Solution for Quiz 9, Math 415

1. $|\lambda I - A| = \begin{vmatrix} \lambda-3 & 1 \\ 1 & \lambda-3 \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4)$

Thus $\lambda_1 = 2$, $\lambda_2 = 4$.

The eigenvectors are $v_1 = (1, 1)'$, $v_2 = (1, -1)'$

$u(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \frac{3}{2} v_1 + \frac{3}{2} v_2$, $c_1 = c_2 = \frac{3}{2}$

Thus $u(t) = \frac{3}{2} e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2} e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, since $A^2 = A^3 = \cdots = A^n = 0$ $n \geq 2$.

Thus $\exp(tA) = I + At = \begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix}$

3. Let $u = (y, y')'$

$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -4y' - 3y \end{bmatrix}$

$\frac{du}{dt} = Au = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} u$

4. Since $f(x, y, z) = 4x^2 + 3y^2 + 6z^2 - 2xy + 4xz + 8yz$

$A = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$

5. $f(x,y) = x^t A x = x^2 + y^2 + 4xy$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = 2x + 4y = 0 \\[2ex] \dfrac{\partial f}{\partial y} = 2y + 4x = 0 \end{array}\right\} \Rightarrow \begin{cases} x = 0 \\ y = 0. \end{cases}$$

The Hessian matrix $H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

~~It's positive definite~~

$$|\lambda I - H| = (\lambda - 2)^2 - 16 = \lambda^2 - 4\lambda - 12 = 0$$

$$\lambda = \frac{4 \pm \sqrt{64}}{2} = \frac{4 \pm 8}{2} \Rightarrow \lambda_1 = 6, \ \lambda_2 = -2.$$

Thus the critical point $(0,0)$ is a saddle point.

6. Refer to the textbook for definition.

We can find its pivots to see if it's positive definite.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

all positive, it's positive definite.