Homework I
Due February 8

Throughout, R is a (unital) ring, M, M_λ and N are (unital) left R modules.

1. Prove that $Hom_R(M, M)$ is a ring via composition.

2. Show that $Hom_R(M, N)$ and $Hom_R(N, M)$ are left and right $Hom_R(M, M)$ modules via composition.

3. Prove that there is a ring isomorphism $Hom_R(M^{\oplus n}, M^{\oplus n}) \cong M_{n\times n}(Hom_R(M, M))$.

4. Recall that if G is a group we can form the group ring $R[G]$ which is also naturally an R–module. Suppose G is finite. Is $Hom_R(R[G], R[G])$ isomorphic to a group ring $R[H]$ for some group H? If not, why not? If so, prove it.

5. Suppose now that M is actually an $R-S$ module (that is, S is a ring and M is an $R-S$ bimodule). Prove that $Hom_R(M, N)$ is naturally a left S–module and that $Hom_R(N, M)$ is naturally a right S–module.

6. Let $I \subseteq R$ be a left ideal. Let $M_I = \{m \in M \mid rm = 0 \text{ if } r \in I\}$. Prove that $Hom_R(R/I, M) \cong M_I$.

7. Show that the following are groups are trivial: $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z})$, and $Hom_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$ when g.c.d.$(m, n) = 1$.

8. Let Λ be a possibly infinite set. Show the following natural maps are generally not isomorphisms by explicit example (be sure to prove your examples):

$$\bigoplus_{\lambda\in\Lambda} Hom_R(M_\lambda, N) \longrightarrow Hom_R(\prod_{\lambda\in\Lambda} M_\lambda, N) \longrightarrow \prod_{\lambda\in\Lambda} Hom_R(M_\lambda, N)$$

$$\bigoplus_{\lambda\in\Lambda} Hom_R(N, M_\lambda) \longrightarrow Hom_R(N, \bigoplus_{\lambda\in\Lambda} M_\lambda) \longrightarrow \prod_{\lambda\in\Lambda} Hom_R(N, M_\lambda)$$