Homework 3
Due March 7

1. Let R be a commutative ring. Recall that in class we showed that $\Lambda^n_R R^{\oplus m}) \cong R^{\oplus \binom{m}{n}}$.

a. Prove that if $f \in \mathrm{Hom}_R(R^{\oplus n}, R^{\oplus n})$ then

$$\Lambda^n(f) = \det(f) = \sum_{\sigma \in \Sigma_n} sgn(\sigma)(\pi_{\sigma(1)} \circ f \circ \eta_1) \circ (\pi_{\sigma(2)} \circ f \circ \eta_2) \circ \cdots \circ (\pi_{\sigma(n)} \circ f \circ \eta_n).$$

We recall that if let $e_i = \overbrace{0 \oplus \cdots \oplus 0}^{i-1} \oplus 1 \oplus 0 \oplus \cdots \oplus 0$ then $\Lambda^n_R R^{\oplus m}$ is the free R-module with basis $e_{i_1} \wedge \cdots \wedge e_{i_n}$ where $i_1 < \cdots < i_n$. Given $f : R^{\oplus m} \to R^{\oplus m}$, then $\Lambda^n_R(f)(e_{i_1} \wedge \cdots \wedge e_{i_n}) = f(e_{i_1}) \wedge \cdots \wedge f(e_{i_n})$. In particular, when $m = n$ we have that $\Lambda_R(f) : R \to R$ is given by multiplication by $f(e_1) \wedge \cdots \wedge f(e_n)$. Since $f(e_i) = \sum_{j=1}^n \pi_j \circ f \circ \eta_i$, by multilinearity we have

$$f(e_1) \wedge \cdots \wedge f(e_n) = \sum_{\alpha \in \mathrm{Hom}(\{n\},\{n\})} \pi_{\alpha(1)} \circ f \circ \eta_1 \wedge \cdots \wedge \pi_{\alpha(n)} \circ f \circ \eta_n$$

Since we get 0 if α is not injective, letting $\Sigma_n = \mathrm{Aut}(\{n\}, \{n\})$ this becomes

$$f(e_1) \wedge \cdots \wedge f(e_n) = \sum_{\sigma \in \Sigma_n} \pi_{\sigma(1)} \circ f \circ \eta_1 \wedge \cdots \wedge \pi_{\sigma(n)} \circ f \circ \eta_n.$$

However, to re-express this with respect to our chosen bases (namely $e_1 \wedge \cdots \wedge e_n$) we need to reorder these to get

$$f(e_1) \wedge \cdots \wedge f(e_n) = \left(\sum_{\sigma \in \Sigma_n)} sgn(\sigma)(\pi_{\sigma(1)} \circ f \circ \eta_1) \circ \cdots \circ (\pi_{\sigma(n)} \circ f \circ \eta_n) \right) (e_1 \wedge \cdots \wedge e_n).$$

b. Prove that $R^{\oplus m} \cong R^{\oplus n}$ as R modules if and only if $m = n$.

c. Prove that if M and N are finitely generated R modules than $M \otimes_R N$ is again a finitely generated R-module. Deduce that if M is a finitely generated R module than $\Lambda^n M$ is also.

Let $R^{\oplus m} \xrightarrow{\alpha} M$ and $R^{\oplus n} \xrightarrow{\beta} N$ be surjective R-module maps. Since tensoring with a fixed module preserves surjections (done in class) we have that the following maps are surjections

$$R^{\oplus m} \otimes_R R^{\oplus n} \xrightarrow{\alpha \otimes 1} M \otimes_R R^{\oplus n} \xrightarrow{1 \otimes \beta} M \otimes_R N$$

Hence the composite is surjective, but $R^{\oplus m} \otimes_R R^{\oplus n} \cong R^{\oplus mn}$ so $M \otimes_R N$ is finitely generated.

2. Let k be a commutative ring. A *k-algebra* R is (unital) ring and a (unital) ring map $k \to R$. Let R and S be commutative k–algebras.

a. Prove that $R\otimes_k S$ is again a k–algebra determined by setting $(r\otimes s)(r'\otimes s') = (rr'\otimes ss')$.

The *key point* missed by all students is to check that this is *well defined*. The second point is that the k-algebra map *is not* the tensor of the k-algebra maps for R and S. The commutative aspect of this problem arises as soon as one checks that the product is well defined. Namely, one needs to know that $(rk \otimes s)(r' \otimes s') = (rkr' \otimes ss') = (rr'k \otimes ss') = (rr' \otimes kss') = (r \otimes ks)(r' \otimes s')$. See where commutative was used? We need it in S for the other factor. If $\alpha : k \to R$ and $\alpha : k \to S$ are the k-algebra structure maps, then the k-algebra structure map for $R \otimes S$ is by $k \mapsto \alpha(k) \otimes 1 = 1 \otimes \beta(k)$.

b. Prove that $R \otimes_k S$ is the coproduct in the category of commutative k–algebras. That is, that we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Comm\,k-Alg}}(R \otimes_k S, T) \cong \mathrm{Hom}_{\mathrm{Comm\,k-Alg}}(R, T) \times \mathrm{Hom}_{\mathrm{Comm\,k-Alg}}(S, T).$$

Given k-algebra maps $\Psi : R \to T$ and $\Phi : S \to T$ then the induced map from $R \otimes_k S$ to T is induced by $(r\otimes s) \mapsto \Psi(r)\Phi(s)$. This is not a ring map in general if T is *not* commutative as well as R and S.

c. Prove the ring isomorphisms $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathbb{Q}$, $\mathbb{Z}[x]\otimes_{\mathbb{Z}} R \cong R[x]$ and $\mathbb{Q}[x]/(f)\otimes_{\mathbb{Q}}\mathbb{Q}[x]/(g) \cong \mathbb{Q}[x, y]/\langle f(x), g(y)\rangle$.

The first follows by the Yoneda lemma applied to

$$Hom_{\mathbb{Z}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) \cong Hom_{\mathbb{Z}}(\mathbb{Q}, Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})) \cong Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$$

as *modules*. The map that induces this equivalence, however, is the one from the universal property of the coproduct applied to the identity maps $(q \otimes q' \mapsto qq')$ and hence a ring map.

We have $\mathbb{Z}$ algebra maps $R \to R[X]$ and $\mathbb{Z}[X] \to R[X]$ (the first is the constant polynomials, the second is induced from the universal $\mathbb{Z}$-algebra structure every ring has). Thus we have a ring map $\mathbb{Z}[X] \otimes_{\mathbb{Z}} R \to R[X]$ $(f(x) \otimes r \mapsto r \cdot f(x))$. This is clearly surjective since $R[X]$ is generated by monomials $(r \cdot x^n)$. Then one needs to show injectivity which is easiest to do by writing down an inverse.

3. Recall that the colimit of $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \cdots$ is $\mathbb{Z}[\frac{1}{p}]$. More generally, let R be a commutative ring and $a \in R$.

a. Prove that the colimit $\left(R \xrightarrow{a} R \xrightarrow{a} R \xrightarrow{a} \cdots\right)$, which we write as $R[\frac{1}{a}]$, is again a commutative ring and in fact an R-algebra.

The nicest way to do this is to observe that the construction of the colimit can be expressed as the cokernel of $R[x] \xrightarrow{1-ax} R[x]$ and hence $R[\frac{1}{a}] = R[x]/(1 - ax)$. Technically, if a is nilpotent (i.e. $a^n = 0$ for some n) then this is the 0 ring, so not a ring with unit if one insists that the unit be distinct from 0.

b. Let $f : R \to S$ be a map of commutative rings (so we can consider S as an R-algebra via f). , Prove that $f(a)$ is invertible in S if and only if there is an R-algebra map $\hat{f} : R[\frac{1}{a}] \to S$.

c. Prove that for any R-module M, $M \otimes_R R[\frac{1}{a}] \cong \text{colim}\ \left(M \xrightarrow{a} M \xrightarrow{a} M \xrightarrow{a} \cdots\right)$.

$$\begin{aligned} M \otimes_R R[\frac{1}{a}] &= M \otimes_R \text{colim}\left(R \xrightarrow{a} R \xrightarrow{a} R \xrightarrow{a} \cdots\right) \\ &\cong \text{colim}\left(M \otimes_R R \xrightarrow{a} M \otimes_R R \xrightarrow{a} M \otimes_R R \xrightarrow{a} \cdots\right) \\ &\cong \left(M \xrightarrow{a} M \xrightarrow{a} M \xrightarrow{a} \cdots\right) \end{aligned}$$

d. Prove that for any R-module M, $Hom_R(R[\frac{1}{a}], M) \cong \lim\ \left(\cdot \xrightarrow{a} M \xrightarrow{a} M \xrightarrow{a} M\right)$.

$$\begin{aligned} Hom_R(R[\frac{1}{a}], M) &= Hom_R(\text{colim}\left(R \xrightarrow{a} R \xrightarrow{a} R \xrightarrow{a} \cdots\right), M) \\ &\cong \lim\left(Hom_R(R, M) \xleftarrow{a^*} Hom_R(R, M) \xleftarrow{a^*} \cdots\right) \\ &\cong \lim\left(M \xleftarrow{a} M \xleftarrow{a} M \xleftarrow{a} \cdots\right) \end{aligned}$$