Solutions to HW 3

3.1) 5. see page 525.

10. (a) Yes. Let $c, d \in \mathbb{R}$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, $v_1 = v_2$, $w_1 = w_2$. Then $c\vec{v} + d\vec{w} = (cv_1 + dw_1, cv_2 + dw_2, cv_3 + dw_3)$
$cv_1 + dw_1 = cv_2 + dw_2$.

(b). No. If $(1, x, y)$ is in the plane, $c(1, x, y)$ is not in the plane if $c \neq 1$.

(c). No. if $\vec{v} = (0, 1, 1)$, $\vec{w} = (1, 0, 1)$, $\vec{v} + \vec{w}$ is not in the subset.

(d). Yes. By definition.

(e). Yes. Notation as in (a). If $v_1 + v_2 + v_3 = 0$, $w_1 + w_2 + w_3 = 0$.
$cv_1 + dw_1 + cv_2 + dw_2 + cv_3 + dw_3 = 0$.

(f). No. If $v = (1, 2, 3)$, then $-v$ is not in the subset.

17(a) This is simply because $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible.

18(a). True. If ~~A = B~~ $A = A^T$, $B = B^T$, then $(cA + dB)^T = cA^T + dB^T$
$= cA + dB$.

19 b. $C(A) = \{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : x_1 \in \mathbb{R} \}$.

$C(B) = \{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} : x_1, x_2 \in R \}$

$C(C) = \{ x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} : x \in \mathbb{R} \}$.

23. $Ax = b$ is solvable if and only if b is in the column space of A. this means that the column space of A is the same with that of $[A\ b]$.

29. $\mathbb{R}^9$. By 23, $[A\ b]$ and A are the same, but b is arbitrary in $\mathbb{R}^9$. Hence $C(A) = \mathbb{R}^9$.

3.2)

9. (a) False. Consider $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

(b) True. Elimination leads to an upper-triangle matrix with nonzero diagonal.

(c),(d) True. The number of pivot variables can not exceed the size of matrix.

13. If column 4 is all zero, then x_4 is a <u>free</u> variable.
Special solution $x = (0, 0, 0, 1, 0)^T$

15. Number of special solution $\underline{n - r}$, because one free variable corresponds to one special solution.

The nullspace contains only $x = 0$ when $\underline{r = n}$

The column space is all of $\mathbb{R}^m$ when $\underline{r = m}$

18. ~~Thus~~ $x = 12 + 3y + z$ so the complete solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

23 The vector in null space has three components. This means the matrix is 3×3. Set

$$\begin{pmatrix} 1 & 0 & x \\ 1 & 3 & y \\ 5 & 1 & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We get $x = -\frac{1}{2}$, $y = -2$, $z = -3$

25. observe $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Adding the frist row of the matrix to 2^{nd} and 3^{rd} row, we get $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & -2 \\ 1 & 0 & 1 & -2 \end{pmatrix}$.

It has 3 pivots. the nullspace is a line.

3.3).

8. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}$. $B = \begin{pmatrix} 3 & 9 & \frac{9}{2} \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{pmatrix}$.

$M = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}$ if $a \neq 0$; If $a = 0$. c has to be 0.

$M = \begin{pmatrix} 0 & b \\ 0 & x \end{pmatrix}$. $x \in \mathbb{R}$.

10. observe the multiple relationship of rows and colums, one can get

$$\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}(1, 2, 2) = \begin{pmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}(1, 1, 3, 2) = \begin{pmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{pmatrix}.$$

16. $(uv^T)\cdot(wz^T) = u(v^T w)z^T = \underline{v^T w}\, u z^T$

rank $(u v^T w z^T) = 1$ unless $\underline{v^T w} = 0$. ($\because u z^T \neq 0$)

17 a. Denote the jth column of A as A_j.

Clearly, $A(B_j) = (AB)_j$.

Suppose $B_j = \sum_{k=1}^{j-1} a_k B_k$

$$(AB)_j = A(B_j) = A\left(\sum_{k=1}^{j-1} a_k B_k\right) = \sum_{k=1}^{j-1} a_k A(B_k)$$
$$= \sum_{k=1}^{j-1} a_k (AB)_k.$$

19. By 17(a), rank$(AB) \leq$ rank (A). But $AB = I$.

Hence, rank$(A) \geq n$. But A is an n by n matrix.

Hence rank$(A) = n$.

So A is invertible and $BA = I$.

20. Since rank AB = rank $I_{2\times 2} = 2$. and rank $AB \leq$ rank A,

rank $A = 2$. Similarly, rank $B = 2$.

So, rank $BA \leq$ rank $A = 2$.

But BA is a 3×3 matrix. Hence $BA \neq I$.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$

$AB = I$. But $BA \neq I$

27. (a) $$R = \begin{pmatrix} \underset{r\times r}{I} & \underset{r\times(n-r)}{F} \\ \underset{(m-r)\times r}{0} & \underset{(m-r)\times(n-r)}{0} \end{pmatrix}$$

(b). Let $B = \begin{pmatrix} I \\ \underset{(n-m)\times m}{0} \end{pmatrix}$. Then $RB = I$

(c). Let $C = \begin{pmatrix} I & \overset{n\times(m-n)}{0} \end{pmatrix}$ Then $CR = I$

3.4)

2. see P527

4. $\begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

So.

$$x_1 = -3x_2 + \tfrac{1}{2}$$

$$x_3 = -2x_4 + \tfrac{1}{2}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

6. see P527.

8. see P527

22. Since $Ax = b$ has infinitely many solutions, A is not of full column or row rank, and rank (A) = rank $(A\ b)$.
So $Ax = B$ has 0 or infinitely many solutions.
$Ax = B$ could have no solution when rank $A <$ rank (A, b)

32. Following the procedure of section 2.6. we get

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = LU$$

$$Ax = b \Rightarrow LUx = b.$$

Let $c = Ux$. Then $Lc = b$.

Solving this system, we get $c = (1, 2, 0, 0)^T$.

Now we have

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 7 & 7 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = 7 - 7x_3$$

$$x_2 = -2 + 2x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}.$$

The same approach works when $b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.