Quiz 4, Math. 415, **Friday, June 26th, 2009**

Explain your answers carefully. Write complete sentences, not just formulas.

1 Consider the matrix $A = \begin{pmatrix} 1 & 2 & 10 \\ 2 & 4 & 20 \\ 4 & 8 & 40 \end{pmatrix}$.

1.a (10 points) What is the rank of A?

$$\begin{pmatrix} 1 & 2 & 10 \\ 2 & 4 & 20 \\ 4 & 8 & 40 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Heme the rank of A is 1.

1.b (10 points) How many free variables has A?

rank $(A) = 1$

So A has $3-1=2$ free variables.

1.c (10 points) Find all solutions of $Az = 0$.

$$x_1 = -2x_2 - 10x_3$$
$$x_2 = x_2$$
$$x_3 = x_3$$

Heme

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -10 \\ 0 \\ 1 \end{pmatrix}.$$

Still $A = \begin{pmatrix} 1 & 2 & 10 \\ 2 & 4 & 20 \\ 4 & 8 & 40 \end{pmatrix}$.

1.d (15 points) Find the condition(s), if any, on $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, so that the system $A\mathbf{x} = \mathbf{b}$ is solvable.

In order that $Ax = b$ admit a solution,

rank $(A, b) = \text{rank}(A)$.

Note that row 2 = 2·(row 1). row 3 = 4·(row 1)

b must satisfy:

$$b_2 = 2b_1$$
$$b_3 = 4b_1 \qquad (*)$$

Clearly, these conditions are also sufficient.

e. (15 points) If $\mathbf{b} = \begin{pmatrix} -3 \\ -12 \\ -24 \end{pmatrix}$, is $A\mathbf{x} = \mathbf{b}$ solvable? If so find all solutions, otherwise explain.

No solution because b does not satisfy condition $(*)$ in (1.d)

2. Let A be a 3×4 matrix (3 rows, 4 columns).
a. (10 points) Can the rank of A be 4? Explain!

No. rank of $A \leq \min(3, 4) = 3$.

That is rank $(A) \leq 3$.

b. (20 points) If the rank of a 3×4 matrix is 3, what can you say about the system $A\mathbf{x} = \mathbf{b}$? Is it solvable for all $\mathbf{b}$? If it is solvable for a given $\mathbf{b}$, is the solution unique?

Since the rank of A is equal to its row.
Whatever b is, the rank of (A, b) has to be 3.
So rank $(A) =$ rank (A, b). Hence $Ax = b$ is always solvable for all b.

For a given b, the solution is not unique. because there is alway a free variable.

In fact, it is the case 2 of the four possibilities on P 159 of the text book.