Homework Solutions (7/29 ~ 7/31)

§7.2

10. The matrix A for T ~~is~~

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Since $T(v) = w_1$, $v = T^{-1} w_1$ (note T is invertible because A^{-1} exists).

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

The coordinate of w_1 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. So the coordinate of v is

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Hence $v = v_1 - v_2$

11. $$A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad A^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$A^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence $T^{-1}(w_1) = v_1 - v_2$.

$T^{-1}(w_2) = v_2 - v_3$.

$T^{-1}(w_3) = v_3$.

14. cf P 543.

24. Let $R = (r_{ij})_{3\times3}$. Since $A = (a_1, a_2, a_3)$, $Q = (q_1, q_2, q_3)$

we know

$$a_1 = r_{11} q_1 + r_{21} q_2 + r_{31} q_3$$

$$a_2 = r_{12} q_1 + r_{22} q_2 + r_{32} q_3$$

$$a_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3$$

By the definition of change of basis matrix, this matrix is R.

§7.3.

1. cf. P 544. $u_2 = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$

2. (a). cf. P 363. $C(A) = \text{span}\{u_1\}$

$C(A^T) = \text{span}\{v_1\}$.

$N(A) = \text{span}\{v_2\}$. $N(A^T) = \text{span}\{u_2\}$

(b). Since $A = (u_1 \ \ u_2) \begin{pmatrix} \sigma_1 & \\ & 0 \end{pmatrix} (v_1 \ \ v_2)^T$,

once we fix u_1, u_2, v_1, v_2.

all we can do is to change σ_1 (we can't change 0 in the diagonal either otherwise the four spaces change).

Hence all the matrixes are of the form tA $t \in \mathbb{R}, t \neq 0$.

5. $$A^TA = \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \underset{\substack{\uparrow \\ v_1}}{\frac{1}{\sqrt{2}}} & \underset{\substack{\uparrow \\ v_2}}{\frac{1}{\sqrt{2}}} \end{pmatrix} \begin{pmatrix} 18 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

So, $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, $\sigma_2 = \sqrt{2}$.

6. $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

7. cf. P 544. if A is of rank r, there is r nonzero entries on the diagonal of Σ. So $A = \sum_{k=1}^{r} \sigma_k u_k v_k^T$.

$\text{rank}(\sigma_k u_k v_k^T) = 1$.

This shows that A is the sum of r matrices of rank 1.

§ 8.1.

1. $$\det(A_0^T C_0 A_0) = \det\begin{pmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{pmatrix}$$

$$= (c_1+c_2)(c_2+c_3)(c_3+c_4) - (c_1+c_2)c_3^2 - c_2^2(c_3+c_4)$$

$$= c_1c_2c_3 + c_1c_2c_4 + c_1c_3^2 + c_1c_3c_4 + c_2^2c_3 + c_2^2c_4 + c_2c_3^2$$

$$+ c_2c_3c_4 - c_1c_3^2 - c_2c_3^2 - c_2^2c_3 - c_2^2c_4$$

$$= c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4.$$

7. By ~~the~~ analysis on P4.1 ~ P4.2.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} c_1 & & & & \\ & c_2 & & & \\ & & c_3 & & \\ & & & c_4 & \\ & & & & c_5 \end{pmatrix}.$$

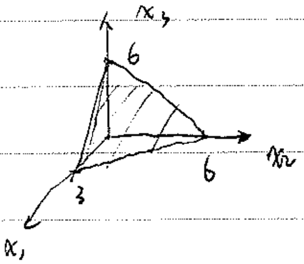
$$K = A^T C A = \begin{pmatrix} c_1+c_2 & -c_2 & & \\ -c_2 & c_2+c_3 & -c_3 & \\ & -c_3 & c_3+c_4 & -c_4 \\ & & -c_4 & c_4+c_5 \end{pmatrix}$$

If $C = I$. then $$K = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

solving $Ku = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, we get

$$u = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

8.



$x = (2, 2, 0)$ does not minimize $c^T x$.

$x^* = (3, 0, 0)$. does