Homework 4
March 31

1. Prove that the following conditions on a ring R are equivalent:

(a) Every R-module is projective.

(b) Every short exact sequence of R-modules is split exact.

(c) Every R-module is injective.

The cases (a) or (c) implies (b) is clear. To show (b) implies (a) you use that every module has a surjection from a free module and direct summands of free modules are projective. To show (b) implies (c) you use that every module has an injection to an injective module and that direct summands of injective modules are again injective.

2. A functor F from R-Mod to S-Mod is called *additive* if $F(0) = 0$ and $F(A \oplus B) \cong F(A) \oplus F(B)$ for all $A, B \in R$-Mod. Prove that every left or right exact functor is additive.

Applying F to the s.e.s $0 \to A = A \to 0 \to 0$ we see that $F(0) \cong 0$. Applying F to the split short exact sequence $0 \to A \to A \oplus B \to B \to 0$ we see that $F(A \oplus B) \cong F(A) \oplus F(B)$.

3. This exercise concerns maps between two short exact sequences of R-modules.

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

Since g is a surjective map, we can find a section, *as sets*, $s : C \to B$. Thus, s is <u>not</u> necessarily a R-module homomorphism but $g \circ s = 1_C$.

(a) Prove that we can define an R-module homomorphism $\partial : \ker(\gamma) \to \operatorname{coker}(\alpha)$ by the formula $\partial(c) = (f')^{-1} \circ \beta \circ s$. In particular, prove that ∂ is independent of the section s chosen and that ∂ is an R-module homomorphism even though s is not necessarily a R-module homomorphism,

If $c \in Ker(\gamma)$, then $g' \circ \beta(s(c)) = \gamma \circ g(s(c)) = \gamma(c) = 0$ so by exactness there is a unique $a' \in A'$ such that $f'(a') = \beta(s(c))$ so the formula makes sense. To show it is well defined, let s and s' be two set sections to g. Since $g(s - s') = 0$, by exactness for any $c \in Ker(\gamma)$ there is a $\bar{a} \in A$ such that $f(\bar{a}) = s(c) - s'(c)$. Let $a, a' \in A'$ be such that $f'(a) = \beta(s(c))$ and $f'(a') = \beta(s'(c))$. Then $f'(\alpha(\bar{a})) = \beta(f(\bar{a})) = \beta(s(c) - s'(c)) = \beta(s(c)) - \beta(s'(c)) = f'(a) - f'(a')$ (since β is a homomorphism). Since f' is injective this implies that $\alpha(\bar{a}) = a - a'$ and so $a \equiv a' \mod Im(\alpha)$ and ∂ is well defined. To show that ∂ is a group homomorphism, note that $(f')^{-1}\beta(s(c + c')) = (f')^{-1}\beta(s(c) + s(c'))$ (by the independence of lift just established).

(b) Show that there is an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\partial} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma) \to 0.$$

(c) Prove: γ injective then α is injective $\iff$ β is injective.

(d) Prove: α surjective then β is surjective $\iff$ γ is surjective.

Part (b) is simply checking the definitions and (c) and (d) follow from the fact that $0 \to X \to 0$ is exact if and only if $X = 0$.

(e) Give an explicit example where α and β are injective but γ is not injective. Give an explicit example where β and γ are surjective but α is not surjective. In both cases identify the groups and morphisms that arise from (b) in your examples.

The commuting diagram

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \to 0 \\ & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle id} & & \downarrow & \\ 0 \to & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \longrightarrow & 0 & \to 0 \end{array}$$

does both cases. The map ∂ is the identity.

4. An R-module M is *flat* if $M \otimes_R \star$ is exact.

(a) Prove that M is flat $\iff$ $M \otimes_R \star$ takes injective R-module homomorphisms to injective maps. Prove that projective modules are flat.

Exact functors preserve injections and conversely, $M \otimes_R \star$ is always right exact, so if it preserves injections as well then the functor is exact. Since R is flat, tensors commutes with direct sums and direct sums of exact sequences are again exact we see that free R modules are flat. Since every projective module P is a direct summand of a free module F, this implies that $A \otimes_R P \to A \otimes_R F$ is injective (it has a retract). Given an injection $A \to B$ we have a commuting diagram with injective vertical maps and the bottom map injective and hence the top horizontal map is injective as well.

$$\begin{array}{ccc} A \otimes_R P & \longrightarrow & B \otimes_R P \\ \downarrow & & \downarrow \\ A \otimes_R F & \longrightarrow & B \otimes_R F \end{array}$$

(b) Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module but that $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module.

If $0 \neq F$ is a free module then $F \otimes_{\mathbb{Z}} \mathbb{Z}/2 \neq 0$ but $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2 = 0$ so $\mathbb{Q}$ is not a free $\mathbb{Z}$-module. It is not generally the case that the tensor product with an injective module is again an injective module. However, given any abelian group A, $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is divisible and hence an injective $\mathbb{Z}$-module. Thus, given an injection $A \xrightarrow{f} B$ there exists a $\mathbb{Z}$-module map $\alpha : B \to A \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\alpha(f) = f(a) \otimes 1$. Thus the composite

$$B \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\alpha \otimes id} A \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{id \otimes \mu} A \otimes_{\mathbb{Z}} \mathbb{Q}$$

(where μ is the product map and an isomorphism by previous homework set) is a section to $f \otimes id_{\mathbb{Q}}$ and hence it is injective.

(c) Prove that if R is a commutative ring and $a \in R$ then $R[\frac{1}{a}]$ is a flat R-module (Warning. The colimit of injective maps is not in general again an injective map.)

You show that $\operatorname{colim}_{\mathbb{N}}$ preserves injective maps (the reason is that $\mathbb{N}$ is filtering). Then since $R[\frac{1}{a}] = \operatorname{colim}_{\mathbb{N}}(R \xrightarrow{a} R \xrightarrow{a} \cdots)$ and colimits commute with tensor products the result follows from the fact that R is a flat R-module.