Math 415 Solutions for 7/20 – 7/22

§ 6.4

$\boxed{3}$ $A-\lambda I = \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix}$

$|A-\lambda I| = 2\cdot 2\lambda + \lambda^2(2-\lambda) + 4\lambda = -(\lambda-4)(\lambda+2)\lambda$

eigenvalues : $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = -2$.

Solve $(A-\lambda_i I)x_i = 0$, we get

$x_1 = (0,-1,1)'$ $x_2 = (2,1,1)'$ $x_3 = (-1,1,1)'$.

Normalize the eigenvectors, we get the unit eigenvectors.

$\hat{x}_1 = \frac{\sqrt{2}}{2}(0,-1,1)'$, $\hat{x}_2 = \frac{\sqrt{6}}{6}(2,1,1)$, $\hat{x}_3 = \frac{\sqrt{3}}{3}(-1,1,1)'$.

$\boxed{4}$ First find the eigenvalues:

$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 - 36 = (\lambda-10)(\lambda+5)$

$\lambda_1 = 10$, $\lambda_2 = -5$.

Thus $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$

Next find the corresponded eigenvectors:

$(A-\lambda_i I)x_i = 0$. implies that

$x_1 = (1,2)'$ $x_2 = (2,-1)'$

Normalize and combine as columns of Q, we have

$Q = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}$ as the orthogonal matrix.

(Note : you only need Q made by eigenvectors rather than unit eigenvectors, which diagonalize A as Λ multiplied by some scalar).

$\boxed{5}$ First find the eigenvectors:

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = -\lambda(\lambda-3)(\lambda+3)$$

Thus $\lambda_1 = -3$, $\lambda_2 = 3$, $\lambda_3 = 0$.

Solve $(A-\lambda_i I)x_i = 0$, we get

$x_1 = (-1,2,2)'$, $x_2 = (2,-1,2)'$, $x_3 = (-2,-2,1)'$.

Thus we can let $Q = [\hat{x}_1, \hat{x}_2, \hat{x}_3]$. i.e.:

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$\boxed{8.}$ $\underline{0}$ (because if we suppose λ as a eigenvalue of A. $Ax = \lambda x$, where x is the eigenvector, thus $A^2x = A(\lambda x) = \lambda Ax = \lambda^2 x$

$A^3x = \lambda^3 x = 0$. which implies $\lambda^3 = 0$)

example: $A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ (upper triangle matrix in this form always have this property).

If A is symmetric, however, it's diagolizable.

we can write $A = Q\Lambda Q^T$, $A^3 = Q\Lambda^3 Q^T = 0$

Thus $\Lambda^3 = 0 \Rightarrow \Lambda = 0 \Rightarrow A = 0$.

$\boxed{9}$ Proof: Note that the eigenvalues are the roots of the polynomial. $\det|\lambda I - A|$

If $a+bi$ is a root of the polynomial, then so is $a-bi$. A 3×3 matrix yields a polynomial of degree 3. Since complex roots come out in pairs. at least one root is real.

$\boxed{15}$ Proof: $|\lambda I - A| = \begin{vmatrix} \lambda - i & -1 \\ -1 & \lambda + i \end{vmatrix} = \lambda^2 + 1 - 1 = \lambda^2.$

Thus the eigenvalues are 0 with multiplicity of 2.

$Ax = 0 \Rightarrow x = a\begin{pmatrix} 1 \\ -i \end{pmatrix}$, $a \in \mathbb{C}$, which means that there is only one line of eigenvectors.

$\boxed{21.}$ (a) F $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) T $\quad A = Q\Lambda Q^T \Rightarrow A^T = (Q\Lambda Q^T)^T = Q\Lambda^T Q^T = Q\Lambda Q^T.$

(c) T. $\quad$ Symmetry $\Rightarrow A = Q\Lambda Q^T \Rightarrow A^{-1} = Q\Lambda^{-1}Q^T$

$\Rightarrow (A^{-1})^T = Q(\Lambda^{-1})^T Q^T = Q\Lambda^{-1}Q^T = A^{-1}.$

(d) F. $\quad$ Refer to the above exercise for examples.

$\boxed{23}$ $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$, i.e. A is a permutation $\Rightarrow$ orthogonal.

$\Rightarrow$ diagonalizable & invertible

The column adds up to 1 $\Rightarrow$ Markov.

$A^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \neq A \Rightarrow$ <u>Not</u> projection.

Possible factorization: $S\Lambda S^{-1}$, $Q\Lambda Q^T$ (because diagonalizable and invertible)

QR (Using Gram-Schmidt).

<u>Not</u> LU (The principal minors of A can be zero).

B: Markov.

$B^2 = \frac{1}{9}\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = B \Rightarrow$ projection.

Symmetric $\Rightarrow$ diagonalizable $\quad$ determinant $= 0 \Rightarrow$ <u>Not</u> invertible

Not orthogonal. $\Rightarrow$ Not permutation.

Since B doesn't have independent columns. Not QR.

(A way of verification) The minors of B are zero, Not LU.

Symmetric $\Rightarrow$ $S\Lambda S^{-1}$, $Q\Lambda Q^T$

[Note: Refer to $P_{564} \sim P_{565}$ of the textbook for different factorizations requirement]

§ 6.5

$\boxed{9}$. $4(x_1-x_2+2x_3)^2 = 4(\underline{x_1^2}-2x_1x_2+\underline{x_2^2}+\underline{4x_3^2}+4x_1x_3-4x_2x_3)$

~~$=4x_1(x_1-2x_2+4x_3)+$~~

Put the coefficients of the squares on the diagonal, and the $\frac{\text{coefficients}}{2}$ of the cross entry on the corresponding position. eg. the $\frac{\text{coefficients}}{2}$ of x_1x_2 as the entry a_{12}.

thus, $A = \begin{bmatrix} 4 & -4 & 8 \\ * & 4 & -8 \\ * & * & 16 \end{bmatrix}$ By symmetry of A, we have

$$A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}.$$

Find pivots: $A \Rightarrow \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank of $A = 1$. (No. of pivots) $\Rightarrow$ Not invertible

$\Rightarrow \det(A) = 0$.

$|\lambda I - A| = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 24.$

$\boxed{15}$ Proof: By definition $x^T A x > 0$, $x^T B x > 0$ for any $x \neq 0$.

Thus $x^T(A+B)x = x^T A x + x^T B x > 0$ for any $x \neq 0$.

$A+B$ is positive definite.

$\boxed{20}$ (a). All eigenvalues are positive, $|\text{determinant}|$ = product of eigenvalues $\neq 0 \Rightarrow$ invertible.

(b). Projection $\Rightarrow P^2 = P$ ~~$\Rightarrow \lambda^2 = \lambda$.~~

~~positive definite $\Rightarrow \lambda = 1 > 0$~~

~~Thus all eigenvalues $= 1$!~~

positive definite $\Rightarrow P$ invertible. $\quad P^2 = P \Rightarrow P(P-I) = 0$

$\Rightarrow P^{-1}P(P-I) = 0 \Rightarrow P - I = 0 \Rightarrow P = I$.

(c). positive diagonal entries are eigenvalues/pivots.

$\Rightarrow$ positive definite.

(d). $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\boxed{29}$ $\dfrac{\partial F_1}{\partial x} = x^3 + 2xy$, $\dfrac{\partial F_1}{\partial y} = x^2 + 2y$, $\dfrac{\partial^2 F_1}{\partial x \partial y} = 2x = \dfrac{\partial^2 F_1}{\partial y \partial x}$.

$\dfrac{\partial^2 F_1}{\partial x^2} = 3x^2 + 2y$, $\dfrac{\partial^2 F_1}{\partial y^2} = 2$. $\Rightarrow H_1 = \begin{bmatrix} 3x^2+2y & 2x \\ 2x & 2 \end{bmatrix}$.

$\begin{cases} \dfrac{\partial F_1}{\partial x} = 0 \\ \dfrac{\partial F_1}{\partial y} = 0 \end{cases} \Rightarrow x^2 + 2y = 0. \quad \Rightarrow H_1 = \begin{bmatrix} 2x^2 & 2x \\ 2x & 2 \end{bmatrix}$. $\det H_1 = 0$.

<u>positive semidefinite</u>.

$$\frac{\partial F_2}{\partial x} = 3x^2 + y - 1, \quad \frac{\partial F_2}{\partial y} = x, \quad \frac{\partial^2 F_2}{\partial x \partial y} = 1 = \frac{\partial^2 F_2}{\partial y \partial x}.$$

$$\frac{\partial^2 F_2}{\partial x^2} = 6x, \quad \frac{\partial^2 F_2}{\partial y^2} = 0 \Rightarrow H_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

$\det H_2 = -1. \Rightarrow$ Not positive definite.

$$\begin{cases} \frac{\partial F_2}{\partial x} = 0 \\ \frac{\partial F_2}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1. \end{cases}$$ saddle point.

$\boxed{31}$ $\frac{\partial z}{\partial x} = 8x + 12y, \quad \frac{\partial z}{\partial y} = 12x + 2cy$

$$\begin{cases} \frac{\partial z}{\partial x} = 0 \\ \frac{\partial z}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} x = ? \\ y = ? \end{cases} \Rightarrow \begin{cases} c = 9: 2x + 3y = 0 \\ c \neq 9: x = y = 0 \end{cases}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 12 = \frac{\partial^2 z}{\partial y \partial x}. \quad \frac{\partial^2 z}{\partial x^2} = 8, \quad \frac{\partial^2 z}{\partial y^2} = 2c.$$

$$H = \begin{bmatrix} 8 & 12 \\ 12 & 2c \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 \\ 12 & 2c \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 2c - 18 \end{bmatrix}$$

Thus, when $2c - 18 > 0$, i.e. $c > 9$. H positive definite.

the graph is a bowl with $(0,0)$ as minimum.

when $c < 9$, H is not positive definite, $\lambda_1 > 0$, $\lambda_2 < 0$.

the graph has a saddle point at $(0,0)$

when $c = 9$, the graph becomes $z = (2x + 3y)^2$. it stays

zero at the line $2x + 3y = 0$ (H is positive semidefinite).

§ 6.6.

$\boxed{6}$. Let's put $A = \begin{bmatrix} a & d \\ c & b \end{bmatrix}$.

$|\lambda I - A| = 0 \Rightarrow \lambda^2 - (a+b)\lambda + ab - dc = 0$.

① $a+b=0 \Rightarrow a=b=0$

i). d or c equals 0

$\lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$.

2 family
- $d=c=0$: zero ~~#~~ matrix
- $d=0, c=1$ or $d=1, c=0$: Jordan from J or J^T

ii). $d=c=1 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$.

One family : one matrix here.

② $a+b=1 \Rightarrow a=1, b=0$ or $a=0, b=1$.

i). d or c equals 0.

$\lambda^2 - \lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 1$

One family, 2×3 matrices. (because cases $a=0$ $b=1$ and $a=1$, $b=0$ are similar).

ii). $d=c=1 \Rightarrow \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$.

One family, 2 matrix.

③ $a+b=2 \Rightarrow a=b=1$.

i). d or c equals 0.

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$.

2 family
- $d=c=0$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $d=0, c=1$ or $d=1, c=0$: Jordan from J or J^T

ii $d=c=1 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$

One family. 1 matrix.

18 Proof: With n independent eigenvectors, we can write $A = Q\Lambda Q^T = B$.

When both have eigenvalues 0, 0 but only one line of eigenvector:

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \quad a \neq b.$$

15 Proof:
$$\begin{aligned} \det(M^{-1}AM - \lambda I) &= \det(M^{-1}AM - \lambda M^{-1} I M) \\ &= \det(M^{-1}(A-\lambda I)M) \\ &= \det(M^{-1})\det(A-\lambda I)\det(M) \\ &= \det(A-\lambda I). \end{aligned}$$

17 a) False. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ similar. (Refer to Ex. 6).

b) T. A invertible MAM^{-1} also invertible.

c) F. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar.

d) T. The eigenvalues are different.

§ 6.7. ~~Wit~~

1. With $A^TA = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. we have

$\lambda_1 = 50$, $\lambda_2 = 0$

$v_1 = \frac{1}{\sqrt{5}}(1,2)'$. $v_2 = \frac{1}{\sqrt{5}}(-2,1)'$.

$\Rightarrow \sigma_1 = \sqrt{\lambda_1} = \sqrt{50}$, $\sigma_2 = \sqrt{\lambda_2} = 0$.

$u_1 = \frac{Av_1}{\sigma_1} = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})'$

$AA^Tu_1 = \lambda_1 \cdot u_1 \Rightarrow u_1$ is indeed an unit eigen ~~vector~~ vector of AA^T.

$u_2 = (\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}})'$.

Thus, SVD $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{50} & \\ & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}_{V}{}^T$.

2. The column space $C(A)$ has only one basis:

$(1,3)/\sqrt{10}$.

The row space $C(A^T)$ also has only one basis:

$(1,2)/\sqrt{5}$.

The last 2-1 column of V spans the $N(A)$.

i.e. $(-2,1)/\sqrt{5}$

The last 2-1 column of U spans the $N(A^T)$

i.e. $(-3,1)/\sqrt{10}$.

$\boxed{4}$ $A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = AA^T$.

$|\lambda I - A^TA| = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2} \Rightarrow \sigma_1^2 = \frac{3+\sqrt{5}}{2},\ \sigma_2^2 = \frac{3-\sqrt{5}}{2}$.

Unit eigenvectors: $v_1 = \frac{2}{5+\sqrt{5}}\left(\frac{1}{2}(1+\sqrt{5}), 1\right)$, $v_2 = \frac{2}{5-\sqrt{5}}\left(\frac{1}{2}(1-\sqrt{5}), 1\right)$

since $\sigma_i \geq 0$,

$\sigma_1 = \frac{1+\sqrt{5}}{2}$, $\sigma_2 = \frac{\sqrt{5}-1}{2}$

$u_1 = \frac{Av_1}{\sigma_1} = v_1$, $u_2 = \frac{Av_2}{\sigma_2} = -v_1$.

$$A = \begin{pmatrix} \frac{(1+\sqrt{5})}{5+\sqrt{5}} & -\frac{(1-\sqrt{5})}{5-\sqrt{5}} \\ 2/_{5+\sqrt{5}} & -2/_{5-\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \\ & \frac{\sqrt{5}-1}{2} \end{pmatrix} \begin{pmatrix} \frac{(1+\sqrt{5})}{5+\sqrt{5}} & \frac{(1-\sqrt{5})}{5-\sqrt{5}} \\ 2/_{5+\sqrt{5}} & 2/_{5-\sqrt{5}} \end{pmatrix}^T$$

$\boxed{6}$. $A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\underline{v}$

$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\underline{u}$

Find the eigenvalues of A^TA and orthonormal eigenvectors

$\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0 \Rightarrow \sigma_1 = \sqrt{3}$, $\sigma_2 = 1$.

$v_1 = \frac{1}{\sqrt{6}}(1,2,1)$, $v_2 = \frac{1}{\sqrt{2}}(-1,0,1)$, $v_3 = \frac{1}{\sqrt{3}}(1,-1,1)$

$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}(1,1)$

$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}}(-1,1)$.

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}^T$$

$\boxed{9}$. $Av_i = u_i$ implies $\sigma_i = 1$

$A = U\Sigma V^T = UIU^T = UV^T$.

i.e, construct A as $[u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

$\boxed{11}$. $A^TA = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$, thus $\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$

$V = [\frac{w_1}{\sigma_1}, \frac{w_2}{\sigma_2} \cdots \frac{w_n}{\sigma_n}]$.

$Av_i = \sigma_i u_i \Rightarrow U = I$

Thus $A = U\Sigma V^T = I \cdot \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} [\frac{w_1}{\sigma_1} \cdots \frac{w_n}{\sigma_n}]^T$,