

**Math 406, Final Practice**  
University of Illinois, December 2004

**Instructions:** Please answer the questions as clearly as possible and ask if you are unsure about what is needed for full credit in your solution. No calculators, cheat sheets, pets, friends, family, enemies or psychic readings can be used to aid you with this exam. Good luck and remember that not all questions are weighted equally.

1. (50 points) The following are short answers and worth 2 points each.

- 1.1) \_\_\_\_\_ is usually credited to have first “proved” that the ratio of the areas of two circles is proportional to the ratio of their squares. He used the same idea to calculate the area of a particular “lune.” Aristotle later claimed that this lune calculation was false.
- 1.2) \_\_\_\_\_ used his method of exhaustion to give the first rigorous proof that the ratio of the areas of two circles is proportional to the ratio of their squares.
- 1.3) Archimedes used both exhaustion and \_\_\_\_\_ to give the first complete proof that for a circle, the ratio of the area to the square of the radius was equal to the ratio of the circumference to the diameter of the circle.
- 1.4) Al-Khwarizmi’s textbook *Al-jabr wa’l muqabalah* has provided us with the modern word \_\_\_\_\_.
- 1.5) \_\_\_\_\_ was the first person to demonstrate an infinite series whose terms go to 0 but which does not converge to any finite number.
- 1.6) The \_\_\_\_\_ in about 1800-1600 b.c. used a paper-cutting styled argument from to prove what we now call Pythagorean’s theorem.
- 1.7) In his work *The Method*, Archimedes indicates a method to find the area of regions by using the “law of the \_\_\_\_\_.”
- 1.8) \_\_\_\_\_ treatise on Physics points out that the infinite presents itself first in the continuous (“what is infinitely divisible is continuous”).
- 1.9) Two length are \_\_\_\_\_ if they are both integral multiples of a common length.
- 1.10) His notion of proportionality allowed one to work with ratios of arbitrary lengths. \_\_\_\_\_
- 1.11) His confusion about exhaustion techniques lead him to claim that in polar coordinates,  $A = \frac{1}{2} \int r ds$  when in fact  $A = \frac{1}{2} \int r^2 d\theta$ . \_\_\_\_\_
- 1.12) He gave a recursive method for determining the coefficients of  $(x + y)^n$  for  $n$  a positive integer. \_\_\_\_\_
- 1.13) \_\_\_\_\_ was the first to solve max-min problems by taking into account the characteristic behavior of a function near its extreme values.
- 1.14) \_\_\_\_\_ used double roots to minimize distance to find the normals to a curve

(and hence the slope of the tangent lines).

1.15) His tables were used by Kepler to establish his third rule of planetary motion.

\_\_\_\_\_

1.16) He was the first to define the *natural* logarithm.

\_\_\_\_\_

1.17) If  $L(xy) = L(x) + L(y)$  for all  $x, y > 0$  and  $L(8) = 3$ , what function is  $L(x)$ ?

\_\_\_\_\_

1.18)  $\lim_{n \rightarrow \infty} \frac{1}{n^{k+1}} \sum_{i=1}^n i^k =$  \_\_\_\_\_.

1.19) Leibniz is one of the inventors of calculus but he is also called  
“The first great German \_\_\_\_\_.”

1.20) The *Acta Eruditorum* published Leibniz’s account of calculus before Newton’s. It is not so surprising that this journal would do this because \_\_\_\_\_.

1.21) At Leibniz’s request, the Royal Society formed a committee to determine who invented calculus. The committee found in favor of Newton, but this is not so surprising because \_\_\_\_\_ wrote the committee’s report.

1.22) One of the main differences in the foundations of Newton’s calculus and that of Leibniz’s was that Newton’s machine had the \_\_\_\_\_ rule built in while Leibniz’s machine had the \_\_\_\_\_ rule built into it.

1.23) Newton’s calculus was based upon what he called the “method of \_\_\_\_\_.”

1.24) Newton’s book *Principia* is recognised as the greatest scientific book ever written. This is a full treatment of Newton’s new physics and its application to \_\_\_\_\_.

1.25) \_\_\_\_\_, in *Treatise on fluxions*, gave the best response to Berkeley’s well founded criticism of Newton’s calculus, using the classical method of exhaustion in an attempt to bring rigorous logical arguments to the subject.

1a.

Discuss the use of invisibles as used by Hippocrates, Archimedes, Kepler, Cavalieri and Leibniz.

1b.

Discuss the derivative of  $x^n$  as proven by Fermat, Hudde, Sluse, and Newton.

1c. Discuss the integral of  $x^n$  as studied by Cavalieri, Pascal and Newton.

1d. Discuss the proofs for the formula for the volume and/or surface area of a sphere of radius  $r$  as done by Archimedes, Cavalieri, and Leibniz.

**1 e.** Discuss how the ideas of the Merton scholars as explained by Orseme were fully realized by the work of Newton.

**1 f.** Discuss the development of the natural logarithm including the roles of Napier, Briggs, Gregory, Mercator and Newton.

**2a.**

a) Give Eudoxus' definition for the ratios  $a : b$  to be *proportional* to  $c : d$  ( $a : b = c : d$ ).

b) What is Archimedes (or what the text calls Eudoxus') Axiom for two positive numbers  $a$  and  $b$ ?

c) Use Archimedes Axiom and Eudoxus' definition of proportionality to show that  $a : c = b : c$  implies  $a = b$  for  $a, b$  and  $c$  positive real numbers.

**2b.** (10 points) (I, 16)

Consider two rectangles with bases  $a$  and  $b$ , area  $A$  and  $B$ , that have the same height. Apply Eudoxus' definitions of proportionality to show that  $A : B = a : b$ .

**3.** (10 points) (II, 23)

Let  $P$  be the segment of a paraboloid obtained by revolving the parabola  $y^2 = x$ ,  $0 \leq x \leq 1$ , about the  $x$ -axis. Use Archimedes' mechanical method to deduce that the volume of  $P$  is one-half that of the circumscribed cylinder  $Z$  obtained by revolving the line segment  $y = 1$ ,  $0 \leq x \leq 1$ , about the  $x$ -axis.

**4a.** (10 points) (IV, 7)

A spherical ring is obtained from a solid sphere by boring out a cylindrical hole whose axis is the vertical diameter of the sphere. Find the volume of the spherical ring by comparing it with a sphere whose diameter is equal to the height of the ring.

**4b.** Use Cavalieri's method of Indivisibles to derive the formula for the volume of a sphere by comparing a hemisphere of radius  $r$  with the solid that is obtained from a cylinder of radius and height  $r$  by removing an inverted cone whose base is the top of the cylinder and whose vertex is the center of the base of the cylinder.

**5.** (10 points)

**a.** (8 points) (V, 5)

Apply the Cartesian circle method using Hudde's rule to show that the slope of the tangent line to  $y = (x^2 + 2)^{3/2}$  is  $3x(x^2 + 2)^{1/2}$ .

**b.** (2 points)

Use Sluse's rule to show that the slope of the tangent line to  $y = (x^2 + 2)^{3/2}$  is  $3x(x^2 + 2)^{1/2}$ .

**6.** (10 points)

Using modern first year calculus results, prove Newton's binomial coefficient theorem (convergence for  $|x| < 1$ ).

**7a.** (10 points)

Use Newton's binomial coefficient theorem, his method for finding roots and his reversion of series to find the first two non-zero terms of a series expansion for  $\tan(x)$  starting from the fact that

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2}$$

**7b.**

Use Newton's binomial coefficient theorem, his method for finding roots and his reversion of series to find the first two non-zero terms of a series expansion for  $e^x - 1$  starting from the fact that

$$\ln(1+x) = \int \frac{dx}{1+x}$$

**8.** (10 points) (XIII, 7)

Using Newton's notation, if  $\dot{y}/\dot{x} = \frac{b}{2\sqrt{x}}\sqrt{a+b\sqrt{x}}$ , substitute  $z = \sqrt{x}$  in this manner to show that

$$y = \square b\sqrt{a+bz}$$

**9a.** (10 points) (IX, 9)

Consider the parabola  $y = \sqrt{x}$ ,  $0 \leq x \leq a$ . Knowing that  $D\sqrt{x} = 1/2\sqrt{x}$ , show that the normal to the parabola is  $n = \frac{1}{2}\sqrt{4x+1}$ . Use Leibniz's characteristic triangle to find that the area of the paraboloid obtained by revolving this parabola about the  $x$ -axis, which is

$$A = \int 2\pi y ds$$

**9b.** (IX, 8)

Use Leibniz's methods to show  $\int_0^x \sin \theta d\theta = 1 - \cos x$ .