Solution for Math 415 7/28 ~ 7/29.

§7.1

3. By definition, linear $\Leftrightarrow$ $T(a\vec{v}+b\vec{w}) = aT(\vec{v}) + bT(\vec{w})$

(a). $T(a\vec{v}+b\vec{w}) = (av_2+bw_2,\ av_1+bw_1)$

$aT(\vec{v})+bT(\vec{w}) = (av_2+bw_2,\ av_1+bw_1)$ ✓

(b). $T(a\vec{v}+b\vec{w}) = (av_1+bw_1,\ av_1+bw_1)$

$aT(\vec{v})+bT(\vec{w}) = (av_1+bw_1,\ av_1+bw_1)$ ✓

(c). $T(a\vec{v}+b\vec{w}) = (0,\ av_1+bw_1)$

$aT(\vec{v})+bT(\vec{w}) = (0,\ av_1+bw_1)$ ✓

(d) X $T(0) \neq 0$.

(e). $T(a\vec{v}+b\vec{w}) = av_1+bw_1-av_2-bw_2$

$aT(\vec{v})+bT(\vec{w}) = a(v_1-v_2)+b(w_1-w_2) = av_1+bw_1-av_2-bw_2$ ✓

(f). $T(a\vec{v}+b\vec{w}) = (av_1+bw_1)(av_2+bw_2)$

$aT(\vec{v})+bT(\vec{w}) = av_1v_2+bw_1w_2$ X

4. (a) $S(T(v)) = S(v) = v$

(b) $S(T(v_1+v_2)) = S(T(v_1)+T(v_2)) = S(T(v_1)) + S(T(v_2))$ linear,

8. (a). Range of T is $\{(x,0) : \forall x \in R\}$.

Kernel of T is $\{(x,y) : x = y\}$.

(b). Range of T is $\{(x,y) : x, y \in R\}$.

Kernel of T is $\{(x,y,z) : x = y = 0\}$.

(c). Range of T is $(0,0)$.

Kernel of T is $\mathbb{R}^2$.

(d) Range of T is $\{(x,x) : x \in \mathbb{R}\}$

Kernel of T is $\{(0,y) : y \in \mathbb{R}\}$.

11. (a) A is m by n

Then input space V is n by 1 vector

output space W is m by 1 vector.

(b) Av is just linear combination of the columns of A.

Thus range of T = column space of A.

(c). $Av=0 \Rightarrow v \in$ nullspace of A.

Thus kernel of T = nullspace of A.

12. (a) $v=(2,2)=2(1,1)$

$T(v)=2T(1,1)=(4,4)$

(b) $v=(3,1)=(1,1)+(2,0)$

$T(v)=(2,2)+(0,0)=(2,2)$

(c). $v=(-1,1)=(1,1)-(2,0)$

$T(v)=(2,2)-(0,0)=(2,2)$

(d) $v=(a,b)=c(1,1)+d(2,0)$ find c and d : $c=b$.

$T(v)=c(2,2)+d(0,0)=c(2,2)=b(2,2)$.

§ 7.2.

1. $Sv_1=0$, $Sv_2=0$, $Sv_3=2=2w_1$ $Sv_4=(3x^2)'=6x=6w_2$.

Thus $B=\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2. $v''=0 \Rightarrow v'=c \Rightarrow v=cx+d$.

kernel of S is {f linear functions}.

The nullspace of B is $\{1, x\}$ = kernel of S.

3. Add a zero row to the first derivative matrix, we get

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

output basis = input basis

4. (a) $$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the transformation matrix for the third derivative with basis $\{1, x, x^2, x^3\}$.

(b) Apply the fourth derivative to the basis, we get zero for all the elements. Thus the transformation matrix B^2 is zero.

5. $T[v_1, v_2, v_3] = [w_2,\ w_1+w_3,\ w_1+w_3]. \Rightarrow$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$T(v_1+2v_2+v_3) = T(v_1) + T(v_2) + T(v_3) = 2w_1 + w_2 + 2w_3.$$

6. $v = \cancel{a-b}\, c(v_2 - v_3)$ (suppose $v = a v_1 + b v_2 + c v_3$, solve $T(v) = 0$ by independence of w_1, w_2, w_3 the coefficients all be zero)

$(0, a, -a)$ are in the nullspace of A

$T(v) = w_2 \Rightarrow aw_2 + (b+c)(w_1+w_3) = w_2$

$\Rightarrow (a-1)w_2 + (b+c)(w_1+w_3) = 0$

$\Rightarrow a = 1, \quad b = -c.$

$v = v_1 + c(v_2 - v_3).$

7. Since the column space are span $\{(0,1,0), (1,0,1)\}$.

The vector not in the column space could be (not unique) $(0,0,1)$.

Correspondingly, we can choose $w = w_3$, it's not in the range of T.