Solution for Math 415 08/03 ~ 08/05.

§10.1.

10. real . imaginary , $|z|^2$, 1

14. $P_4 = \begin{bmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}$, $P_4 - \lambda I = \begin{bmatrix} -\lambda&0&0&1\\ 1&-\lambda&0&0\\ 0&1&-\lambda&0\\ 0&0&1&-\lambda \end{bmatrix}$

$\det(P_4-\lambda I) = \lambda^4 + 1(-1)^3 = \lambda^4 - 1$

$\lambda = \pm 1, \pm i$

$\lambda_1 = 1$. $V_1 = (1,1,1,1)'$

$\lambda_2 = -1$, $V_2 = (-1,1,-1,1)'$

$\lambda_3 = i$, $V_3 = (-i,-1,i,1)'$

$\lambda_4 = -i$, $V_4 = (i,-1,-i,1)'$.

16. $A^T = -A$, $\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ symmetric matrix.
The eigenvalues must be real, So λ is pure imaginary.

21. $e^{3i\theta} = \cos 3\theta + i\sin 3\theta = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = (\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta)\cdot(\cos\theta + i\sin\theta)$
$= (\cos^3\theta - 3\sin^2\theta\cos\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Thus by comparing real and imaginary parts, we get

$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

22. $\bar{z} = 1/z \Rightarrow z\bar{z} = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$.
The numbers are points on the unit circle.

§ 10.2.

3. $\begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} z = 0$. By elimination method,

we get $z = a(1+i, 1+i, -2)^T$, $a \in \mathbb{C}$.

$A^H = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$ $\quad z^H \cdot A^H = a\begin{pmatrix} 1-i \\ 1-i \\ -2 \end{pmatrix}^T \begin{pmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{pmatrix} = 0.$

Thus z is orthogonal to the column of A^H.
It's easy to verify that z is <u>not</u> orthogonal to the column of A^T.

4. $C(A)$, $N(A)$, $C(A^H)$, $N(A^H)$

5. Proof: (a). $(A^H A)^H = A^H (A^H)^H = A^H A$
Hermitian by definition.

(b) $Az = 0 \Rightarrow A^H A z = 0 \Rightarrow z^H A^H A z = 0$

$\Rightarrow (Az)^H Az = 0$ Thus $Az = 0$. So $Az = 0$
(parallel to $\bar{z} \cdot z = 0 \Rightarrow z = 0$) $\Leftrightarrow A^H A z = 0$

The nullspaces of A and $A^H A$ are the same from the deduction above.

18. <u>Unitary</u> Suppose $z = a_1 v_1 + \cdots + a_n v_n$

then by the fact that v_i are orthonormal,
$v_i^H z = a_i$.
Thus $z = (v_1^H z) v_1 + \cdots + (v_n^H z) v_n$.

29. $A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} = A^H$. Hermian. real eigenvalues.

$\det(\lambda I - A) = \lambda^2 - 5\lambda + 6 - 2 = (\lambda-1)(\lambda-4)$

$\lambda_1 = 1$, $\lambda_2 = 4$. $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$v_1 = (i-1, 1)'$, $v_2 = (1, 1+i)'$

Normalize v_1, v_2, we get

$S = \frac{1}{\sqrt{3}} \begin{pmatrix} i-1 & 1 \\ 1 & 1+i \end{pmatrix}$

§ 10.3.

11. Easy to verify that $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = i^2 = -1$, $\lambda_4 = i^3 = -i$.

12. $\Lambda = \begin{bmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{bmatrix}$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ has eigenvalues = cube roots of 1.