**Quiz 10, Math. 415,** **Friday, July 24th, 2009**

Explain your answers carefully. Write complete sentences, not just formulas.

**1** (15 points) Consider the curve in the $x$-$y$-plane, with equation

$$f(x,y) = \mathbf{x}^t A \mathbf{x} = 1, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Is the curve an ellipse or an hyperbola? Explain you answer.

Since $1 > 0$, $\det A = 1 \times 1 - 2 \times 2 = -3 < 0$.

$A$ is an indefinite matrix. Hence $f(x,y)$ is a hyperbola.

Another way to solve this problem

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0.$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = -1.$$

**2.** (15 points) Calculate the singular values of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

$$B = AA^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 1$$

So the singular values of the matrix are.

$$\sigma_1 = \sqrt{3}, \ \sigma_2 = 1$$

**3.a** (10 points) Explain when two $n \times n$ matrices $A, B$ are *similar.*

$A \cdot B$ are similar if and only if there exists an invertible matrix $M$, such that $A = M^{-1} B M$.

**3.b** (10 points) Show that $\det(A) = \det(B)$ if $A, B$ are similar.

$A = M^{-1} B M$ for some invertible matrix, $M$.

We know $\det(M^{-1}) = (\det M)^{-1}$. So,

$\det(A) = \det(M^{-1} B M) = \det(M^{-1}) \det(B) \det(M)$

$= \det(B)$.

**3.c** (10 points) If $A, B$ are similar, and $A$ has eigenvector $\mathbf{x}$ and eigenvalue $\lambda$, find an eigenvector and eigenvalue for $B$. Explain.

Since $A = M^{-1} B M$, $A x = \lambda x$,

$A x = M^{-1} B M x = \lambda x$.

So, $B M x = \lambda M x$.

And $M x$ is an eigenvector for $B$ with $\lambda$ as its eigenvalue.

**4** You know that a rank one matrix $A$ can always be written as $A = \mathbf{x}\mathbf{y}^t$ for some column vectors $\mathbf{x}, \mathbf{y}$.

**a.** (15 points) If $A = \mathbf{x}\mathbf{y}^t$ is an $m \times n$ matrix of rank one, what are the sizes of $\mathbf{x}$ and $\mathbf{y}$? So the questions is to find $p, q$ such that $\mathbf{x} \in \mathbb{R}^p, \mathbf{y} \in \mathbb{R}^q$.

$p = m \quad q = n.$

$A = x_{m\times 1} (y^t)_{1\times n}$ is then a $m\times n$ matrix.

**b.** (15 points) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find $\mathbf{x}, \mathbf{y}$ such that $A = \mathbf{x}\mathbf{y}^t$.

Let $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Then

$x y^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1\ 2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$

[ Note rank $A = 1$ if $A = xy^t$. To find $x, y$;

Simply write down the first column ( or another column) as $x$.

then determine ~~the~~ $y$. Or first reduce $A$ to a ~~one row~~ matrix with only one non-zero row. This row is a ~~the~~ candidate for $y^t$ )