Homework 6
April 30

**1.** Compute the kernel and cokernel of the following maps:

a) $\alpha : \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 3}$ given by

$$\begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$$

b) $\beta : \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 3}$ given by

$$\begin{pmatrix} 5 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

**2.** Consider the matrix $A \in M_3(\mathbb{Z})$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

a) Thinking of $A$ as a linear transformation of $\mathbb{Q}$ vector spaces, find the characteristic and minimal polynomials for $A$. Find the rational canonical form for $A$ (over $\mathbb{Q}$).

b) Repeat part a) but now consider $A$ as a linear transformation of $\mathbb{Z}/7$ and $\mathbb{Z}/2$ respectively.

c) For which fields is $A$ diagonalizable?

**3.**

a) Let $F$ be a field and $f \in F[x]$ of degree $k$. Prove that if $f$ splits in a Galois extension of $F$ of degree $n$ such that gcd(k,n) = 1 then $f$ has a root in $F$.

b) Let $\gamma : V \to V$ be a $F$-module homomorphism of a vector space of dimension $k$ such that the minimal polynomial of $\gamma \otimes_F F'$ splits for $F'$ a Galois extension of $F$ of degree $n$ with gcd(k,n) =1. Prove that $\gamma$ has an eigenvector.

c) Let $F$ be a field whose algebraic closure is a finite Galois extension of degree $n$ (for example, $\mathbb{R}$). Let $G$ be a finite group whose order is divisible in $F$. Let $M$ be a simple left $F[G]$ module which has dimension $k$ prime to $n$. Prove that $Hom_{F[G]}(M, M) \cong F$. (Hint, Schur's lemma, strong form).

d) As in c), we know that $F[G]$ is ring isomorphic to a finite product of matrix rings of division algebras, say $M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$, each of which is an $F$-algebra. Prove that if $dim_F(D_i)$ is prime to $n$, then $D_i \cong F$ in this decomposition.

**4** Suppose we have a condition $\mathcal{P}$ which the modules of a ring $R$ may or may not satisfy (for example, $\mathcal{P}$ could be the condition "the module is finitely generated", then $M \in \mathcal{P}(R)$ means that $M$ is a finitely generated $R$-module). Suppose that for every short exact sequence of $R$-modules,

$$0 \to A \to B \to C \to 0$$

$$(p) \qquad A \in \mathcal{P}(R) \text{ and } C \in \mathcal{P}(R) \implies B \in \mathcal{P}(R)$$

(a) Suppose in addition that we assume $B \in \mathcal{P}(R) \implies C \in \mathcal{P}(R)$. Prove that in this case, $R \in \mathcal{P}(R) \iff M \in \mathcal{P}(R)$ for every finitely generated $R$-module $M$.

(b) Prove that the statement $\mathcal{P} =$ "every quotient object is projective" satisfies (p) + (d). Prove that in this case $R \in \mathcal{P}(R) \iff R$ is semi-simple.

(c) Prove that the statement $\mathcal{P} =$ "every quotient object is injective" satisfies (p) + (d). Prove that a domain $R$ satisfies this condition if and only if it is a field.

(d) Suppose we now assume the different additional hypothesis that $B \in \mathcal{P}(R) \implies A \in \mathcal{P}(R)$. Prove that in this case, $R \in \mathcal{P}(R) \iff$ every submodule of a finitely generate free $R$-module is in $\mathcal{P}(R)$.

(e) Prove that the statement $\mathcal{P} =$ "every submodule is free" satisfies (a) + (p). Prove that a Noetherian domain $R$ satisfies this condition $\iff R$ is a PID.

(f) Prove that the statement $\mathcal{P} =$ "every submodule is injective" satisfies (a)+ (p). Prove that a ring $R$ satisfies $\mathcal{P} \iff R$ is semi-simple.