Quiz 6, Math. 415, **Wednesday, July 8th, 2009**

Explain your answers carefully. Write complete sentences, not just formulas.

1 (15 points) Let P^2 be the vector space of all polynomials of degree 2 or less and let $V \subset P^2$ be the subspace of polynomials $f(x)$ with $f(0) = 0$. (This is a subspace, but you don't need to prove this.)

Find a basis for V and its dimension .

A basis of P^2 is $\{1, x, x^2\}$.

The general form of polynomials in P^2 is

$$f(x) = a_0 + a_1 x + a_2 x^2.$$

Now we have $f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0 = 0$.

so the complete solution is

$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The vectors describe the coordinate of a basis ~~for~~ for V. Hence a basis for V is $\{x, x^2\}$.

$\dim V = 2$

2 (15 points) Let $\mathtt{Mat}_{3\times 3}$ be the vector space of 3×3 matrices, and let $\mathcal{U}$ be the subspace of *upper triangular* matrices: elements of $\mathcal{U}$ have the form $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$.

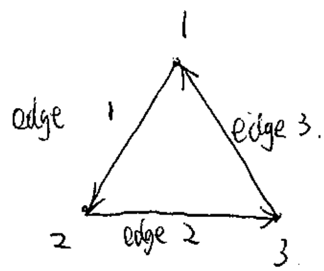
Find a basis for $\mathcal{U}$ and its dimension.

A basis for $\mathcal{U}$ is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$\dim \mathcal{U} = 6$.

3 (15 points) Consider the *incidence* matrix $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$. Draw the network corresponing to A. Carefully label the edges and nodes, and indicate the direction of the edges.



4 (15 points) Consider the matrix $A = \begin{pmatrix} 2 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. You know that $\mathbf{x}$ can be wrtitten as $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ where $\mathbf{x}_r$ is in the row space of A and $\mathbf{x}_n$ in the null space. Find $\mathbf{x}_r$ and $\mathbf{x}_n$.

Clearly, the row space of A has dimension 1.
a basis is $\{ (1, -1) \}$.

The nullspace of A has dimension $2-1 = 1$.
By observation (or you can solve the linear system), a basis for $N(A)$ is $\{ (1, 1) \}$.

Now write $x = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
By observation (or you can solve $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$), we get $a = b = \frac{1}{2}$. Hence.

$$x = x_r + x_n = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

5 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

a. (15 points) Find the projection $\mathbf{p}$ of $\mathbf{b}$ on the line L through $\mathbf{a}$.

$$p = \frac{a \cdot b}{a \cdot a} a = \frac{a^T b}{a^T a} \cdot a = \frac{5}{3} a$$

$$= \begin{pmatrix} \frac{5}{3} \\ \frac{5}{3} \\ \frac{5}{3} \end{pmatrix}$$

b. (15 points) If L is still the line through $\mathbf{a}$, what is the point on L that is closest to $\mathbf{b}$?

It is the projection $p = \begin{pmatrix} \frac{5}{3} \\ \frac{5}{3} \\ \frac{5}{3} \end{pmatrix}$.

because $\hat{x} = \dfrac{a \cdot b}{a \cdot a}$ minimizes $E = \| b - a \cdot x \|^2$