mathcal{E}^\varepsilon(-(\sqrt{\varepsilon}/\delta_\varepsilon)^2, z/\delta_\varepsilon, x/\delta_\varepsilon)}{(\sqrt{\varepsilon}/\delta_\varepsilon)^2} \right)^{d/2} \\ &\quad \times \exp \left[-\mathcal{E}^\varepsilon(-(\sqrt{\varepsilon}/\delta_\varepsilon)^2, z/\delta_\varepsilon, x/\delta_\varepsilon) + K(\sqrt{\varepsilon}/\delta_\varepsilon)^2 \left(\frac{\delta_\varepsilon}{\varepsilon} + 1 \right) \right] dz. \end{aligned}$$

for all $\varepsilon > 0$. A simple calculation (which uses the boundedness of Γ) reveals that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \mathcal{E}^\varepsilon(-(\sqrt{\varepsilon}/\delta_\varepsilon)^2, z/\delta_\varepsilon, x/\delta_\varepsilon) = \bar{\mathcal{J}}^{(3)}(z - x), \quad (4.18)$$

and that this limit is uniform for z in any compact subset of \mathbb{R}^d . Another simple calculation shows that for some positive α' and β' ,

$$\varepsilon \mathcal{E}^\varepsilon(-(\sqrt{\varepsilon}/\delta_\varepsilon)^2, z/\delta_\varepsilon, x/\delta_\varepsilon) \geq \alpha' \|z - x\|^2 - \beta' \quad (4.19)$$

for all z in \mathbb{R}^d , and all $\varepsilon > 0$. A Laplace-type argument using (4.18) and (4.19) shows that

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{X_1^{x, \varepsilon, \delta_\varepsilon} \in F\} &\leq - \inf_{z \in F} \bar{\mathcal{J}}^{(3)}(z - x) + \overline{\lim}_{\varepsilon \rightarrow 0} K \left(\frac{\varepsilon}{\delta_\varepsilon} + \left(\frac{\varepsilon}{\delta_\varepsilon} \right)^2 \right) \\ &= - \inf_{z \in F} \bar{\mathcal{J}}^{(3)}(z - x) \end{aligned}$$

which is exactly (4.17). \square

Combining the results of Lemma 4.2 Propositions 4.3 and 4.5, we get the large deviations result of Theorem 4.1 when $T = 1$ for any fixed $x \in \mathbb{R}^d$. Again, we get the result for any fixed $T > 0$ and $x \in \mathbb{R}^d$ by rescaling.

We also have

Proposition 4.6. *For any fixed $T > 0$, $x \in \mathbb{R}^d$, and $\alpha \in (0, 1/2)$,*

$$\overline{\lim}_{L \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \|X^{x, \varepsilon, \delta_\varepsilon}\|_{C^\alpha([0, T]; \mathbb{R}^d)} \geq L \right\} = -\infty.$$

Proof. Exactly the same as the proof of Proposition 3.2. Here we replace (3.3) with $\overline{\lim}_{\varepsilon \rightarrow 0} \|B^{\varepsilon, \delta_\varepsilon}\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)} \leq \|B_1\|_{C_p(\mathbb{R}^d; \mathbb{R}^d)}$. \square

The reader will naturally wonder if there is an alternate proof of Theorem 4.1 which uses Theorem 0.1. The reader will also wonder if $g_{T, x}$ here in Regime 3 is given by (assuming that $g_{T, x}$ exists) taking the limit of (3.1) in Regime 2 as c of (0.3) tends to infinity (see also [2, Theorem 1.25]). One can indeed show that for all $T > 0$ and $x \in \mathbb{R}^d$, $g_{T, x}$ and is given by $g_{T, x}(\theta) = \langle \theta, x \rangle + T \bar{\mathcal{J}}^{(3)}(\theta)$ where

$$\begin{aligned} \bar{\mathcal{J}}^{(3)}(\theta) \stackrel{\text{def}}{=} \inf_{\phi \in C^\infty(\mathfrak{X})} \sup_{\mu \in \mathcal{P}(\mathfrak{X})} \int_{\mathfrak{X}} \left\{ \frac{1}{2} \sum_{l=1}^d \left(\langle \hat{A}_l(z), \theta \rangle - (\hat{\sigma}_l \phi)(z) \right)^2 \right. \\ \left. + \left(\langle \hat{B}_1(z), \theta \rangle - (\hat{\beta}_1 \phi)(z) \right) \right\} \mu(dx). \quad \theta \in \mathbb{R}^d \end{aligned}$$

Unfortunately, we were not able to verify hypothesis *iii*) of Theorem 0.1, nor were we able to verify any weaker conditions (such as an analysis under the tilted measure) under which the conclusions of Theorem 0.1 hold. By [5, Lemma 2.2.9], we do, however, know that the Legendre-Fenchel transform of $\mathcal{J}^{(3)}$ must be $\bar{\mathcal{J}}^{(3)}$, and by now appealing to the convexity result of Lemma 4.2, we indeed have that $\mathcal{J}^{(3)}$ is the Legendre-Fenchel transform of $\bar{\mathcal{J}}^{(3)}$.

§5. LARGE DEVIATIONS IN PATH SPACE

The natural next step in our study of $X^{x,\varepsilon,\delta_\varepsilon}$ is an analysis of path-space large deviations; i.e., we need to look at the $C([0, T]; \mathbb{R}^d)$ -valued random variables $\{\{X_t^{x,\varepsilon,\delta_\varepsilon} : 0 \leq t \leq T\} : \varepsilon > 0\}$ for any $T > 0$. The large deviations in functional space should easily follow from the calculations of §2–4, the Markov property, and some regularity conditions. Throughout, we assume that $x \in \mathbb{R}^d$ and $T > 0$ are fixed and that we are in Regime i for some $i \in \{1, 2, 3\}$. If we are in Regime 1, we assume that Assumption 2.1 holds.

We start with the following obvious result:

Proposition 5.1. *Fix any positive integer m . Then the family*

$$\{(X_{T/m}^{x,\varepsilon,\delta_\varepsilon}, X_{2T/m}^{x,\varepsilon,\delta_\varepsilon}, \dots, X_T^{x,\varepsilon,\delta_\varepsilon}) : \varepsilon > 0\}$$

of $(\mathbb{R}^d)^m$ -valued random variables has a large deviations principle with rate function

$$I_m((y_1, y_2, \dots, y_m)) \stackrel{\text{def}}{=} \sum_{l=1}^m \mathcal{J}^{(i)}\left(\frac{y_l - y_{l-1}}{T/m}\right)(T/m), \quad (y_1, y_2, \dots, y_m) \in (\mathbb{R}^d)^m$$

where $y_0 \stackrel{\text{def}}{=} x$ in this formula.

Proof. We need to show two things (see [5, Exercise 2.1.14]):

a) that for every $(y_1, y_2, \dots, y_m) \in (\mathbb{R}^d)^m$,

$$\lim_{\eta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \sup_{1 \leq l \leq m} \|X_{lT/m}^{x,\varepsilon,\delta_\varepsilon} - y_l\| \leq \eta \right\} = -I((y_1, y_2, \dots, y_m))$$

b) that $\{(X_{T/m}^{x,\varepsilon,\delta_\varepsilon}, X_{2T/m}^{x,\varepsilon,\delta_\varepsilon}, \dots, X_T^{x,\varepsilon,\delta_\varepsilon}) : \varepsilon > 0\}$ is exponentially tight in $(\mathbb{R}^d)^m$.

Let's first prove a). By an easy calculation involving the Markov property,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \sup_{1 \leq l \leq m} \|X_{lT/m}^{x,\varepsilon,\delta_\varepsilon} - y_l\| \leq \eta \right\} = -I((y_1, y_2, \dots, y_m)) + \omega(\eta),$$

where

$$|\omega(\eta)| \leq \sum_{l=1}^m \sup_{\|z\| \leq 2\eta} \left| \mathcal{J}^{(i)}\left(\frac{y_l - y_{l-1} + z}{T/m}\right) - \mathcal{J}^{(i)}\left(\frac{y_l - y_{l-1}}{T/m}\right) \right|(T/m).$$

Clearly, $\lim_{\eta \rightarrow 0} \omega(\eta) = 0$ since $\mathcal{J}^{(i)}$ is continuous (Lemma A.1) This finishes the proof of a).

To prove b), note that for any $K > 0$,

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \sup_{1 \leq l \leq m} \|X_{lT/m}^{x,\varepsilon,\delta_\varepsilon}\| \geq K \right\} &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \left(\sum_{l=1}^m \mathbb{P} \left\{ \|X_{lT/m}^{x,\varepsilon,\delta_\varepsilon}\| \geq K \right\} \right) \\ &\leq \max_{1 \leq l \leq m} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P} \left\{ \|X_{lT/m}^{x,\varepsilon,\delta_\varepsilon}\| \geq K \right\}. \end{aligned}$$

Since $\lim_{\|z\| \rightarrow \infty} \mathcal{J}^{(i)}(z) = \infty$ (Lemma A.1), the family $\{X_t^{x,\varepsilon,\delta_\varepsilon} : \varepsilon > 0\}$ of \mathbb{R}^d -valued random variables is exponentially tight, and thus b) is true. \square

We can now easily guess at the path-space large deviations principle. We provide the proof in Appendix A, using exponential approximations.

Theorem 5.2. Fix $x \in \mathbb{R}^d$ and $T > 0$, and assume that we are in Regime i for some $i \in \{1, 2, 3\}$. If we are in Regime 1, we assume that Assumption 2.1 holds. The family $\{\{X_t^{x,\varepsilon,\delta_\varepsilon} : 0 \leq t \leq T\} : \varepsilon > 0\}$ of $C([0, T]; \mathbb{R}^d)$ -valued random variables has a large deviations principle with rate function

$$S_{0,T}^{(i)}(\varphi) \stackrel{\text{def}}{=} \begin{cases} \int_0^T \mathcal{J}^{(i)}(\dot{\varphi}(s)) ds & \text{if } \varphi \text{ is differentiable and } \varphi(0) = x \\ \infty & \text{else.} \end{cases} \quad (5.1)$$

for all $\varphi \in C([0, T]; \mathbb{R}^d)$.

Remark 5.3. The reader may wonder if we could also prove Theorem 5.2 by using projective limits (see [4, Ch. 4.6 and Theorem 5.1.2]). We were unable to do this since we were unable to show that $\{\{X_t^{x,\varepsilon,\delta_\varepsilon} : 0 \leq t \leq T\} : \varepsilon > 0\}$ is exponentially tight in $C([0, T]; \mathbb{R}^d)$ in Regime 1. In Regimes 2 and 3, Propositions 3.2 and 4.6 would have worked for the projective limit approach.

§6. WAVEFRONT PROPAGATION

Using the results of §2–4, we can at last consider the wavefront problem for (0.2). Again, we assume that we are in Regime i for some $i \in \{1, 2, 3\}$, and that if we are in Regime 1, we assume that Assumption 2.1 holds. Let V be defined by (0.4) with $S_{0,t}^{(i)}$ of (5.1) playing the role of $S_{0,t}$. Notice that since $\mathcal{J}^{(i)}$ is convex,

$$\inf_{\substack{\varphi \in C^1([0,t]; \mathbb{R}^d) \\ \varphi(0)=x \\ \varphi(t)=y}} \int_0^t \mathcal{J}^{(i)}(\dot{\varphi}(s)) ds = \mathcal{J}^{(i)}\left(\frac{y-x}{t}\right) \quad (6.1)$$

for all $t > 0$, x and y in \mathbb{R}^d . That the right side dominates the left side is obvious; take $\varphi(s) = x + \frac{s}{t}(y-x)$ for all $s \in [0, t]$. To see the opposite inequality, let α be any element of the subdifferential of $\mathcal{J}^{(i)}$ at $(y-x)/t$. Then for any $\varphi \in C^1([0, t]; \mathbb{R}^d)$ such that $\varphi(0) = x$ and $\varphi(t) = y$,

$$\int_0^t \mathcal{J}^{(i)}(\dot{\varphi}(s)) ds \geq t \mathcal{J}^{(i)}\left(\frac{y-x}{t}\right) + \int_0^t \left\langle \alpha, \dot{\varphi}(s) - \left(\frac{y-x}{t}\right) \right\rangle ds,$$

and the integral in the last expression is zero since $\varphi(0) = x$ and $\varphi(t) = y$. Thus, (6.1) holds. Since the minimizers of (6.1) are straight lines, it follows that for any $t > 0$, and $x \in \mathbb{R}^d$ such that $V(t, x) < 0$,

$$V(t, x) = \sup \left\{ f'(0)t - S_{0,t}^{(i)}(\varphi) : \varphi \in C([0, t]; \mathbb{R}^d), \varphi(0) = x, \varphi(t) \in G_0, \right. \\ \left. V(t-s, \varphi(s)) < 0 \text{ for all } 0 < s < t \right\}.$$

This allows us to copy the proof of Theorem 2.1 of [10, Ch. 6] to see that (0.5) holds. The analysis also shows us that (0.5) holds uniformly on the set

$$\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d : 0 \leq t \leq T, \|x\| \leq T, |V(t, x)| \geq \eta\}.$$

for any $T > 0$ and $\eta > 0$.

§7. SYSTEMS OF RDE'S

A. One can also consider homogenization and the wavefront propagation problem for systems of RDE's with periodic coefficients. For the sake of brevity, let us consider a system of the form

$$\begin{aligned} \frac{\partial u_k}{\partial t} &= \mathcal{L}_k u_k + \sum_{j=1}^n c_{k,j}(x)(u_j - u_k) + c_k(u_k)u_k & t > 0, x \in \mathbb{R}^d \\ & & 1 \leq k \leq n \end{aligned} \quad (7.1)$$

$$u_k(0, \cdot) = g_k.$$

Here, for each $1 \leq k, j \leq n$, $c_{k,j}(x) > 0$ for all $x \in \mathbb{R}^d$ and the strongly elliptic operators \mathcal{L}_k are given by

$$\mathcal{L}_k \stackrel{\text{def}}{=} \frac{1}{2} \sum_{1 \leq i, j \leq n} a_k^{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{1 \leq i \leq n} b_k^i(x) \frac{\partial}{\partial x_i}. \quad 1 \leq k \leq n$$

The nonlinear terms are assumed to be of KPP type; for all $1 \leq k \leq n$, c_k is continuous, $c_k(u) > 0$ for $u < 1$, $c_k(u) < 0$ for all $u > 1$, and $c_k(0) = \max_{u \geq 0} c_k(u)$; set $\mathfrak{c}_k \stackrel{\text{def}}{=} c_k(0)$ for all $1 \leq k \leq n$. The coefficients of the \mathcal{L}_k 's and the $c_{k,j}$'s are all assumed to be in $C_p^\infty(\mathbb{R}^d)$. We assume that for all $1 \leq k \leq n$, the set $G_0^k \stackrel{\text{def}}{=} \text{supp}(g_k)$ is compact and that its closure coincides with the closure of its interior. We set $G_0 \stackrel{\text{def}}{=} \cup_{k=1}^n G_0^k$. We can understand existence and uniqueness of these PDE's via an iterative Feynman-Kac formula, just as in [10].

Let $\{(X_t, \nu_t) : t \geq 0\}$ be the Markov process on $\mathbb{R}^d \times \{1, 2, \dots, n\}$ corresponding to the linear part of the system (7.1) (see [6]); i.e.,

$$\begin{aligned} dX_t &= \sum_{l=1}^d A_{l, \nu_t}(X_t) dW_t^l + B_{\nu_t}(X_t) dt & (7.2) \\ \mathbb{P}\{\nu_{t+\Delta} = j | \nu_t = i, X_t = x\} &= c_{i,j}(x)\Delta + o(\Delta) \quad \Delta \searrow 0 \end{aligned}$$

where for all $1 \leq k \leq n$, $\sum_{1 \leq l \leq d} A_{l,k}^i(x) A_{l,k}^j(x) = a_k^{i,j}(x)$ for all $1 \leq i, j \leq d$, and $B_k(x) = (b_k^1(x), b_k^2(x) \dots b_k^d(x))$. Then we can write down an equation for the solution of (7.1):

$$u_k(t, x) = \mathbb{E} \left[g_{\nu_t^{x,k}}(X_t^{x,k}) \exp \left[\int_0^t c_{\nu_s^{x,k}}(u_{\nu_s^{x,k}}(t-s, X_s^{x,k})) ds \right] \right],$$

for each $t > 0$ and $x \in \mathbb{R}^d$, where $\{(X_t^{x,k}, \nu_t^{x,k}) : t \geq 0\}$ is the solution of (7.2) which starts at the point (x, k) .

To describe the asymptotic behavior of the solution of (7.1) as t tends to infinity, define a family of operators $\{\mathcal{L}_k^\theta : \theta \in \mathbb{R}^d, 1 \leq k \leq n\}$, on $C_p^\infty(\mathbb{R}^d)$, as

$$(\mathcal{L}_k^\theta \varphi)(x) = (\mathcal{L}_k \varphi)(x) - \langle a_k(x)\theta, \nabla \varphi(x) \rangle, \quad \varphi \in C_p^\infty(\mathbb{R}^d), x \in \mathbb{R}^d$$

Consider the eigenvalue problem

$$\begin{aligned} (\mathcal{L}_k^\theta \varphi)(x, k) + \left[c_k(x) - \langle B_k(x), \theta \rangle + \frac{1}{2} \langle a_k(x)\theta, \theta \rangle \right] \varphi(x, k) \\ + \sum_{j=1}^n c_{k,j}(x)(\varphi(x, j) - \varphi(x, k)) = \lambda(\theta)\varphi(x, k), \quad 1 \leq k \leq n, x \in \mathbb{R}^d \end{aligned}$$

where θ is a parameter. Let $\lambda(\theta)$ be the eigenvalue corresponding to the positive eigenfunction. Such an eigenvalue exists and is simple thanks to the Frobenius theorem applied to the positive semigroup $\{\mathbf{Q}_t^\theta : t \geq 0\}$ on $C_p(\mathbb{R}^d)$ defined as

$$(\mathbf{Q}_t^\theta \psi)(x, k) \stackrel{\text{def}}{=} \mathbb{E} \left[\psi(X_t^{x,k}, \nu_t^{x,k}) \exp \left[\int_0^t \left[c_{\nu_s^{x,k}}(X_s^{x,k} - \langle B_{\nu_s^{x,k}}(X_s^{x,k}), \theta \rangle + \frac{1}{2} \langle a_{\nu_s^{x,k}}(X_s^{x,k}) \theta, \theta \rangle \right] ds \right] \right]$$

(compare this with Lemma 7.2.1 in [10]). The function λ is convex and it follows from standard perturbation theory that it is differentiable. Denote by H its Legendre-Fenchel transform;

$$H(z) \stackrel{\text{def}}{=} \sup_{\theta \in \mathbb{R}^d} \{ \langle z, \theta \rangle + \lambda(\theta) \}. \quad z \in \mathbb{R}^d$$

The following result can be proved in the same way as Theorem 7.3.1 of [10]:

Theorem 7.1. *For any compact subset F of $\{z \in \mathbb{R}^d : H(z) > 0\}$, we have that $\lim_{t \rightarrow \infty} u_k(t, tz) = 0$ uniformly for $z \in F$ and all $1 \leq k \leq n$. For any compact subset K of $\{z \in \mathbb{R}^d : H(z) < 0\}$, $\lim_{t \rightarrow \infty} u_k(t, tz) = 1$ uniformly for $z \in K$ and all $1 \leq k \leq n$.*

One can, of course, rescale the space and time as $x \rightarrow \frac{x}{\varepsilon}$ and $t \rightarrow \frac{t}{\varepsilon}$. Then Theorem 7.1 can be reformulated to describe the limiting behavior as ε tends to zero of the solution of a corresponding Cauchy problem with a small parameter. This case corresponds to Regime 2 considered in §3. One can consider the other regimes as well. Note, however, that in the case of systems, there are more ways to introduce several parameters, since the small diffusion parameter and the homogenization parameter may tend to zero at different rates for each equation of the system. This may, for example, be the natural setting in which to consider chemical reactions or diffusions which occur at different rates.

B. One can also consider RDE's with slowly-varying periodic coefficients. In the case of one equation without drift, this means that after the proper space-time rescaling, the RDE has the form

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2} \sum_{1 \leq i, j \leq d} a^{i,j} \left(\frac{x}{\varepsilon}, x \right) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{1}{\varepsilon} c(u^\varepsilon) u^\varepsilon \quad t > 0, x \in \mathbb{R}^d \quad (7.3)$$

$$u^\varepsilon(0, \cdot) = g,$$

where g and c satisfy the same assumptions as in §0, but now the $a^{i,j}$'s are functions on $\mathbb{R}^d \times \mathbb{R}^d$ and $x \mapsto a^{i,j}(x, x')$ is in $C_p(\mathbb{R}^d)$ for every $x' \in \mathbb{R}^d$. Again, the theory of these PDE's can be developed through the Feynman-Kac formula.

This situation should be compared with the corresponding results in [10, Ch. 6] and [12]; we will only indicate what the results should be. The solution u^ε of (7.3) tends to a step function with values 0 and 1 as ε tends to zero. The motion of the interface between where it tends to 0 and where it tends to 1 can again be described by a Huygen's principle (see, for example, [10, Ch. 6]). This Huygen's principle, is homogeneous and isotropic in a certain Finsler metric. Note that in the simplest

case, when $a^{i,j}(\frac{x}{\varepsilon}, x)$ in (7.3) in fact depends only on the second argument, then this Huygen's principle is with respect to the Riemannian metric corresponding to the form $ds^2 = \sum_{1 \leq i, j \leq d} a_{i,j}(x) dx^i dx^j$, where $(a_{i,j}(x)) = (a^{i,j}(x))^{-1}$. To be more precise, consider the eigenvalue problem

$$\begin{aligned} \frac{1}{2} \sum_{1 \leq i, j \leq d} a^{i,j}(x, y) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) - \langle a(x, y) \theta, \nabla \varphi(x) \rangle \\ + \left[c(x) + \frac{1}{2} \langle a(x, y) \theta, \theta \rangle \right] \varphi(x) = \lambda(y, \theta) \varphi(x), \quad x \in \mathbb{R}^d \end{aligned}$$

with y and θ in \mathbb{R}^d being parameters. Let $\lambda(y, \theta)$ be the eigenvalue corresponding to the positive eigenfunction, and set $H_y(z) \stackrel{\text{def}}{=} \sup_{\theta \in \mathbb{R}^d} \{ \langle z, \theta \rangle + \lambda(y, \theta) \}$ for all y and θ in \mathbb{R}^d . Let d be the Finsler metric on \mathbb{R}^d such that the unit sphere \mathfrak{S}_y at y is defined by $\mathfrak{S}_y \stackrel{\text{def}}{=} \{ z \in \mathbb{R}^d : H_y(z - y) = 0 \}$ for each $y \in \mathbb{R}^d$. Then $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x) = 1$ if $d(x, G_0) < t$ and $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x) = 0$ if $d(x, G_0) > t$.

It is interesting to note that if the nonlinear term also depends on the spatial variable, the motion of the front can be much more complicated. For instance, if the nonlinear term has the form $c(\frac{x}{\varepsilon}, u)u$ and $c(\frac{x}{\varepsilon}) \stackrel{\text{def}}{=} c(\frac{x}{\varepsilon}, 0)$ depends on $\frac{x}{\varepsilon}$, the front can have jumps. Some other generalizations could easily follow from the techniques of [12].

APPENDIX A. THE PROOF OF THEOREM 5.2.

Here we provide the details of passing from the finite-dimensional LDP of Proposition 5.1 to the path-space large deviations principle of Theorem 5.2. Essentially, we will approximate $X^{x, \varepsilon, \delta_\varepsilon}$ by piecewise-linear approximations. We will use [4, Theorem 4.2.16] to coordinate all of the details.

Let's start by collecting in one place some technical results about the $\mathcal{J}^{(i)}$'s; we used some of these results in proving Proposition 5.1.

Lemma A.1. *The function $\mathcal{J}^{(i)}$ is convex and continuous and there is an $\alpha > 0$ and $\beta > 0$ such that*

$$\frac{1}{\alpha} \|\theta\|^2 - \beta \leq \mathcal{J}^{(i)}(\theta) \leq \alpha \|\theta\|^2 + \beta. \quad \theta \in \mathbb{R}^d \tag{A.1}$$

Proof. The convexity of $\mathcal{J}^{(i)}$ in Regimes 1 and 2 was already pointed out; in those regimes $\mathcal{J}^{(i)}$ is defined as a Legendre-Fenchel transform. The convexity of $\mathcal{J}^{(3)}$ is contained in Lemma 4.2. Continuity follows from convexity. In Regime 1, (A.1) follows from the second equality of (2.1); in Regime 2, it follows from (1.6) and the fact that $\mathcal{J}^{(2)}$ is the Legendre-Fenchel transform of $g_{1,0}$. Lemma 4.2 contains the bounds of (A.1) for Regime 3. \square

Let's now get back to Theorem 5.2. Let's define, for each positive integer m , the family $\{\mathbf{X}^{m, \varepsilon} : \varepsilon > 0\}$ of $(\mathbb{R}^d)^m$ -valued random variables by

$$\mathbf{X}^{m, \varepsilon} \stackrel{\text{def}}{=} \left(X_{T/m}^{x, \varepsilon, \delta_\varepsilon}, X_{2T/m}^{x, \varepsilon, \delta_\varepsilon}, \dots, X_T^{x, \varepsilon, \delta_\varepsilon} \right). \quad \varepsilon > 0$$

Proposition 5.1 tells us that $\{\mathbf{X}^{m,\varepsilon} : \varepsilon > 0\}$ has a large deviations principle with action functional I_m . For each positive integer m , now define $\Xi_m : (\mathbb{R}^d)^m \rightarrow C([0, T]; \mathbb{R}^d)$ by piecewise-linear interpolation;

$$\begin{aligned} (\Xi_m((y_1, y_2 \dots y_m)))(t) &\stackrel{\text{def}}{=} \\ &y_{\lfloor tm/T \rfloor + 1} \left(\frac{tm}{T} - \left\lfloor \frac{tm}{T} \right\rfloor \right) + y_{\lfloor tm/T \rfloor} \left(1 - \frac{tm}{T} + \left\lfloor \frac{tm}{T} \right\rfloor \right), \\ &0 \leq t \leq T, (y_1, y_2 \dots y_d) \in (\mathbb{R}^d)^m \end{aligned}$$

where $y_0 \stackrel{\text{def}}{=} x$ and $\lfloor \cdot \rfloor$ is the integer floor function. Clearly Ξ_m is continuous for each m . For each positive integer m , let's now define $PL_m([0, T]; \mathbb{R}^d)$ as the collection of piecewise-linear elements of $C([0, T]; \mathbb{R}^d)$ which have vertices at $\{T/m, 2T/m \dots T\}$; i.e., $PL_m([0, T]; \mathbb{R}^d) = \text{Range}(\Xi_m)$. The contraction principle now yields

Proposition A.2. *For each positive integer m , the family $\{\mathbf{X}^{m,\varepsilon} : \varepsilon > 0\}$ of $C([0, T]; \mathbb{R}^d)$ -valued random variables has a large deviations principle with action functional*

$$\tilde{S}_m(\varphi) \stackrel{\text{def}}{=} \begin{cases} \sum_{j=1}^m \mathcal{J}^{(i)} \left(\frac{\varphi(jT/m) - \varphi((j-1)T/m)}{T/m} \right) (T/m). & \text{if } \varphi \in PL_m([0, T]; \mathbb{R}^d) \\ \infty & \text{else} \end{cases} \quad (\text{A.2})$$

for all $\varphi \in C([0, T]; \mathbb{R}^d)$

Proof. Use the contraction principle and the easily-verifiable fact that $\tilde{S}_m(\varphi) = \inf\{I_m(y) : \Xi_m(y) = \varphi\}$ for all $\varphi \in C([0, T]; \mathbb{R}^d)$. \square

Let's next show that $\Xi_m(\mathbf{X}^{m,\varepsilon})$ is sufficiently "close" to $X^{x,\varepsilon,\delta_\varepsilon}$ for large m and small ε . We have

Proposition A.3. *For each $\eta > 0$,*

$$\lim_{m \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\rho_{C([0, T]; \mathbb{R}^d)}(\Xi_m(\mathbf{X}^{m,\varepsilon}), X^{x,\varepsilon,\delta_\varepsilon}) \geq \eta\} = -\infty. \quad (\text{A.3})$$

Proof. The proof is a modification of some of the calculations of [13, Theorem 2.2 of Ch. 3 and Lemma 2.1 of Ch. 5]. A simple calculation shows that

$$\rho_{C([0, T]; \mathbb{R}^d)}(\Xi_m(\mathbf{X}^{m,\varepsilon}), X^{x,\varepsilon,\delta_\varepsilon}) \leq \sup_{1 \leq l \leq m} \sup_{(l-1)T/m \leq t \leq lT/m} \|X_t^{x,\varepsilon,\delta_\varepsilon} - X_{(l-1)T/m}^{x,\varepsilon,\delta_\varepsilon}\|.$$

If we are in Regimes 2 or 3, fix $\alpha \in (0, 1/2)$ and note that then by Propositions 3.2 and 4.6,

$$\begin{aligned} &\lim_{m \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\{\rho_{C([0, T]; \mathbb{R}^d)}(\Xi_m(\mathbf{X}^{m,\varepsilon}), X^{x,\varepsilon,\delta_\varepsilon}) \geq \eta\} \\ &\leq \lim_{m \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}\left\{\|X^{x,\varepsilon,\delta_\varepsilon}\|_{C^\alpha([0, T]; \mathbb{R}^d)} \geq 2\eta \left(\frac{m}{T}\right)^\alpha\right\} = -\infty. \end{aligned}$$

If we are in Regime 1, we calculate that

$$\begin{aligned} &\mathbb{P}\{\rho_{C([0, T]; \mathbb{R}^d)}(\Xi_m(\mathbf{X}^{m,\varepsilon}), X^{x,\varepsilon,\delta_\varepsilon}) \geq \eta\} \\ &\leq \sum_{l=1}^m \mathbb{P}\left\{\sup_{(l-1)T/m \leq t \leq lT/m} \|X_t^{x,\varepsilon,\delta_\varepsilon} - X_{(l-1)T/m}^{x,\varepsilon,\delta_\varepsilon}\| \geq \eta\right\} \\ &\leq 2m \sup_{\substack{y \in \mathbb{R}^d \\ 0 < t \leq T/m}} \mathbb{P}\{\|X_t^{y,\varepsilon} - y\| \geq \eta/2\}. \end{aligned}$$

We now use Proposition 2.2 and (A.1) to see that for any $\gamma > 0$, there is an $\varepsilon' > 0$ such that for all $\varepsilon \in (0, \varepsilon']$,

$$\begin{aligned}
& \sup_{\substack{y \in \mathbb{R}^d \\ 0 < t \leq T/m}} \mathbb{P} \{ \|X_t^{y, \varepsilon} - y\| \geq \eta/2 \} \\
& \leq \sup_{\substack{y \in \mathbb{R}^d \\ 0 < t \leq T/m}} \exp \left[-\frac{1}{\varepsilon} \left\{ \inf_{\|z-y\| \geq \eta/2} I_{t,y}^{(1)}(z) - \gamma \right\} \right] \\
& \leq \sup_{0 < t \leq T/m} \exp \left[-\frac{1}{\varepsilon} \left\{ \inf_{\|z\| \geq \eta/2} (\alpha t \|z/t\|^2 - \beta t) - \gamma \right\} \right] \\
& = \exp \left[-\frac{1}{\varepsilon} \left\{ \frac{m\alpha\eta^2}{4T} - \frac{\beta T}{m} - \gamma \right\} \right].
\end{aligned}$$

This gives us (A.3) in Regime 1. \square

We also need to show that the \tilde{S}_m 's of (A.2) are sufficiently “regular” and close to $S_{0,T}$ of (5.1). To get around the fact that \tilde{S}_m is finite only on $PL_m([0, T]; \mathbb{R}^d)$, let's define, for each positive integer m , $\Gamma_m : C([0, T]; \mathbb{R}^d) \rightarrow PL_m([0, T]; \mathbb{R}^d)$ by linear interpolation; set $\Gamma_m(\varphi) \stackrel{\text{def}}{=} \Xi_m(\varphi(T/m), \varphi(2T/m) \dots \varphi(T))$ for all $\varphi \in C([0, T]; \mathbb{R}^d)$. Thus, although $\tilde{S}_m(\varphi) = \infty$ if $\varphi \in C([0, T]; \mathbb{R}^d) \setminus PL_m([0, T]; \mathbb{R}^d)$, $\tilde{S}_m(\Gamma_m(\varphi)) < \infty$ and is the natural approximation of $S_{0,T}(\varphi)$. For any fixed $\varphi \in C([0, T]; \mathbb{R}^d)$, a simple calculation shows that $\rho_{C([0, T]; \mathbb{R}^d)}(\Gamma_m(\varphi), \varphi) \leq 2\omega_\varphi(T/m)$ for all m , where ω_φ is the modulus of continuity of φ . The following technical regularity results about $S_{0,T}$ and the \tilde{S}_m 's are what we need to proceed:

Lemma A.4. *We have that*

- i) $S_{0,T}$ is a good rate function.*
- ii) For each $\varphi \in C([0, T]; \mathbb{R}^d)$, $\lim_{m \rightarrow \infty} \tilde{S}_m(\Gamma_m(\varphi)) = S_{0,T}(\varphi)$.*
- iii) For any positive integer m , $\tilde{S}_m \circ \Gamma_m = S_{0,T} \circ \Gamma_m$.*
- iv) For any closed $F \subset C([0, t]; \mathbb{R}^d)$, $\inf_{\varphi \in F} S_{0,T}(\varphi) \leq \underline{\lim}_{m \rightarrow \infty} \inf_{\varphi \in F} \tilde{S}_m(\varphi)$.*

Proof. Let's begin with *i*). We need to show that for each $s \geq 0$, the set

$$\Phi(s) \stackrel{\text{def}}{=} \{ \varphi \in C([0, T]; \mathbb{R}^d) : S_{0,T}(\varphi) \leq s \}$$

is a compact subset of $C([0, T]; \mathbb{R}^d)$. Since $\mathcal{J}^{(i)}$ is convex (Lemma A.1), $\Phi(s)$ is closed. In light of (A.1),

$$\Phi(s) \subset \left\{ \varphi \in C([0, T]; \mathbb{R}^d) : \dot{\varphi} \text{ is well-defined and } \int_0^T \|\dot{\varphi}(s)\|^2 ds \leq \alpha(s + \beta T) \right\}.$$

The set on the right is a compact subset of $C([0, T]; \mathbb{R}^d)$, so we are done with claim *i*). Claim *ii*) is left to the reader. When $S_{0,T}(\varphi) < \infty$, the proof uses the convexity of the $\mathcal{J}^{(i)}$'s (Lemma A.1) and (A.1). When $S_{0,T}(\varphi) = \infty$, the proof uses (A.1). Claim *iii*) of Lemma A.4 is obvious.

Let's finally prove *iv*). The result is trivial if the right-hand side is infinity, so let's assume that $K \stackrel{\text{def}}{=} \underline{\lim}_{m \rightarrow \infty} \inf_{\varphi \in F} \tilde{S}_m(\varphi)$ is finite. Then there is a sequence

$\{m_n : n = 1, 2, \dots\}$ tending to infinity such that $\inf_{\varphi \in F} \tilde{S}_{m_n}(\varphi) < K + 1$. for all n . Using the fact that the level sets of \tilde{S}_{m_n} are compact, we conclude that for each n , there is a $\varphi_{m_n} \in F \cap PL_m([0, T]; \mathbb{R}^d)$ such that $\tilde{S}_{m_n}(\varphi_{m_n}) = \inf_{\varphi \in F} \tilde{S}_{m_n}(\varphi)$. Using now claims *i*) and *iii*), we know that there is a $\varphi^* \in F$ such that $S_{0,T}(\varphi^*) \leq \lim_{n \rightarrow \infty} S_{0,T}(\varphi_{m_n}) = \lim_{n \rightarrow \infty} \tilde{S}_{m_n}(\varphi_{m_n})$. Clearly, $S_{0,T}(\varphi^*) \geq \inf_{\varphi \in F} \tilde{S}_{0,T}(\varphi)$, so, combining all of this, we finally get exactly *iv*). \square

We now have all of the tools to prove Theorem 5.2.

Proof of Theorem 5.2. According to [4, Thm. 4.2.16], we only need to prove that $S_{0,T} = I$, where $I : C([0, T]; \mathbb{R}^d) \rightarrow [0, \infty]$ is defined as

$$I(\varphi) \stackrel{\text{def}}{=} \sup_{\delta > 0} \liminf_{m \rightarrow \infty} \inf\{\tilde{S}_m(\psi) : \rho_{C([0,T];\mathbb{R}^d)}(\psi, \varphi) < \delta\}. \quad \varphi \in C([0, T]; \mathbb{R}^d)$$

The groundwork for applying [4, Thm 4.2.16] involves Proposition A.3 and claims *i*) and *iv*) of Lemma A.4. To prove that $S_{0,T} = I$, let's fix $\varphi \in C([0, T]; \mathbb{R}^d)$.

Let's first show that $S_{0,T}(\varphi) \geq I(\varphi)$. Fix $\delta > 0$. Recall that $\Gamma_m(\varphi)$ is within distance $2\omega_\varphi(T/m)$ of φ (in the $\rho_{C([0,T];\mathbb{R}^d)}$ norm); thus $\rho_{C([0,T];\mathbb{R}^d)}(\Gamma_m(\varphi), \varphi) < \delta$ for m sufficiently large. Thus, $\inf\{\tilde{S}_m(\psi) : \rho_{C([0,T];\mathbb{R}^d)}(\psi, \varphi) < \delta\} \leq \tilde{S}_m(\Gamma_m(\varphi))$. Using claim *ii*) of Lemma A.4, we see that

$$\liminf_{m \rightarrow \infty} \inf\{\tilde{S}_m(\psi) : \rho_{C([0,T];\mathbb{R}^d)}(\psi, \varphi) < \delta\} \leq S_{0,T}(\varphi).$$

Taking the supremum of the left side over $\delta > 0$, we indeed have that $S_{0,T}(\varphi) \geq I(\varphi)$.

We next show that $S_{0,T}(\varphi) \leq I(\varphi)$. Again, fix $\delta > 0$. Then by using *iv*) of Lemma A.4, we see that

$$\liminf_{m \rightarrow \infty} \inf\{\tilde{S}_m(\psi) : \rho_{C([0,T];\mathbb{R}^d)}(\psi, \varphi) < \delta\} \geq \inf\{S_{0,T}(\psi) : \rho_{C([0,T];\mathbb{R}^d)}(\psi, \varphi) \leq \delta\}.$$

Thus, $I(\varphi) \geq \lim_{\delta \rightarrow 0} \inf\{S_{0,T}(\psi) : \rho_{C([0,T];\mathbb{R}^d)}(\psi, \varphi) \leq \delta\} = S_{0,T}(\varphi)$, the last equality following from [5, Lemma 2.1.2] and the fact that $S_{0,T}$ is good (claim *i*) of Lemma A.4). Thus, we must have that $S_{0,T}(\varphi) \leq I(\varphi)$. \square

APPENDIX B. THE PROOF OF LEMMA 4.4.

Here we give the proof of Lemma 4.4. The idea is to compare \check{X} to a $BESQ^{\kappa_6}(0)$ process and then bound the law of \check{X} with a κ_6 -dimensional Wiener process. Although most of the calculations constitute something of a folk theorem, we have been unable to find a reference to cite. Thus, we shall outline the proof. We begin by rewriting (4.6) with simpler notation. Define $\check{A}_l(t) \stackrel{\text{def}}{=} A_l\left(\frac{X_t^{x, \varepsilon, \delta_\varepsilon}}{\delta_\varepsilon}\right)$ for all $0 \leq t \leq 1$ and $l = 1, 2, \dots, d$. Then $\check{X}_t = \sum_{l=1}^d \int_0^t \check{A}_l(s) d\check{W}_s$ for all $0 \leq t \leq 1$. Let's now define $Z_t \stackrel{\text{def}}{=} \|\check{X}_t\|^2$ for all $0 \leq t \leq 1$. The evolution of Z is given by

$$Z_t = 2 \sum_{l=1}^d \int_0^t \langle \check{X}_s, \check{A}_l(s) \rangle d\check{W}_s^l + \sum_{l=1}^d \int_0^t \|\check{A}_l(s)\|^2 ds$$

for $0 \leq t \leq 1$. Set $\gamma(t) \stackrel{\text{def}}{=} \sum_{l=1}^d \|\check{A}_l(t)\|^2$ and $\alpha(t) \stackrel{\text{def}}{=} \sum_{j=1}^d \langle \check{X}_t, \check{A}_j(t) \rangle^2 / \|\check{X}_t\|^2$ for all $0 \leq t \leq 1$. The nondegeneracy of the A_l 's implies that \hat{X} spends zero time at the origin ($\check{\mathbb{P}}$ -a.s.), so α is well-defined. Letting M_z^* denote the adjoint of M_z (given in (4.3) for each $z \in \mathbb{R}^d$ and letting $\{\mathbf{e}_1, \mathbf{e}_2 \dots \mathbf{e}_d\}$ be the standard basis for \mathbb{R}^d , we observe that

$$\begin{aligned}
\gamma(t) &= \sum_{l=1}^d \|M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}(\mathbf{e}_l)\|^2 \leq d\kappa_4 \\
\sum_{l=1}^d \langle \check{X}_t, \check{A}_l(t) \rangle^2 &= \sum_{l=1}^d \langle \check{X}_t, M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}(\mathbf{e}_l) \rangle^2 = \left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^*(\check{X}_t) \right\|^2 \\
\|\check{X}_t\|^2 &\leq \left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^{-1}(\check{X}_t) \right\| \left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^*(\check{X}_t) \right\| \\
&\leq \sqrt{\kappa_3} \|\check{X}_t\| \left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^*(\check{X}_t) \right\| \\
\left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^*(\check{X}_t) \right\|^2 &\leq \left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}(M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^*(\check{X}_t)) \right\| \|\check{X}_t\| \\
&\leq \sqrt{\kappa_4} \left\| M_{X_t^{x,\varepsilon,\delta_\varepsilon/\delta_\varepsilon}}^*(\check{X}_t) \right\| \|\check{X}_t\| \\
\frac{1}{\kappa_3} &\leq \alpha(t) \leq \kappa_4.
\end{aligned} \tag{B.1}$$

We have used here some of the bounds of (4.4). The second line gives us an alternate representation of the numerator of $\alpha(t)$. Using this and the third and fourth lines, we conclude the last line.

Next rescale time. Define $\tau : [0, \infty) \rightarrow [0, \infty)$ as a solution of the random ODE $\dot{\tau}(t) = \frac{1}{\alpha(\tau(t))}$ for $0 \leq \tau(t) \leq 1$, with initial condition $\tau(0) = 0$ ($\tau(t)$ will be uniquely defined until $\tau(t) = 1$, which is sufficient for our needs). By the last line of (B.1), $\tau(t) \geq t/\kappa_4$ for all $t \leq \tau^{-1}(1)$ and thus $\tau^{-1}(1) \leq \kappa_4$ ($\tau^{-1}(1)$ is well-defined since the last line of (B.1) implies that τ is strictly increasing and thus invertible).

We now define $\check{\beta}_t \stackrel{\text{def}}{=} \sum_{l=1}^d \int_0^{\tau(t)} \frac{\langle \check{X}_s, \check{A}_l(s) \rangle}{\|\check{X}_s\|} d\check{W}_s^l$ for all $0 \leq t \leq \tau^{-1}(1)$. It is easy to see that $\check{\beta}$ is a Brownian motion on $[0, \tau^{-1}(1)]$. Continuing with this rescaling, define $\check{\zeta}_t \stackrel{\text{def}}{=} Z_{\tau(t)}$ for all $0 \leq t \leq \tau^{-1}(1)$. Then

$$\check{\zeta}_t = 2 \int_0^t \sqrt{\check{\zeta}_s} d\check{\beta}_s + \int_0^t \frac{\gamma(\tau(s))}{\alpha(\tau(s))} ds.$$

From the first and last lines of (B.1), we see that $\frac{\gamma(t)}{\alpha(t)} \leq d\kappa_3\kappa_4 \leq \kappa_6$ for all $0 \leq t \leq 1$.

Let's now define a Bessel process with which to compare $\check{\zeta}$. Let ζ be the unique solution of $\zeta_t = 2 \int_0^t \sqrt{\zeta_s} d\check{\beta}_s + (d\kappa_3\kappa_4)t$ for $t \geq 0$.

Then ζ is a $BESQ^{d\kappa_3\kappa_4}(0)$ process (with respect to $\check{\mathbb{P}}$). The relation between $\check{\zeta}$ and ζ is given by the following coupling result:

Lemma B.1. *We have that $\check{\mathbb{P}} \{ \check{\zeta}_t \leq \zeta_t \text{ for all } 0 \leq t \leq \tau^{-1}(1) \} = 1$.*

Proof. The proof is very similar to some calculations of [27, Sec. 3 of Chap. IX]. Define $U_t \stackrel{\text{def}}{=} \check{\zeta}_t - \zeta_t$ for all $0 \leq t \leq \tau^{-1}(1)$. Then, letting $z^+ \stackrel{\text{def}}{=} \max\{z, 0\}$ for all

$z \in \mathbb{R}$, we have that $U_t^+ = \int_0^t \chi_{\{U_s > 0\}} dU_s + \frac{1}{2} L_t^0(U)$ for all $0 \leq t \leq \tau^{-1}(1)$, where $L^0(U)$ is the local time of U at 0. Clearly

$$dU_t = 2 \left(\sqrt{\check{\zeta}_t} - \sqrt{\zeta_t} \right) d\beta_t + \left(\frac{\gamma(\tau(t))}{\alpha(\tau(t))} - \kappa_6 \right) dt. \quad 0 \leq t \leq \tau^{-1}(1)$$

Since

$$\int_0^t \frac{1}{|U_s|} d\langle U \rangle_s = 4 \int_0^t \frac{1}{|U_s|} \left(\sqrt{\check{\zeta}_s} - \sqrt{\zeta_s} \right)^2 ds \leq 4 \int_0^t \frac{1}{|U_s|} |\check{\zeta}_s - \zeta_s| ds \leq 4t$$

for all $0 \leq t \leq \tau^{-1}(1)$ and since $\int_{0+} (1/u) du = \infty$, standard calculations (Lemma 3.3 of [27, Sec. 3 of Chap. IX]) imply that $L_t^0(U) = 0$ for all $0 \leq t \leq \tau^{-1}(1)$. Thus,

$$\check{\mathbb{E}} \left[U_{t \wedge \tau^{-1}(1)}^+ \right] = \check{\mathbb{E}} \left[\int_0^{t \wedge \tau^{-1}(1)} \chi_{\{U_s > 0\}} \left(\frac{\gamma(\tau(s))}{\alpha(\tau(s))} - \kappa_6 \right) ds \right] \leq 0.$$

This almost immediately implies the result. \square

Thus we have that

$$\begin{aligned} \check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq 1} \|\check{X}_t\| \leq \bar{L} \right\} &= \check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq 1} Z_t \leq \bar{L}^2 \right\} = \check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq \tau^{-1}(1)} \check{\zeta}_t \leq \bar{L}^2 \right\} \\ &\geq \check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq \tau^{-1}(1)} \zeta_t \leq \bar{L}^2 \right\} \geq \check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq \kappa_4} \zeta_t \leq \bar{L}^2 \right\} \geq \check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq 1} \zeta_t \leq \bar{L}^2 / \kappa_4 \right\} \end{aligned}$$

for each $\bar{L} \geq 0$.

The rest is easy. A $BESQ^{\kappa_6}(0)$ process has the same law as a κ_6 -dimensional Wiener process. Using the equivalence of the l^2 and l^∞ norms on \mathbb{R}^{κ_6} and letting V be a \mathbb{P} -Wiener process, we have that

$$\check{\mathbb{P}} \left\{ \sup_{0 \leq t \leq 1} \zeta_t \leq \bar{L}^2 / \kappa_4 \right\} \geq \left(\mathbb{P} \left\{ \sup_{0 \leq s \leq 1} |V_s| \leq \bar{L} / \sqrt{\kappa_4 \kappa_6} \right\} \right)^{\kappa_6}.$$

By a celebrated result of Levy [27, Theorem VI.2.3], [20, Problem 2.8.8], we furthermore have that

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq s \leq 1} |V_s| \leq \bar{L} / \sqrt{\kappa_4 \kappa_6} \right\} &\geq \mathbb{P} \left\{ \sup_{0 \leq s \leq 1} \left(\sup_{0 \leq r \leq s} (V_r - V_s) \right) \leq \bar{L} / \sqrt{\kappa_4 \kappa_6} \right\} \\ &\geq \mathbb{P} \left\{ \sup_{0 \leq s \leq 1} V_s \leq \bar{L} / \sqrt{\kappa_4 \kappa_6} \right\} \geq \frac{2}{\sqrt{2\pi}} \int_0^{\bar{L} / \sqrt{\kappa_4 \kappa_6}} e^{-b^2/2} db \\ &\geq \frac{2}{\sqrt{2\pi}} \int_0^{1 \wedge (\bar{L} / \sqrt{\kappa_4 \kappa_6})} e^{-b^2/2} db \geq \frac{2e^{-1/2}}{\sqrt{2\pi}} \min\{1, \bar{L} / \sqrt{\kappa_4 \kappa_6}\}. \end{aligned}$$

This completes the proof of Lemma 4.4.

REFERENCES

1. P. Baldi, *Large deviations for diffusion processes with homogenization and applications*, *Annals of Probability* **19** (1991), 509–524.
2. P.H. Baxendale and D.W. Stroock, *Large deviations and stochastic flows of diffeomorphisms*, *Probab. Th. Rel. Fields* **80** (1988), 169–215.
3. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis of Periodic Structures*, North-Holland, New York, 1978.
4. A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Jones and Bartlett, Boston, 1993.
5. J.-D. Deuschel and D. W. Stroock, *Large Deviations*, Academic Press, New York, 1989.
6. A. Eizenberg and M. Freidlin, *On the Dirichlet problem for a class of second order PDE systems with small parameter*, *Stochastics and Stochastics Reports* **33** (1990), 111–148.
7. R. S. Ellis, *Large deviations for a general class of random vectors*, *Ann. Probab.* **12** (1984), 1–12.
8. A. Fannjiang and G. Papanicolaou, *Convection enhanced diffusion for periodic flows*, *SIAM J. Appl. Math* **54** (1994), 333–408.
9. M. I. Freidlin, *On wave propagation in periodic media*, *Stochastic Analysis and Applications* (M. A. Pinsky, ed.), Marcel Dekker, New York, 1984.
10. ———, *Functional Integration and Partial Differential Equations*, Princeton University Press, Princeton, 1985.
11. ———, *Semi-linear PDE's and limit theorems for large deviations*, *Ecole d'Eté de Probabilités de Saint Flour XX-1990* (P. L. Hennequin, ed.), *Lecture Notes in Mathematics* 1527, Springer Verlag, New York, 1992.
12. M. I. Freidlin and T.-Y. Lee, *Wave front propagation and large deviations for diffusion-transmutation processes*, *Probab. Theory Related Fields* **106** (1996), 39–70.
13. M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*; Joseph Szücs, Springer, New York, 1984.
14. J. Gärtner, *On large deviations from the invariant measure*, *Th. Prob. Appl.* **22** (1977), 24–39.
15. J. Gärtner and M. I. Freidlin, *On the propagation of concentration waves in periodic and random media*, *Dokl. Acad. Nauk SSSR* **249** (1979), 521–525. (Russian)
16. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, New York, 1981.
17. T. Kato, *Perturbation Theory for Linear Operators*, (2nd ed.), Springer Verlag, New York, 1984.
18. A. Kolmogorov, I. Petrovskii, and N. Piskunov, *Étude de l'équation de la diffusion avec croissance de la matière et son application a un problème biologique*, *Moscow Univ. Bull. Math.* **1** (1937), 1–25.
19. S. M. Kozlov and A. L. Pyatnitskii, *Averaging on a background of vanishing viscosity*, *Math. USSR Sbornik* **70** (1991), 241–261.
20. I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer Verlag, New York, 1988.
21. S. Makhno, *Large deviations for solutions of stochastic equations*, *Theory of Probability and its Applications* **40** (1995), 660–678.
22. A. Majda and P. Souganidis, *Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales*, *Nonlinearity* **7** (1994), 1–30.
23. J. R. Norris, *Heat kernel bounds and homogenization of elliptic operators*, *Bulletin London Mathematical Society* **26** (1994), 75–87.
24. ———, *Long time behaviour of heat flow: global estimates and exact asymptotics*, *Arch. Rat. Mech. Anal* **140** (1997), 161–195.
25. J. Norris and D. W. Stroock, *Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients*, *Proceedings London Mathematical Society* **62** (1991), 373–402.
26. G. Papanicolaou and X. Xin, *Reaction-diffusion fronts in periodically layered media*, *Journal of Statistical Physics* **63** (1991), 915–931.
27. D. Revuz, and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, New York, 1991.
28. R. Sowers, *Large deviations for a reaction-diffusion equation with non-Gaussian perturbations*, *Annals of Probability* **20** (1992), 504–537.

29. J. X. Xin, *Existence of multidimensional travelling waves in the transport of reactive solutes through periodic porous media*, Arch. for Ratl. Mech. and Anal. **128** (1995), 74–103.

DEPARTMENT OF MATHEMATICS UNIVERSITY OF MARYLAND COLLEGE PARK, MD 20742

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN ST., URBANA,
IL 61801

E-mail address: `mif@math.umd.edu` `r-sowers@math.uiuc.edu`