Homework 2–partial solutions
Due February 22

**1.** Recall that if $A$ is a finitely generated abelian group, then $A \cong \mathbb{Z}^{\oplus r} \oplus T$ where $T$ is a torsion group and the number $r$ is called the *rank* of $A$. Compute the colimit of

$$A \xrightarrow{2} A \xrightarrow{3} A \xrightarrow{4} A \xrightarrow{5} A \xrightarrow{6} \cdots$$

in terms of the rank of $A$. Hint: It may be useful to first do the case when $A = \mathbb{Z}$.

Ans: $\mathbb{Q}^{\oplus r}$

We define $\alpha_n : A \longrightarrow \mathbb{Q}^{\oplus r}$ by $\alpha_n(a_1, \ldots, a_r; t) = (\frac{a_1}{n!}, \frac{a_2}{n!}, \ldots, \frac{a_r}{n!})$. The $\alpha_n$ are compatible with the colimit maps and so by the universal property of colimits there is a unique homomorphism $\alpha$ from $\mathrm{colim}_n A$ to $\mathbb{Q}^{\oplus r}$. We see that $\alpha$ is surjective since given an $r$-tuple of rational numbers, $(q_1, \ldots, q_r)$ there is some $n$ and integers $(z_1, \ldots, z_r)$ such that $q_j = \frac{z_j}{n!}$ for all $1 \le j \le r$. To show that $\alpha$ is injective, suppose $\alpha(\overline{a_1, \ldots, a_r; t}) = 0$. This implies $(\frac{a_1}{n!}, \frac{a_2}{n!}, \ldots, \frac{a_r}{n!}) = (0, \ldots, 0)$ so that all the $a_i$ are 0. Now we note that $\overline{(0, \ldots, 0; t)} = \overline{(0, \ldots, 0, nt)}$ for all $n \in \mathbb{N}$ (not 0) and since $t \in T$ is torsion there is some $n$ such that $nt = 0$ and hence $\alpha$ is injective.

**2.** Let $R$ be a ring. Let $\mathcal{D}$ be a small category. Let $D$ and $D'$ be two $\mathcal{D}$ diagrams of $R$-modules and $f : D \to D'$ a map of $\mathcal{D}$ diagrams. Prove that if $f_d$ is injective (i.e. one-to-one) for all $d \in \mathcal{D}$, then $\lim_{\mathcal{D}} f$ is injective and give an example to show that $\mathrm{colim}_{\mathcal{D}} f$ need not be injective. Dually, prove that if $f_d$ is surjective (i.e. onto) for all $d \in \mathcal{D}$, then $\mathrm{colim}_{\mathcal{D}} f$ is surjective and give an example to show that $\lim_{\mathcal{D}} f$ need not be surjective.

By construction, the limit is obtained by a kernel of a map between products. Since the product of injective maps is an injective map, one simply needs to observe that in general if we have a commuting diagram of modules

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow{f'} & B' \end{array}$$

with $\alpha$ and $\beta$ injective, then the induced map from $ker(f)$ to $ker(f')$ is also injective. Similarly, since colimits are formed as the cokernel of direct sums of modules. We simply need to observe that a direct sum of surjective maps is again surjective and that if $\alpha$ and $\beta$ are surjective then the induced map $coker(f) \longrightarrow coker(f')$ is surjective also.

For counterexamples, consider the examples

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} \\ \downarrow{\scriptstyle (0,1)} & A & \downarrow{\scriptstyle 1} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{1+p} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ \downarrow{\scriptstyle 1} & B & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/p \end{array}$$

In diagram A, the vertical maps are injective but the map of cokernels is not. Since a cokernel is a colimit this is an example of colimits failing to preserve injectives. In diagram B, the vertical maps are surjections but the map on kernels is not. Since kernels are limits this is an example of limits failing to preserve surjections.

**3.** Let $R$ be a ring. Let $\mathcal{D}$ be a small category and $D$ a $\mathcal{D}$ diagram of $R$-modules. Let $N$ be any $R$-module. Prove that

$$\mathrm{Hom}_R(N, \mathrm{lim}_{\mathcal{D}} D) \;\cong\; \mathrm{lim}_{\mathcal{D}} \mathrm{Hom}_R(N, D)$$

and show that there is a natural map

$$(*) \qquad \mathrm{colimit}_{\mathcal{D}} \mathrm{Hom}_R(N, D)) \;\to\; \mathrm{Hom}_R(N, \mathrm{colimit}_{\mathcal{D}} D)$$

but that it is not an isomorphism in general.

For the first part, we need only use that $Hom(N, \star)$ preserves products and kernels, that is, that $Hom(N, ker(f)) = ker(Hom(N, (f))$. The universal property of colimits gives us the natural map and to show it is not an isomorphism in general we need only show an example where $cokernel(Hom(N, f) \to Hom(N, coker(f))$ is not an isomorphism. Let $N = \mathbb{Z}/p$ and $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$. Then $cokernel Hom_{\mathbb{Z}}(\mathbb{Z}/p, p) = 0$ but $Hom_{\mathbb{Z}}(\mathbb{Z}/p, cokernel(p)) = Hom_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$.

Recall that we let $\mathcal{D}^{op}$ be the *opposite* category to $\mathcal{D}$. That is, $\mathcal{D}$ has the same objects as $\mathcal{D}$ but $\mathrm{Hom}_{\mathcal{D}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(Y, X)$ and composition is defined in reverse order. Prove that

$$\mathrm{Hom}_R(\mathrm{colim}_{\mathcal{D}} D, N) \;\cong\; \mathrm{lim}_{\mathcal{D}^{op}} \mathrm{Hom}(D, N)$$

and show that there is a natural map

$$\mathrm{colim}_{\mathcal{D}^{op}} \mathrm{Hom}_R(D, N) \;\to\; \mathrm{Hom}_R(\mathrm{lim}_{\mathcal{D}} D, N)$$

but that it is not an isomorphism in general.

For the first part we need only use that $Hom(\star, N)$ takes direct sums to products and reflects cokernels to kernels, that is, that $Hom(cokernel(f), N) \cong ker Hom(f, N)$. The universal property of colimits gives us the natural map and to show it is not an isomorphism in general we need only show an example where the natural map $cokernel Hom(f, N) \to Hom(ker(f), N)$ is not an isomorphism. Let $N = \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$. Then we see that $cokernel Hom(p, \mathbb{Z}) = \mathbb{Z}/p$ while $Hom(ker(p), \mathbb{Z}) = 0$.

**4.** An $R$-module $M$ is *finitely generated* if there is a surjective $R$-module map from $R^{\oplus r}$ to $M$ for some natural number $r$. Prove that if $M$ is a finitely generate $R$-module then for any collection of $R$-modules $\{N_x\}_{x \in X}$,

$$\mathrm{Hom}_R(M, \bigoplus_{x \in X} N_x) \;\cong\; \bigoplus_{x \in X} \mathrm{Hom}_R(M, N_x)$$

but give an example of a finitely generated module $M$ such that the natural map in $(*)$ above is still not an isomorphism.

The counter example is given in problem 3.

Since the natural map $\bigoplus_{x\in X} \mathrm{Hom}_R(M, N_x) \xrightarrow{\phi} \mathrm{Hom}_R(M, \bigoplus_{x\in X} N_x)$ is injective, it is surjectivity we must show. Let $\pi : R^{\oplus r} \xrightarrow{\phi} M$ be an $R$-module surjection and $\eta_i : R \to R^{\oplus r}$ the colimit structure maps for $1 \leq i \leq r$. Let $\alpha \in \mathrm{Hom}_R(M, \bigoplus_{x\in X} N_x)$. Then $\alpha \circ \pi \circ \eta_i$ is determined by its image at $1 \in R$. In particular, $\alpha \circ \pi \circ \eta_i$ lands in $\bigoplus_{u\in U_i} N_u \subseteq \bigoplus_{x\in X} N_x$ for some *finite* subset $U_i \subseteq X$. Let $U = \bigcup_{i=1}^{r} U_i$, a finite subset of $X$. Then $\alpha \circ \pi = \sum_{i=1}^{r} \alpha \circ \pi \circ \eta_i$ and hence its image lands in $\bigoplus_{u\in U} N_u$. Since $\pi$ is surjective, this implies that the image of $\alpha$ is contained in $\bigoplus_{u\in U} N_u$ also and hence an element of $\bigoplus_{u\in U} Hom_R(M, N_u) \subseteq \bigoplus_{x\in X} Hom_R(M, N_x)$. Since $\alpha$ was arbitrary, the map $\phi$ is surjective.

**5.** For this problem we let $S$ be a commutative ring and $I$ a (2-sided, proper) ideal of $S$. The *$I$-adic completion* of $S$ is written as $S_I^\wedge$ and can be defined as the inverse limit (as $S$-modules) of

$$\cdots \longrightarrow S/I^4 \longrightarrow S/I^3 \longrightarrow S/I^2 \longrightarrow S/I.$$

**a.** Prove that $S_I^\wedge$ is again a commutative ring and that the the natural map from $S$ to $S_I^\wedge$ (induced by the quotient maps $S \to S/I^n$) is a ring map.

If we represent an element of $S_I^\wedge$ by $(s_n)$ where $s_n \equiv s_m \bmod I^m$ if $n \geq m$ then the product is defined by $(s_n)(t_n) = (s_n t_n)$ which is a well defined ring structure because the maps $S \longrightarrow S/I^n$ are ring maps for all $n$. The map $S \xrightarrow{\phi} S_I^\wedge$ is given by $s \mapsto (s_n = s)$ (i.e. each $s_n = s$).

**b.** Let $I = (x)$ in $\mathbb{Q}[x]$. Prove that $\mathbb{Q}[x]_I^\wedge \cong \mathbb{Q}[[x]]$.

The maps $\mathbb{Q}[[x]] \longrightarrow \mathbb{Q}[x]/x^n$ assemble to give a homomorphism from $\mathbb{Q}[[x]]$ to $\mathbb{Q}[x]_{(x)}^\wedge$. The map is easily checked to be both injective and surjective.

**c.** Fill in the blank and prove the following:

$S_I^\wedge$ is a domain if and only if the ideal $I$ satisfies the condition $\star\star\star$.

I apologize. I'm not sure what $\star\star\star$ is.

**d.** Give an example of a ring $S$ and a prime ideal $I$ such that $S_I^\wedge$ is not a domain.

$\mathbb{Z}/4$ with the ideal $(2)$.

**e.** Prove that $\mathbb{Z}_p^\wedge$ is a domain.

Suppose $(s_n)(t_n) = 0 \Rightarrow p^n | s_n t_n$ for all $n$. Suppose $(t_n) \neq 0$. Let $k$ be maximal such that $t_{k-1} \neq 0$. Thus, $p^{k-1} | t_k$ but $p^k \not| t_k$. Since $p^{k+n} | s_{k+n} t_{k+n}$ but $p^k \not| t_{k+n}$ we have that $p^{n+1} | s_{k+n} \Rightarrow s_{n+1} = 0$. Since $n$ is arbitrary, $(s_n) = 0$.