Homework, Math. 241
Tuesday, November 10, 2009

1. Recall that a function f from $\mathbf{R}$ to $\mathbf{R}^3$ is *periodic* if there is some number P such that $f(t) = f(t+P)$ for all time t. Let T be the map from $\mathbf{R}^2$ to $\mathbf{R}^3$ defined by

$$\begin{aligned} T(\theta,\phi) &= (x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) \\ x(\theta,\phi) &= (a + b\cos\theta)\cos\phi \\ y(\theta,\phi) &= (a + b\sin\theta)\sin\phi \\ z(\theta,\phi) &= b\sin\phi \end{aligned}$$

Let ρ be a path of constant speed that traces out a line in $\mathbf{R}^2$. That is, $\rho(t) = (\theta(t), \phi(t))$ with $\rho'(t) = (q,p)$. Show that if $q \neq 0$ then ρ is periodic if and only if $\frac{p}{q}$ is rational.

2. Let $S = \{\mathbf{a} \in \mathbf{R}^3 \mid \|\mathbf{a}\| = 1\}$ and $n = (0,0,1)$ (the North pole of the sphere). Consider $\mathbf{R}^2 \subseteq \mathbf{R}^3$ as the xy-plane (i.e. the points whose z coordinate is 0).

2a. For each point $(a,b) \in \mathbf{R}^2$, the line through n and $(x,y,0)$ intersects S in exactly one other point, $(x(a,b), y(a,b), z(a,b))$. Find explicit formulas for $x(a,b)$, $y(a,b)$ and $z(a,b)$.

2b. Let $X(a,b) = (x(a,b), y(a,b), z(a,b))$ be the map from $\mathbf{R}^2$ to S minus the north pole, n. Show that X is one-to-one and C^1.

2c. Show that X takes lines in $\mathbf{R}^2$ to the circle in S passing through the north pole n obtained by intersecting the plane containing the north pole and the line with the sphere S.

2d. For a number $-1 < h < 1$, let $S_h = \{(x,y,z) \in S \mid z \leq h\}$. Show that a region D in $\mathbf{R}^2$ is bounded if and only if $X(D) \subseteq S_h$ for some h (which depends upon D).

2e. The function X (and its inverse from S minus n) are called *stereo graphic* projection. We sometimes refer to S as the *one point compactification* of $\mathbf{R}^2$ and think of n as the point "∞." With this idea, why might a mathematician say that all lines intersect at ∞ and two lines are not parallel when they intersect in two points? If one allows n as ∞ to be a point on a "line" on S, how many "points" are now needed to determine a line?