Homework 5
April 11

1.

(a) Let M be a Noetherian A-module and $\alpha : M \to M$ a module homomorphism. Prove that if u is surjective then u is an isomorphism.

Consider chain of submodules of M:

$$ker(\alpha) \subseteq ker(\alpha^2) \subseteq ker(\alpha^3) \subseteq \cdots$$

Since M is Noetherian this stabilizes at some finite stage, say n. Thus, $ker(\alpha^n) = ker(\alpha^{n+1})$. Thus, $\alpha^{n+1}(x) = 0 \implies \alpha^n(x) = 0$. If $\alpha(y) = 0$, then since α^n is surjective there is an $x \in M$ such that $\alpha^n(x) = y$. But then $0 = \alpha(y) = \alpha(\alpha^n(x)) = \alpha^{n+1}(x)$ implies $0 = \alpha^n(x) = y$ so α is injective. Part (b) is simply the "dual" of part (a).

(b) Let N be an Artinian A-module and $u : N \to N$ a module homomorphism. Prove that if u is injective then u is an isomorphism.

2. Let A be a commutative ring and M a Noetherian A-module. Let I be the annihilator of M in A (i.e. $I = \{a \in A \mid a \cdot m = 0 \text{ for all } m \in M\}$. Prove that A/I is a Noetherian ring (or 0).

M Noetherian implies it is finitely generated, say by $\{m_1, \ldots, m_n\}$. We have an A-module map $A \xrightarrow{\phi} \oplus^n M$ by $\phi(a) = (am_1 \oplus \cdots \oplus am_n)$. The kernel of ϕ is I so $A/I \xrightarrow{\phi} \oplus^n M$ is an injective A-module homomorphism. Since M is Noetherian, a finite direct sum of M's is again Noetherian which implies A/I is Noetherian as an A-module which implies is it Noetherian as an A/I module.

3. Let A be a commutative ring in which the zero ideal is a product $M_1 \cdots M_n = 0$ of (not necessarily distinct) maximal ideals. Prove that A is Noetherian if and only if A is Artinian.

Start by considering the chain of ideals

$$0 = M_1 \cdots M_n \subseteq M_1 \cdots M_{n-1} \subseteq \cdots \subseteq M_1 M_2 \subseteq M_1 \subseteq A.$$

Each factor $M_1 \cdots M_i / M_1 \cdots M_{i+1}$ is a A/M_{i+1} vector space. Thus each factor is Noetherian if and only if it is Artinian, i.e. when the factor is finite dimensional as a A/M_{i+1} vector space. Since we have that for any short exact sequence $0 \to X \to Y \to X \to 0$, X and Z are Noetherian/Artinian iff and only if Y is Noetherian/Artinian, the result follows by finite induction up the chain.

4. Let A be an Artinian commutative ring. Prove that every prime ideal is maximal.

Let $P \subseteq A$ be a prime ideal. Then $R = A/P$ is Artinian and an integral domain. Let $0 \neq x \in R$. Since R is Artinian, the descending chain of ideals $\cdots \subseteq (x^3) \subseteq (x^2) \subseteq (x)$

must stabilize at some finite stage, say $(x^{n+1}) = (x^n)$. Thus, $x^{n+1}y = x^n$ for some y. This implies $x^n(1 - xy) = 0$ and since R is a domain and $x \neq 0$ this implies $1 = xy$ and so x was invertible in R.

5. Let M be a finitely generated module over a Principal Ideal Domain (PID) R. Suppose that the the annihilator of M is generated by α and that $\alpha = p_1 \cdots p_r$ is a product of *distinct* primes. Prove that M is semi-simple. (Hint: I used the Chinese Remainder Theorem).

The important observation to first make is that if $\alpha : R \to S$ is a *surjective* ring map, then for every (semi-) simple S module M, $\alpha^*(M)$ (M considered as an R-module via α) is a (semi-) simple R module. If we let I be the annihilator of M, then $M = \alpha^* M$ for $\alpha : R \to R/I$ and so it suffices to prove that R/I is semi-simple as an R-module. By the Chinese Remainder Theorem we have that

$$R/I \cong R/(p_1) \times \cdots \times R/(p_r).$$

Since R is a PID, each (p_i) is a maximal ideal and so R/I is a product of fields and hence semi-simple.

6. This problem is from *Introduction to Commutative Algebra* by Atiyah MacDonald. Let A be a commutative Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \ \in A[[x]]$. Prove that f is nilpotent (i.e. $f^k = f \cdot f \cdots f = 0$ for some k) if and only if each a_n is nilpotent.

Thanks to Lucas Wiman for telling me how to do this one. Let $Nil(A)$ be the ideal of nilpotent elements of A. If $I \subseteq Nil(A)$ is finitely generated, say by $\{a_1, \ldots, a_k\}$, then $I^m = 0$ for all m such that $a_i^m = 0$ for all $1 \leq i \leq k$. For $f \in A[[x]]$, let I_f be the ideal of A generated by the coefficients of f. Then $I_{f^n} \subseteq (I_f)^n$ because if we let $f = \sum_{i=0}^{\infty} a_k x^k$ then $f^n = \sum_{i=0}^{\infty} \left(\sum_{j_1 + \cdots + j_n = i} a_{j_1} a_{j_2} \cdots a_{j_n} \right) x^i$. Thus, if A is Noetherian, $Nil(A)$ is finitely generated and hence nilpotent as an ideal. In particular, if the coefficients of f all lie in $Nil(A)$ then $I_{f^n} \subset Nil(A)^n = 0$ for some n and hence $f^n = 0$.

Conversely, assume that $f^k = 0$. This implies that $a_0^k = 0$ and hence nilpotent. Assume now that the coefficients $a_0, \ldots, a_n$ are all nilpotent. If we let $g = \sum_{i=0}^{n} a_i x^i$ then g is nilpotent (since $I_{g^n} \subseteq (I_g)^n$ and I_g is nilpotent since it is a finitely generated submodule of $Nil(A)$). Since $Nil(A[[x]])$ is an ideal of $A[[x]]$, $f - g = \sum_{i=n+1}^{\infty} a_i x^i$ is nilpotent which implies that a_{n+1} must be nilpotent and hence by induction every coefficient of f is nilpotent. Note, this direction did not use the hypothesis that A was Noetherian).