Math 415 Solution 07-06 ~ 07-08

§ 4.3.

#1. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Solve $A^T A \hat{x} = A^T b$.

You can input the command = linear fit{{0,0},{1,8},{3,8},{4,20}} in Wolframalpha.com (Or using Matlab with the command (A'*A)\(A'*b))

The best-fit line is $1+4t$. So $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$, $e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$, $E = \|e\|^2 = 44$.

#4. $E = \|Ax-b\|^2 = \left\| \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} - \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} C \\ C+D-8 \\ C+3D-8 \\ C+4D-20 \end{bmatrix} \right\|^2$

$= C^2 + (C+D-8)^2 + (C+3D-8)^2 + (C+4D-20)^2$.

$\partial E / \partial C = 2C + 2(C+D-8) + 2(C+3D-8) + 2(C+4D-20) = 0$

$\partial E / \partial D = 2(C+D-8) + 6(C+3D-8) + 8(C+4D-20) = 0$.

Divide the equations by 2. we get

$$\begin{cases} 4C + 8D - 36 = 0 \\ 8C + 26D - 112 = 0. \end{cases} \quad (*)$$

$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$

$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$

So (*) implies $A^T A \hat{x} = A^T b$.

#12. Sol: (a) $a^T a = (\underbrace{1, \cdots, 1}_{m})(\underbrace{1, \cdots, 1}_{m})^T = m$.

$a^T b = (1, \cdots, 1)(b_1, \cdots, b_m)^T = \sum_{i=1}^{m} b_i$

Hence by solving $a^T a \hat{x} = a^T b$, we get

$$\hat{x} = \frac{1}{m}\sum_{i=1}^{m} b_i$$

(There ~~are~~ commands such as "linear fit" and "mean{...}" in Wolframalpha for your reference).

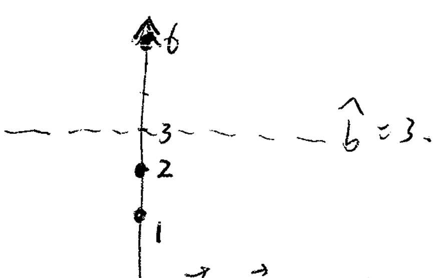
(b). $e = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \frac{1}{m}\sum_{i=1}^{m} b_i = \begin{pmatrix} \frac{\sum_{i=2}^{m}(b_1 - b_i)}{m}, \\ \vdots \\ \sum_{i=1}^{m-1}(b_m - b_i) \end{pmatrix}$

$\cancel{\|e\|^2 = \frac{1}{m^2}\cdot 2 \cdot \sum_{\substack{i \neq j \\ 1 \le i < j \le m}} (b_i - b_j)^2}$ $\|e\|^2 = \frac{1}{m^2}\left(\sum_{k=1}^{m}\left(\sum_{i \neq k}^{m}(b_k - b_i)\right)^2\right)$

(Use command "Variance {x, y, z}". Unfortunately, the number of variable must be specific values.)

$$\|e\| = \frac{1}{m}\left(\sum_{k=1}^{m}\left(\sum_{i \neq k}(b_k - b_i)^2\right)\right)^{\frac{1}{2}}.$$

(c). the mean of b is 3. Hence the horizontal line $\hat{b} = 3$ is closest to b.



$\vec{e} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$

$\vec{p} \cdot \vec{e} = 3 \cdot (-2) + 3 \cdot (-1) + 3 \cdot 3 = 0$. $\left(\vec{p} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}\right)$

So $\vec{p}$ is perpendicular to $\vec{e}$.

Projection Matrix $P = \frac{p p^T}{p^T p} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

#13 & #14: Refer to the back of the textbook P_{532}.

#15. $\sigma^2(A^TA)^{-1} = \sigma^2\left[(1,1,1,1)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right]^{-1} = \sigma^2 \cdot 4^{-1} = \frac{1}{4}\sigma^2.$

#22. Input "linear fit {{-2,4}, {-1,2}, {0,-1}, {1,0}, {2,0}}"

in Wolfram-Alpha.

The result is $1-1t$.

§4.4

#1. (a) Only independent.

(b) Only orthogonal

(c) orthonormal.

#4c. $q_1 = (1,1,1)/\sqrt{3}$. Note: the answer is not unique.

$q_2 = (1,0,-1)/\sqrt{2}$ The general idea is using Gram-Schmidt.

$q_3 = (-1,2,-1)/\sqrt{6}$

#10. (a) Vector proof: Suppose $\vec{q}_1 \cdots \vec{q}_n$ are orthonormal vectors.

assume $c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots + c_n\vec{q}_n = 0$ $(*)$

multiply $\vec{q}_i$ on both sides of $(*)$, we have

$c_i = 0$. Thus $\vec{q}_1 \cdots \vec{q}_n$ are independent.

(b) Matrix proof: suppose $Q = [\vec{q}_1 \cdots \vec{q}_n]$

$Q^TQx = 0$. Note that Q^TQ is invertible,

we conclude $x = 0$.

Thus $\vec{q}_1 \cdots \vec{q}_n$ are independent.

#11. Refer to the back of the textbook P533.

($Q = [\vec{q}_1, \vec{q}_2]$)

#13 - #14. #15 (As above: P533 for answer).

(Input "qr decomposition {matrix}" in Wolfram-Alpha).

#22. (As above)

(You can input "orthogonalize $\{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n\}$" in Wolfram-Alpha for the answer, the outputs are orthonormal vectors).

#24. (a) $\mathbb{R}^4$, with one constraint, the basis is consist of 3 vectors in $\mathbb{R}^4$.

(Not unique: $(1,-1,0,0), (1,0,-1,0), (1,0,0,1)$)

(b) $\dim(S^\perp) = 4 - \dim(S) = 1$.

$(1, 1, 1, -1)$

(c). $b = (1, 1, 1, 1)$

$b_2 = (1, 1, 1, -1)$

$b_1 = b - b_2 = (0, 0, 0, 2)$ (again, not unique decomposition).

§ 8.5.

#2. Proof. $\int_{-1}^{1} 1\cdot x\,dx = 0$, $\int_{-1}^{1} 1\cdot(x^2-\frac{1}{3})\,dx = \frac{x^3}{3}\Big|_{-1}^{1} - \frac{1}{3}x\Big|_{-1}^{1}$

$= \frac{2}{3} - \frac{2}{3} = 0$

$$\int_{-1}^{1} x\cdot(x^2-\tfrac{1}{3})\,dx = \int_{-1}^{1} x^3\,dx - \tfrac{1}{3}\int_{-1}^{1} x\,dx = 0.$$

So they're orthogonal.

$$f(x) = 2x^2 = 2(x^2-\tfrac{1}{3}) + \tfrac{2}{3}\cdot 1$$

#3. Sol: $\vec{w} = (w_1, w_2, w_3, \ldots)$, $\vec{v} = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$

$$\vec{w}\cdot\vec{v} = 0 \Rightarrow \sum_{i=1}^{\infty} w_i \cdot \frac{1}{2^{i-1}} = 0. \qquad (*)$$

Let $w_1 = -1$, $w_2 = w_3 = \cdots = 1$, this satisfies $(*)$, but the length $\|w\|$ is infinite, doesn't work.

Refine the idea. Let

$$w_i = \frac{1}{2^{i-1}} \quad \text{for } i \geq 2.$$

$$\sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \cdot \frac{1}{2^{i-1}} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}.$$

Let $w_1 = -\frac{1}{3}$.

Thus $\vec{w}$ can be $(-\frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \ldots)$

$$\|\vec{w}\| = \sqrt{\frac{1}{9} + \frac{1}{3}} = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

(Not unique!).

#4. Refer to P547 for the answer.

#8b (As above).