Homework solutions (7/15 ~ 7/17)

6.1)

3. see P536. For an invertible matrix A, if $\lambda \neq 0$ and
$Ax = \lambda x$, then $\lambda A^{-1} x = x$.
Hence $A^{-1} x = \frac{1}{\lambda} x$. ($\frac{1}{\lambda}$ is an eigenvalue of A^{-1})

8 see P537

10. Computing the eigenvalues and eigenfunctions of A yields

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

That is
$$A \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}.$$

Let $T = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. Then $T^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$.

Let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$. Then $AT = T\Lambda$,

$$A = T \Lambda T^{-1}$$

$$A^{100} = (T\Lambda T^{-1})^{100} = \underbrace{(T\Lambda T^{-1})(T\Lambda T^{-1}) \cdots (T\Lambda T^{-1})}_{100 \text{ times}}$$
$$= T \Lambda^{100} T^{-1}.$$

However, $\Lambda^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 0.4^{100} \end{pmatrix}$.

Hence,
$$A^{100} T = T \Lambda^{100}$$

From here, one can see the eigenvalues and eigenfunctions of A^{100}.

16. See P537. This problem explains why $\det A = \lambda_1 \cdots \lambda_n$.

17 see P537. The ~~trace~~ trace equals the sum of eigenvalues. It is a general rule, not only for 2×2 matrix.

21. For a square matrix B, we know $\det B = \det B^T$ (see P249).
Now, $(A-\lambda I)^T = A^T - \lambda I$.
Hence, $\det(A-\lambda I) = \det(A^T - \lambda I)$.

26. By Problem 25 in section 25,

$$\det(A-\lambda I) = \det\begin{pmatrix} B-\lambda I & C \\ 0 & D-\lambda I \end{pmatrix} = \det((B-\lambda I)(D-\lambda I))$$

$$= \det(B-\lambda I)\cdot\det(D-\lambda I)$$

But we know $\det(B-\lambda I) = (\lambda-1)(\lambda-2)$.

$\det(D-\lambda I) = (\lambda-3)(\lambda-4)$.

So $\det(A-\lambda I) = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$

and the eigenvalues of A is $1, 2, 3, 4$.

29. $\det(A-\lambda I) = (1-\lambda)(4-\lambda)(6-\lambda) = 0$

So the eigenvalues of A is $1, 4, 6$.

$$\det(B-\lambda I) = \det\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{pmatrix}$$

Big formula

$$= \lambda^2(2-\lambda) - 3(2-\lambda)$$

$$= (\lambda^2-3)(2-\lambda).$$

Hence the eigenvalues of B is 2 and $\pm\sqrt{3}$.

$$\det(C-\lambda I) = \det\begin{pmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix}$$

$$= -\begin{vmatrix} 2 & 2 & 2-\lambda \\ 2 & 2-\lambda & 2 \\ 2-\lambda & 2 & 2 \end{vmatrix}$$

$$= -8\begin{vmatrix} 1 & 1 & 1-\frac{\lambda}{2} \\ 1 & 1-\frac{\lambda}{2} & 1 \\ 1-\frac{\lambda}{2} & 1 & 1 \end{vmatrix}$$ (pull out 2 for each row).

$$= -8\begin{vmatrix} 1 & 1 & 1-\frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & \frac{\lambda}{2} \\ 0 & \frac{\lambda}{2} & -(1-\frac{\lambda}{2})^2+1 \end{vmatrix}$$ (multiply 1st row by -1, then add to 2nd row. Multiply 1st row by $1-\frac{\lambda}{2}$, then add to 3rd row).

$$= -8\begin{vmatrix} 1 & 1 & 1-\frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & \frac{\lambda}{2} \\ 0 & 0 & 1+\frac{\lambda}{2}-(1-\frac{\lambda}{2})^2 \end{vmatrix}$$

$$= -x^2(\lambda-6)$$

So the eigenvalue of C is 0, 0, and 6.

[. By using WolframAlpha.com, one can simply input "eigenvalue" followed by a matrix to compute the eigenvalues of a matrix.]

6.2

2. $A = S\Lambda S^{-1}$ where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 2 & \\ & 5 \end{pmatrix}$.

So $S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$$

3. See 5.37. In general, if $A = S\Lambda S^{-1}$, $\Lambda = S^{-1}AS$,

so $\Lambda + kI = S^{-1}AS + kI = S^{-1}AS + kS^{-1}S$

distribution law $\rightsquigarrow = S^{-1}(A+kI)S$

$A + kI = S(\Lambda + kI)S^{-1}$ for $k \in R$.

4. See P 537. We need n independent eigenvectors to ~~diagolize~~ diagonalize a matrix. But diagonalizable matrix may have 0 as its eigenvalue. In ~~this~~ case the matrix has determinant 0 thus not invertible.

13 See P 538. Once we know $\det A = 25$, trace $A = 10$, and A is a 2×2 matrix, we can write the characteristic equation of A as $\lambda^2 - 10\lambda + 25 = 0$ because we know $\lambda_1 + \lambda_2 = -10$ $\lambda_1\lambda_2 = 25$. So the char. eqn. is $(\lambda-5)^2=0$.
By using big formula. One can find the missing entry in the matrix e.g. $\det \underset{\uparrow}{A_2} = 9\times 1 - 4x = 25 \Rightarrow x = -4$.
the second A in the problem

16. By using the same approach of problem 10 in section 6.1.
Write A_1 as

$$A_1 = S\Lambda S^{-1} = \begin{pmatrix} \frac{9}{4} & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{10} \end{pmatrix}\begin{pmatrix} \frac{4}{13} & \frac{4}{13} \\ -\frac{4}{13} & \frac{9}{13} \end{pmatrix}$$

$$\Lambda^k = \begin{pmatrix} 1 & 0 \\ 0 & (-\frac{3}{10})^k \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ as } k \to 0.$$

$$\lim_{k\to\infty} S\Lambda^k S^{-1} = S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1} = \begin{pmatrix} \frac{9}{4} & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{4}{13} & \frac{4}{13} \\ -\frac{4}{13} & \frac{9}{13} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{4} & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{4}{13} & \frac{4}{13} \\ -\frac{4}{13} & \frac{9}{13} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{13} & \frac{9}{13} \\ \frac{4}{13} & \frac{4}{13} \end{pmatrix}.$$

The sum of ~~every co~~ all entries in each column is 1.

17. Similar to Problem 10 in section 6.1. We find

$$\Lambda = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.3 \end{pmatrix} \quad S = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = S \Lambda S^{-1}$$

So. $A_2^{10} S = S \Lambda^{10}$

$u_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is the ~~eigenfun~~ eigenvector corresponding to 0.9

So. $A_2^{10} u_0 = (0.9)^{10} u_0$.

Similarly, if $u_0' = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ then $A_2^{10} u_0 = (0.3)^{10} u_0$.

If $u_0'' = u_0 + u_0' = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$.

then,

$$A_2^{10} u_0'' = A_2^{10} u_0 + A_2^{10} u_0' = (0.9)^{10} u_0 + (0.3)^{10} u_0'$$

18. Computing the eigenvalues and eigenvectors yields.

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So, $S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} \end{pmatrix}$

$$A = S \Lambda S^{-1},$$

$$A^k = S \Lambda^k S^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$= \frac{1}{2} \begin{pmatrix} -3^k & 1 \\ 3^k & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 3^k + 1 & 1 - 3^k \\ 1 - 3^k & 3^k + 1 \end{pmatrix}.$$

19. The approach is the same with that of Promblem 18. see P_{538}.

25. 1) Let $A = (c_1, \cdots, c_n)$, where $c_1 \cdots c_n$ are column vectors.

Then $A^2 = A \Leftrightarrow A(c_1, \cdots, c_n) = (c_1, \cdots, c_n)$

or, $A c_k = c_k \quad k = 1, \cdots, n$.

That means each c_k is an eigenvector with $\lambda = 1$.

On the other hand, if v is an eigenvector with $\lambda = 1$,

then $Av = v$.

Write $v = (v_1, \cdots, v_n)^T$. Then

$$v = v_1 c_1 + v_2 c_2 + \cdots + v_n c_n$$

This means v is in the column space of A.

So the column space of A contains all the eigenvectors with $\lambda = 1$.

We have shown $C(A) = \{$ eigenvectors with $\lambda = 1 \}$.

2). if $Av = 0 \cdot v = 0$, then $v \in N(A)$.

On the other hand. $\forall x \in N(A)$, $Ax = 0 = 0 \cdot x$.

So, $N(A) = \{$ eigenvectors with $\lambda = 0 \}$.

3). Remember (see Section 3.6) if the rank of A is r.

$\dim N(A) = n - r$, $\quad \dim C(A) = r$.

So, $\dim N(A) + \dim C(A) = n$.

This means there are n independent eigenvectors of A.

So, A is diagonalizable.

Remark: this problem shows that a matrix with $A^2 = A$ is always diagonalizable and its ~~eigen~~ eigenvalues are nothing but 0 and 1.

26. see P538. The key points out two problems.

As an example, consider $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in the intersection of $N(A)$ and $C(A)$. (The case of problem 1).

$\lambda = 0$ is the only eigenvalue of A.

But $\dim N(A) = 2 - \text{rank } A = 1$.

A is not diagonalizable.

Problem 2 happens if we let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

$\lambda = 1$ is the only eigenvalue.

$\dim C(A) = 2$. But $A - \lambda I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

Solving $(A - \lambda I) x = 0$ shows that

there is only one eigenvector with $\lambda = 1$.

So A is not diagonalizable.

<u>Remark.</u> Compare Problem 25 and this problem.

$N(A) = \{\text{eigenvectors with } \lambda = 0\}$ is always true

for general square matrix A.

However, generally we only have

$C(A) \supseteq \{\text{eigenvectors with } \lambda \neq 0\}$. $(*)$

That is why we need to argue in "both directions" in problem 25.

Since we know $\dim N(A) + \dim C(A) = n$, if the converse inclusion in $(*)$ is true, then A is diagonalizable. (The two problems cannot happen then. why?)

6.3)

4. See P539 for the answer and P313 for the approach.

5. Let $B = -A$. The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 0$.
and corresponding eigenvectors are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Since $\begin{pmatrix} 30 \\ 10 \end{pmatrix} = -10\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 20\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

the solution is $\quad v(t) = 20 + 10e^{2t}$

$w(t) = 20 - 10e^{2t}$.

$v(t) \to \infty$ as $t \to \infty$.

7. Let $a = (1, 1)^T$. Then

$$P = \frac{aa^T}{a^Ta} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues, $\lambda_1 = 1$. $\lambda_2 = 0$.
The eigenvalues of $-P$ are: $\lambda_1 = -1$ $\lambda_2 = 0$
The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Since $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

$$u(t) = 2e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-t}+1 \\ 2e^{-t}-1 \end{pmatrix}.$$

$u(t) \to \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as $t \to \infty$.

8 See P539.

19. $B = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$.

$$e^{Bt} = I + Bt + \frac{(Bt)^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + 0$$

$$= \begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix}.$$

$$B e^{Bt} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}.$$

From the expansion of e^{Bt}, we know

$$(e^{Bt})' = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$$

So, $(e^{Bt})' = Be^{Bt}$.

21, 22, 24 See P539. They are straight forward.

In Problem 24,

$$(e^{At})' = \begin{pmatrix} e^t & \frac{3}{2}e^{3t} - \frac{1}{2}e^t \\ 0 & 3e^{3t} \end{pmatrix}$$

$$(e^{At})'\big|_{t=0} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = A.$$