

# STABILITY OF A STOCHASTIC TWO-DIMENSIONAL NON-HAMILTONIAN SYSTEM

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ABSTRACT. We study the largest Lyapunov exponent of the response of a two dimensional non-Hamiltonian system driven by additive white noise. The specific system we consider is the third-order truncated normal form of the unfolding of a Hopf bifurcation. We show that in the small-noise limit the top Lyapunov exponent always approaches zero from below (and is thus negative for noise sufficiently small); we also show that there exist large sets of parameters for which the top Lyapunov exponent is positive. Thus the two-point motion can be either stable or unstable, while the one-point motion is always stable.

## 1. INTRODUCTION

The primary concern in the analysis of nonlinear dynamical systems is the determination and prediction of steady-state or stationary motions and their corresponding stability. No theorem has had so direct and powerful an influence upon the study of stochastic stability of noisy dynamical systems as the multiplicative ergodic theorem (MET) of Oseledets [1], which established the existence of (typically) finitely many deterministic exponential growth rates called *Lyapunov exponents*. The stability of linear stochastic systems based on the MET has been well established [2, 3] and the top Lyapunov exponent can be evaluated explicitly with relative ease when the noisy perturbations and dissipation are weak [4, 5]. The challenge has been to extend the existing techniques in order to explicitly evaluate the top Lyapunov exponent of nonlinear systems with noise, and in particular additive white noise.

It has been shown that the two point motion of a one-dimensional nonlinear stochastic system is unique. More precisely, if we consider the noisy one-dimensional equation,

$$(1) \quad \dot{x}_t = f(x_t) + g(x_t)\xi(t), \quad \mathbf{E} [\xi(t)\xi(t')] = \delta(t - t'),$$

it has the stationary invariant measure

$$(2) \quad p(x) = \frac{N}{g^2(x)} \exp \left\{ \int^x \frac{2f(\eta)}{g^2(\eta)} d\eta \right\}.$$

Provided  $p(x)$  is normalizable, then as in Arnold [3], the Lyapunov exponent is easily derived as

$$(3) \quad \lambda = -2 \int_0^\infty \left[ \frac{f(x)}{g(x)} \right]^2 p(x) dx.$$

Thus the Lyapunov exponent is always negative provided  $f(x) \neq 0$  (this is a well-known result for one-dimensional SDE [6]).

Schimansky-Geier and Herzog [7] were the first to consider numerically the Lyapunov exponents of a two dimensional nonlinear system under additive noise. Their work was devoted to the effect of noise on the Kramers Oscillator

$$(4) \quad \ddot{x}_t + \epsilon \dot{x}_t + U'(x_t) = \sqrt{2\epsilon}\xi(t) \quad U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4, \quad a, b > 0$$

with a double-well potential. This is the first two dimensional example where it was shown that the top Lyapunov exponent is asymptotically positive, i.e., it was shown that  $\lambda(\epsilon) > 0$  for  $\epsilon$  positive but sufficiently small. Hence, an additive noise induces an unstable stationary measure.

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This remarkable observation was proved analytically by Arnold *et al.* [8, 9] using a perturbation approach. However, only the first term of the asymptotic expansion was analytically evaluated. Based on this, it was shown that the top Lyapunov exponent is positive, and for small values of noise intensity  $\sqrt{\epsilon}$  and dissipation  $\epsilon$ , the exponent is proportional to  $\epsilon^{\frac{1}{3}}$ . The underpinning of the method presented in [8, 9] is a separation of scales; there is a coordinate for the unperturbed nonlinear system which slowly varies (often thought of as the energy) and a coordinate which quickly varies (typically thought of as an angular component). (Stated differently, we are studying the slowly-varying adiabatic system in the action-angle variables.) Similar results are also reported by Baxendale and Goukasian [10] for the multiplicative case, where calculations were done using the coordinates suggested by Sowers [11].

Most of the results in this area have been for Hamiltonian dynamics; however, many important systems in science and engineering are non-Hamiltonian. The purpose of this paper is to consider a noisy non-Hamiltonian problem, and to this end we consider the effects of additive noise on the normal form associated with a Hopf bifurcation. One hopes that the reduced model (normal form) encapsulates the salient features of the original system. We pick this reduced model for the calculation of the top Lyapunov exponent for two reasons. First, it is well-known that in the absence of noise the system exhibits a stable limit cycle. Second, in the presence of additive noise and under some mild additional assumptions, the system has a stationary density function, as we show below. Although such measures are known to exist, contrary to the one-dimensional case or the two-dimensional weakly-perturbed Hamiltonian system, there are no concrete results on the sign of the top Lyapunov exponents corresponding to these stationary measures.

The challenge, in general, is to explicitly evaluate the top Lyapunov exponent of these stationary measures, and the results of this paper are to establish a series of estimates and approximations — both asymptotic and numerical — which give good bounds on this exponent in certain parameter regimes. In particular, we show that for small noise, the system has a Lyapunov exponent which is negative and approaches zero as noise goes to zero. Moreover, we also show the existence of a “shear instability” — namely, that if the twist parameter in the Hopf normal form is sufficiently large — implies that the top Lyapunov exponent is positive. This is reminiscent of the work of Lin and Young [12], who considered the Lyapunov exponents of a two-dimensional linear shear flow with a hyperbolic limit cycle and periodic kicking:

$$(5) \quad \begin{aligned} \dot{I}_t &= -\mu I_t + A \sin(2\pi\theta) \sum_{n=0}^{\infty} \delta(t - nT), \quad I_0 = I \in \mathbb{R}, \quad T > 0, \\ \dot{\theta}_t &= \omega(I_t) \stackrel{\text{def}}{=} 1 + bI_t, \quad \theta_0 = \theta \in \mathbf{S}, \end{aligned}$$

where the kicks were not along the foliation or perpendicular to the stable manifold. The quantity

$$L \stackrel{\text{def}}{=} \frac{b}{\mu} \cdot A,$$

where  $b$  represents the shear,  $\mu$  represents the contraction rate and  $A$  denotes the “kick” amplitude, was used as a parameter to classify various phenomena; it was shown that for sufficiently large  $L$  the top Lyapunov exponent was positive implying chaos. We show the same phenomenon exists in this system: for sufficiently large shearing, the Lyapunov exponent becomes positive.

One motivation for computing Lyapunov exponents is that they are useful in a qualitative description of a stochastic dynamical system. We will show below that the system we study is neighborhood positive recurrent, and thus the stationary measure can be viewed as the occupation measure, i.e., there is an invariant measure  $\mu(dx, dy) = p(x, y) dx dy$ , and the proportion of time the system spends in a set  $A$  is equal to  $\int_A \mu$ . In short, the “one-point motion” is stable: if we watch any single trajectory for a long enough time, it will eventually settle down to the occupation measure [13, Chapter 1]. In contrast, the Lyapunov exponents are determined by the behavior of two neighboring orbits (the “two-point motion”). In this context, the positivity of the top Lyapunov exponent means that even though every solution trajectory asymptotically approaches the stationary measure, the distance between any two initial conditions will grow, on average, at an exponentially fast rate, meaning that the dynamics on the attractor are mixing. In this paper, we will thus show that the noisy Hopf bifurcation is, for some parameter values, mixing on the attractor.

The remainder of this paper is organized as follows: In Section 2 we describe the precise problem we study. In Section 3 we derive a formula for the maximum Lyapunov exponent of our system using the Furstenberg–Khasminskii formula; however, as is typical, this is given in the form of the expectation of a complicated

expression with respect to an unknown probability measure. In Section 4 we study this expression, using various analytic, asymptotic, and numerical techniques, to determine the sign of the top Lyapunov exponent.

## 2. STATEMENT OF THE PROBLEM

We consider the cubic normal form for a Hopf bifurcation:

$$(6) \quad \begin{aligned} \dot{x}_t &= \mu x_t - \omega y_t + (-ax_t - by_t)(x_t^2 + y_t^2) \\ \dot{y}_t &= \omega x_t + \mu y_t + (-ay_t + bx_t)(x_t^2 + y_t^2) \end{aligned}$$

where  $a, b, \omega$  are constants and  $\mu$  is the bifurcation parameter. The reduced model (6) encapsulates the salient features of any system that exhibits a local Hopf bifurcation [14, 15]. In polar coordinates  $x = r \cos \phi$   $y = r \sin \phi$ , (6) reduces to

$$(7) \quad \begin{aligned} \dot{r}_t &= \mu r_t - ar_t^3, \\ \dot{\phi}_t &= \omega + br_t^2. \end{aligned}$$

For  $\mu < 0$  the trivial solution is asymptotically stable, but if  $\mu, a > 0$ , then this system has a stable non-trivial limit cycle. The amplitude of the limit cycle is given by

$$r_0 = \sqrt{\frac{\mu}{a}}, \quad \mu, a > 0.$$

The purpose of this paper is to examine the asymptotic sample stability of this nonlinear system under random additive perturbations:

$$(8) \quad \begin{aligned} dx_t &= \{\mu x_t - \omega y_t + (-ax_t - by_t)(x_t^2 + y_t^2)\}dt + \sigma \circ dW_t^1 \\ dy_t &= \{\omega x_t + \mu y_t + (-ay_t + bx_t)(x_t^2 + y_t^2)\}dt + \sigma \circ dW_t^2 \end{aligned}$$

*Lemma 2.1.* (Stationary measure): The stationary density of the Fokker-Planck equation associated with (8) is

$$(9) \quad p(x, y) = Z \exp\left(\frac{2}{\sigma^2} \left[\frac{\mu}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2\right]\right),$$

provided  $a > 0$ , where

$$(10) \quad Z = \frac{2}{\sigma \sqrt{\frac{\pi}{2a}} \operatorname{erfc}\left(\frac{-\mu}{\sigma \sqrt{2a}}\right)}.$$

**Proof:** Direct computation shows  $p(x, y)$  as defined in (9) satisfies

$$L^*p = -\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)p - f\frac{\partial p}{\partial x} - g\frac{\partial p}{\partial y} + \frac{\sigma^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p,$$

where

$$f = \mu x_t - \omega y_t + (-ax_t - by_t)(x_t^2 + y_t^2), \quad g = \omega x_t + \mu y_t + (-ay_t + bx_t)(x_t^2 + y_t^2),$$

and  $Z$  is chosen so that  $\int \int p(x, y) dx dy = 1$ . □

## 3. STABILITY OF STATIONARY SOLUTIONS

Our aim is to determine the top Lyapunov exponent of the random dynamical system described by (8). For this purpose we have to study its linearization. Denoting the linearized variables by  $(u, v)$ , we have

$$(11) \quad \begin{bmatrix} du_t \\ dv_t \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix} dt,$$

where

$$\begin{aligned} A_{11} &= \mu - a(3x^2 + y^2) - 2bxy, \\ A_{12} &= -\omega - b(x^2 + 3y^2) - 2axy, \\ A_{21} &= \omega + b(3x^2 + y^2) - 2axy, \\ A_{22} &= \mu - a(x^2 + 3y^2) + 2bxy. \end{aligned}$$

We now decompose (11) into its radial and angular parts. This provides the most convenient setting for the description of the top Lyapunov exponent by means of the Furstenberg–Khasminskii formula. Writing

$$u = \exp(\rho) \cos \theta, \quad v = \exp(\rho) \sin \theta,$$

the linear equations reduce to

$$(12) \quad d\rho_t = Q(\theta_t, x_t, y_t) dt, \quad d\theta_t = h(\theta_t, x_t, y_t) dt$$

where

$$Q(\theta, x, y) = \mu - 2a(x^2 + y^2) + \sqrt{a^2 + b^2} \{(x^2 - y^2) \sin(2\theta + \chi_0) - 2xy \cos(2\theta + \chi_0)\}$$

$$h(\theta, x, y) = \omega + 2b(x^2 + y^2) + \sqrt{a^2 + b^2} \{(x^2 - y^2) \cos(2\theta + \chi_0) + 2xy \sin(2\theta + \chi_0)\}$$

and  $\chi_0 = \arccos(\frac{b}{\sqrt{a^2 + b^2}})$ . In order to examine the Lyapunov exponent of the stationary solution  $(x_t, y_t)$ , we must consider the coupled equations (8) and (12). Since the normal form equations governing the stationary solution  $x_t, y_t$  has an  $S^1$  symmetry, letting  $x = r \cos \phi, y = r \sin \phi$ , in polar coordinates, (8) reduces to

$$(13) \quad dr_t = \left\{ \mu r_t - ar_t^3 + \frac{\sigma^2}{2r_t} \right\} dt + \sigma dW_t^r$$

$$(14) \quad d\phi_t = \left\{ \omega + br_t^2 \right\} dt + \frac{\sigma}{r_t} dW_t^\phi,$$

where  $W^r, W^\phi$  are independent Wiener processes. Then the angular component described by the process  $\theta_t, t \in \mathbf{R}$ , satisfies the random differential equation

$$d\theta_t = h^\theta(\phi_t, r_t, \theta_t) dt = \left\{ \omega + 2br_t^2 + \sqrt{a^2 + b^2} r_t^2 \cos(2\theta_t + \chi_0 - 2\phi_t) \right\} dt$$

while the radial part of the linearization simplifies to

$$(15) \quad Q(\phi_t, r_t, \theta_t) = \mu - 2ar_t^2 + \sqrt{a^2 + b^2} r_t^2 \sin(2\theta_t + \chi_0 - 2\phi_t).$$

**3.1. Expression for the Maximum Lyapunov Exponent.** To represent Lyapunov exponents, we shall make use of the Furstenberg–Khasminskii formula and integrate the functional  $Q(\phi, r, \theta)$  with respect to the invariant measure governed by

$$dr_t = \left\{ \mu r_t - ar_t^3 + \frac{\sigma^2}{2r} \right\} dt + \sigma dW_t^r$$

$$d\phi_t = \left\{ \omega + br_t^2 \right\} dt + \frac{\sigma}{r_t} dW_t^\phi$$

$$d\theta_t = \left\{ \omega + 2br_t^2 + \sqrt{a^2 + b^2} r_t^2 \cos(2\theta_t + \chi_0 - 2\phi_t) \right\} dt$$

Choosing the fictitious angle  $\psi = 2\theta - 2\phi + \chi_0$  as the difference of the two phase processes, the equations reduce to a simpler form which allows us to obtain more detailed information on the top Lyapunov exponent without seeking for scaling parameters. The Lyapunov exponents are indeed functions of this angle  $\psi$  and the  $r(t)$  process of the nonlinear system, i.e.,

$$(16) \quad dr_t = \left\{ \mu r_t - ar_t^3 + \frac{\sigma^2}{2r_t} \right\} dt + \sigma dW_t^r$$

$$d\psi_t = 2\{b + \sqrt{a^2 + b^2} \cos \psi\} r_t^2 dt - \frac{2\sigma}{r_t} dW_t^\phi$$

and the functional of the radial part of the linearization reduces to

$$(17) \quad Q(r, \psi) = \left\{ \mu - 2ar^2 + \sqrt{a^2 + b^2} r^2 \sin(\psi) \right\}$$

Due to the regularity properties of our vector fields, there exists an invariant density  $p(r, \psi)$ . The formula of Furstenberg-Khasminskii (see Arnold [3]) states that the top Lyapunov exponent  $\lambda$  of our system satisfies

$$(18) \quad \lambda = \mathbf{E}[Q] = \int_{[0, 2\pi] \times \mathbf{R}_+} Q(r, \psi) p(r, \psi) d\psi dr.$$

Note that  $p$  is the invariant measure of (16), **not** the original system!

In order to evaluate (18), we need to the solution  $p(r, \psi)$  to

$$L^* p(r, \psi) = 0,$$

where

$$(19) \quad Lf(r, \psi) \stackrel{\text{def}}{=} \left( \mu r - ar^3 + \frac{\sigma^2}{2r} \right) \frac{\partial f}{\partial r}(\psi, r) + \left( 2 \left\{ b + \sqrt{a^2 + b^2} \cos \psi \right\} r^2 \right) \frac{\partial f}{\partial \psi}(\psi, r) \\ + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial r^2}(\psi, r) + \frac{2\sigma^2}{r^2} \frac{\partial^2 f}{\partial \psi^2}(\psi, r)$$

**3.2. Marginals of the invariant measure.** It is difficult to solve for  $p$  exactly, so we study the marginal distributions of  $p$ , namely

$$\nu_\psi(\psi) := \int_0^\infty p(r, \psi) dr, \quad \nu_r(r) := \int_0^{2\pi} p(r, \psi) d\psi.$$

Since the equation for  $r(t)$  is independent of  $\psi$  in (16), the  $r(t)$  process of the nonlinear system is by itself a Markov diffusion process, and its stationary density is

$$(20) \quad \nu_r(r) = \frac{2r \exp\left(-\frac{a}{2\sigma^2} \left[r^2 - \frac{\mu}{a}\right]^2\right)}{\sqrt{\frac{\pi}{2\sigma^2 a}} \operatorname{erfc}\left(\frac{-\mu}{\sigma\sqrt{2a}}\right)},$$

which is integrable, provided  $a > 0$ . Deriving an analogous formula for  $\nu_\psi$  seems difficult, the chief complication being that the equation for  $\psi(t)$  in (16) depends on both  $r$  and  $\psi$ . However, consider

$$(21) \quad w_{r_0}(\psi) := p(\psi | r_0),$$

and then  $w(\psi)$  solves

$$(22) \quad -\frac{\partial}{\partial \psi} \left( \left\{ 2 \left( b + \sqrt{a^2 + b^2} \cos \psi \right) r_0^2 \right\} w(\psi) \right) + \frac{2\sigma^2}{r_0^2} \frac{\partial^2 w}{\partial \psi^2}(\psi) = 0,$$

along with the periodicity and the normalization conditions

$$w(\psi) = w(\psi + 2\pi) \quad \text{and} \quad \int_0^{2\pi} w(\psi) d\psi = 1.$$

*Proposition 3.1.* Denote

$$D_0 = \frac{r_0^4}{\sigma^2} b, \quad D_c = \frac{r_0^4}{\sigma^2} \sqrt{a^2 + b^2}, \quad \text{and} \quad \kappa = \frac{1}{4\pi^2 e^{-\pi D_0} I_{iD_0}(D_c) I_{-iD_0}(D_c)},$$

where  $I_{\pm iD_0}(D_c)$  are the Bessel functions of imaginary argument and imaginary order. Then

$$(23) \quad w(\psi) = \kappa \int_\psi^{\psi+2\pi} \exp\{-D_0(r_0) [\phi - \psi] - D_c(r_0) [\sin \phi - \sin \psi]\} d\phi$$

satisfies (22).

**Proof:** It is obvious from (22),

$$\frac{\partial w}{\partial \psi}(\psi) - D \left( b + \sqrt{a^2 + b^2} \cos \psi \right) w(\psi) = C,$$

where  $D = \frac{r_0^4}{\sigma^2}$  and the general solution is

$$w(\psi) = C \int_\beta^\psi \exp\{-Db[\phi - \psi] - D\sqrt{a^2 + b^2}[\sin \phi - \sin \psi]\} d\phi.$$

Further simplification using the periodicity condition  $w(\psi) = w(\psi + 2\pi)$  yields

$$(24) \quad w(\psi) = \frac{C}{(\exp\{-2\pi D_0\} - 1)} \int_\psi^{\psi+2\pi} \exp\{-D_0[\phi - \psi] - D_c[\sin \phi - \sin \psi]\} d\phi.$$

Using the normalization condition  $\int_0^{2\pi} w(\psi) d\psi = 1$  and making the change of variables  $\phi = \psi - \chi$  yields

$$K = \frac{\exp\{-2\pi D_0\} - 1}{C} = \int_0^{2\pi} \int_0^{2\pi} \exp\{-D_0\chi - D_c[\sin(\psi + \chi) - \sin\psi]\} d\chi d\psi,$$

or

$$K = \int_0^{2\pi} \exp\{-D_0\chi\} \int_0^{2\pi} \exp\{-D_c[2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\cos\psi - 2\sin^2\left(\frac{\chi}{2}\right)\sin\psi]\} d\psi d\chi.$$

From [16, 3.338] we obtain  $\int_0^{2\pi} \exp\{b\sin x + c\cos x\} dx = 2\pi I_0(\sqrt{b^2 + c^2})$ , which in turn yields

$$K = 2\pi \int_0^{2\pi} \exp\{-D_0\chi\} I_0\left(2D_c \sin\left(\frac{\chi}{2}\right)\right) d\chi.$$

Then using [16, 6.681] results in

$$\int_0^{2\pi} \exp\{-D_0\chi\} I_0\left(2D_c \sin\left(\frac{\chi}{2}\right)\right) d\chi = 2\pi I_{iD_0}(D_c) I_{-iD_0}(D_c) \exp\{-D_0\pi\}.$$

Hence,

$$C = \frac{e^{-2\pi D_0} - 1}{4\pi^2 e^{-\pi D_0} I_{iD_0}(D_c) I_{-iD_0}(D_c)},$$

and (24) gives  $\kappa$ . □

#### 4. ESTIMATION OF THE TOP LYAPUNOV EXPONENT

In this section, we present a series of estimates and bounds for the top Lyapunov exponent  $\lambda$ . We take a few approaches: we first demonstrate useful upper and lower bounds for  $\lambda$ ; we then show that in the small noise limit, the invariant measure for (16) becomes localized in  $(r, \psi)$  and use this localization to asymptotically approximate  $\lambda$ ; finally, we give scaling arguments and numerical evidence that, holding all other parameters fixed, and taking  $b$  sufficiently large, the top Lyapunov exponent becomes positive.

The results of this section can be summarized in the following two statements, one a proposition and one an observation:

*Proposition 4.1.* (1) If  $b < \sqrt{3}a$  and  $\mu$  is sufficiently small, then  $\lambda < 0$ .  
(2) Define  $\epsilon = \sigma^2 a^2 / \mu^2$ . As  $\epsilon \rightarrow 0$ ,  $\lambda = C\epsilon + O(\epsilon^2)$  with  $C < 0$ .

**Proof:** We prove items 1 and 2 in Sections 4.1 and 4.2, respectively. □

Finally, in Section 4.3, we give scaling arguments and numerical evidence showing that the Lyapunov exponent becomes positive when the “shear”  $b$  is large.

These can all be summarized as such: “Small shear and small noise give stability, but large shear gives instability.”

**4.1. Bounds for  $\lambda$ .** We first derive naive bounds for  $\lambda$  as follows. If we define

$$Q^\pm(r, \psi) = \mu + r^2(-2a \pm \sqrt{a^2 + b^2}),$$

$$\lambda^\pm = \mathbf{E}[Q^\pm] = \int_{[0, 2\pi] \times \mathbf{R}_+} Q^\pm(r, \psi) p(r, \psi) d\psi dr,$$

then  $Q^- \leq Q \leq Q^+$  and since  $p \geq 0$ ,  $\lambda^- \leq \lambda \leq \lambda^+$ . Depending on where the bulk of the measure  $p(r, \psi)$  lies, one or both of these estimates may be coarse. However, for now we are only interested in the sign of  $\lambda$ , so even a coarse bound may be useful, and, moreover, it is easy to compute  $\lambda^\pm$  explicitly:

$$\lambda^\pm = \mu + \mathbf{E}[r^2](-2a \pm \sqrt{a^2 + b^2}),$$

and using (20) we have

$$(25) \quad \mathbf{E}[r^2] = \frac{\mu}{a} + \sigma \sqrt{\frac{2}{\pi a}} \frac{e^{-\mu^2/2a\sigma^2}}{\operatorname{erfc}(-\mu/\sqrt{2a\sigma^2})}.$$

It turns out that  $\lambda^- < 0$  for any choice of  $a, b, \mu, \sigma$ . To see this, notice that  $\mathbf{E}[r^2] \geq \mu/a$ , and thus

$$\lambda^- \leq \mu(1 - (2a + \sqrt{a^2 + b^2})/a) \leq -2\mu < 0,$$

so this gives no useful lower bound on the sign.

However, we can use the other bound to show that the Lyapunov exponent is negative for some parameters. For illustration, fix  $a, b, \sigma$  and consider  $\lambda^+$  as a function only of  $\mu$ . Expanding  $\lambda^+$  in a Taylor series near  $\mu = 0$ , we obtain

$$(26) \quad \lambda^+ = (\sqrt{a^2 + b^2} - 2a)\sigma\sqrt{\frac{2a}{\pi}} + \left( \frac{(\pi + 2a)\sqrt{a^2 + b^2} - a(\pi + 4a)}{\pi a} \right) \mu + O(\mu^2).$$

The  $O(1)$  term is negative if  $b < \sqrt{3}a$ , and therefore  $\lambda \leq \lambda^+ < 0$  for  $\mu$  small enough. This proves part (1) of Proposition 4.1. We also remark that as  $\mu \rightarrow \infty$ ,  $\mathbf{E}[r^2] \approx \mu/a$ , so that

$$\lambda^+ \approx \left( \sqrt{1 + (b/a)^2} - 1 \right) \mu,$$

and  $\lambda^+$  becomes positive for  $\mu$  sufficiently large. One could use (26) or other techniques to get good approximations where  $\lambda^+$  crosses the axis and get an idea how large a  $\mu$ -domain guarantees  $\lambda$  negative.

**4.2. Asymptotics for  $\lambda$  in the small noise limit.** We consider the solution to  $L^*p = 0$ , where

$$(27) \quad L^*p = -\partial_r \left( \left( \mu r - ar^3 + \frac{\sigma^2}{2r} \right) p \right) - \partial_\psi (2(b + \sqrt{a^2 + b^2} \cos \psi)r^2 p) + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial r^2} + \frac{2\sigma^2}{r^2} \frac{\partial^2 p}{\partial \psi^2},$$

and we want to compute

$$\lambda = \int_0^\infty \int_0^{2\pi} (\mu - 2ar^2 + \sqrt{a^2 + b^2}r^2 \sin \psi) p(r, \psi) d\psi dr.$$

Rescaling (27) with the variable  $r = r_0 \rho$  gives

$$0 = -\partial_\rho \left( \left( a(\rho - \rho^3) + \frac{\sigma^2}{2\rho r_0^4} \right) p \right) - \partial_\psi (2\rho^2(b + \sqrt{a^2 + b^2} \cos \psi)p) + \frac{\sigma^2}{2r_0^4} \frac{\partial^2 p}{\partial \rho^2} + \frac{2\sigma^2}{\rho^2 r_0^4} \frac{\partial^2 p}{\partial \psi^2}.$$

From this we see the natural small parameter is  $\epsilon = \sigma^2/r_0^4$ , and this gives

$$(28) \quad 0 = -\partial_\rho \left( \left( a(\rho - \rho^3) + \frac{\epsilon}{2\rho} \right) p \right) - \partial_\psi (2\rho^2(b + \sqrt{a^2 + b^2} \cos \psi)p) + \frac{\epsilon}{2} \frac{\partial^2 p}{\partial \rho^2} + \frac{2\epsilon}{\rho^2} \frac{\partial^2 p}{\partial \psi^2}.$$

Notice that this equation is singularly-perturbed (the  $\epsilon$  appears in front of the highest-order terms) and so in the limit  $\epsilon \rightarrow 0$  we expect to get a singular solution. Of course, we do: as  $\epsilon \rightarrow 0$ , the distribution limits on a delta function at the attracting fixed point  $r = \sqrt{\mu/a}, \psi = \psi^*$ , where  $\cos(\psi^*) = -b/\sqrt{a^2 + b^2}$ , and  $\sin(\psi^*) = a/\sqrt{a^2 + b^2}$ . This gives a leading-order term of 0, as we showed above. We note, as it will become important below, that  $\sin \psi^* > 0$  because we are choosing an *attracting* fixed point of the dynamics.

Now rescale

$$(29) \quad \rho = 1 + \epsilon^\alpha \xi, \quad \psi = \psi^* + \epsilon^\alpha \eta,$$

and then

$$\partial_\xi = \epsilon^{-\alpha} \partial_x, \quad \partial_\eta = \epsilon^{-\alpha} \partial_y.$$

Writing (28) as

$$0 = -(f^\rho(\rho)p + O(\epsilon))_\rho - (f^\psi(\rho, \psi)p)_\psi + \epsilon \sigma^\rho \frac{\partial^2 p}{\partial \rho^2} + \epsilon \sigma^\psi \frac{\partial^2 p}{\partial \psi^2},$$

and using the expansion in (29) gives

$$\begin{aligned} 0 &= -(f_\rho^\rho \epsilon^\alpha \xi + O(\epsilon))_\rho - (\epsilon^\alpha (f_\rho^\psi \xi + f_\psi^\psi \eta))_\psi \\ &\quad + \epsilon (\sigma^\rho(1, \psi^*) + O(\sqrt{\epsilon})) p_{\rho\rho} + (\sigma^\psi(1, \psi^*) + O(\sqrt{\epsilon})) p_{\psi\psi} \\ &= -(f_\rho^\rho \xi + O(\epsilon^{1-\alpha}))_\xi - (f_\rho^\psi \xi + f_\psi^\psi \eta)_\eta \\ &\quad + \epsilon^{1-2\alpha} (\sigma^\rho(1, \psi^*) + O(\sqrt{\epsilon})) p_{\rho\rho} + \epsilon^{1-2\alpha} (\sigma^\psi(1, \psi^*) + O(\sqrt{\epsilon})) p_{\psi\psi}, \end{aligned}$$

where we denote  $f_\rho^\rho$ , etc., the values of the derivatives at the point  $(1, \psi^*)$ . If we choose  $\alpha = 1/2$ , then each of the four terms has a leading order term of  $O(1)$  and a higher order term, so looking only at terms of  $O(1)$  gives

$$0 = -(f_\rho^\rho \xi)_\xi - (f_\rho^\psi \xi + f_\psi^\psi \eta)_\eta + \sigma^\rho(1, \psi^*) p_{\xi\xi} + \sigma^\psi(1, \psi^*) p_{\eta\eta}.$$

Computing

$$f_\rho^\rho = -2a, \quad f_\rho^\psi = 0, \quad f_\psi^\psi = -2a, \quad \sigma^\rho(1, \psi^*) = \frac{1}{2}, \quad \sigma^\psi(1, \psi^*) = 2,$$

and we have

$$(30) \quad 0 = (2a\xi p)_\xi + (2a\eta p)_\eta + \frac{1}{2} p_{\xi\xi} + 2p_{\eta\eta}.$$

This equation can be solved explicitly:

$$p(\xi, \eta) = Z^{-1} \exp(-2a\xi^2 - a/2\eta^2),$$

or

$$p(\rho, \psi) = Z^{-1} \exp\left(-\frac{1}{\varepsilon}(2a(x-1)^2 + \frac{a}{2}(\psi - \psi^*)^2)\right),$$

or

$$(31) \quad p(r, \psi) = Z^{-1} \exp\left(-\varepsilon^{-1} \left(\frac{2a^2}{\mu} \left(r - \sqrt{\frac{\mu}{a}}\right)^2 + \frac{a}{2}(\psi - \psi^*)^2\right)\right).$$

The strength of this representation is that it can be written as a product:

$$p(r, \psi) = \frac{p^r(r)p^\psi(\psi)}{Z_r Z_\psi},$$

where

$$(32) \quad p^r(r) = e^{-\frac{2a^2}{\varepsilon\mu}(r - \sqrt{\mu/a})^2}, \quad Z_r = \int_0^\infty p(r) dr,$$

$$(33) \quad p^\psi(\psi) = e^{-\frac{a}{2\varepsilon}(\psi - \psi^*)^2}, \quad Z_\psi = \int_0^{2\pi} p(\psi) d\psi.$$

Now we need to compute

$$\lambda := \int_{\mathbb{R}^2} Q(r, \psi) p(r, \psi) dr d\psi = \int_{\mathbb{R}^2} (\mu - 2ar^2 + \sqrt{a^2 + b^2} r^2 \sin \psi) p(r, \psi) dr d\psi,$$

which we will write as three integrals  $I_1 + I_2 + I_3$  in the obvious way. Then we have

$$\begin{aligned} I_1 &= \mu, \\ I_2 &= -2aZ_r^{-1} \int_0^\infty r^2 p^r(r) dr, \\ I_3 &= \sqrt{a^2 + b^2} Z_r^{-1} Z_\psi^{-1} \int_0^\infty r^2 p^r(r) dr \int_0^{2\pi} \sin \psi p^\psi(\psi) d\psi. \end{aligned}$$

Now, first note that in the limit  $\varepsilon \rightarrow 0$ ,

$$p^r(r) \rightarrow \delta_{r=\sqrt{\mu/a}}, \quad p^\psi(\psi) \rightarrow \delta_{\psi=\psi^*}$$

in the sense of distributions, so that as  $\varepsilon \rightarrow 0$ , we have

$$\lambda \rightarrow \mu - 2a(\mu/a) + \sqrt{a^2 + b^2}(\mu/a) \frac{a}{\sqrt{a^2 + b^2}} = \mu - 2\mu + \mu = 0,$$

agreeing with the previous section.

We can replace the domain of every integration by  $(-\infty, \infty)$  and only add an error  $\sim O(e^{-1/\sqrt{\varepsilon}})$ : the variance of each the Gaussians scales like  $\varepsilon$ , and thus the error terms introduced by extending the domain

of integration are all of the form  $\text{erf}(C\varepsilon^{-1/2})$ . Extending the domains in this fashion allows us to solve for  $I_k$  exactly. For example,

$$\int_{-\infty}^{\infty} r^2 p^r(r) dr = \sqrt{\frac{\varepsilon\pi}{2}} \mu^{3/2} \frac{4a + \varepsilon}{4a^3},$$

$$Z_r = \frac{\sqrt{\pi\varepsilon\mu}}{a\sqrt{2}},$$

so that we have

$$(34) \quad I_2 = -\frac{(4a + \varepsilon)\mu}{2a} = -2\mu - \varepsilon \frac{\mu}{2a}.$$

We cannot evaluate  $I_3$  exactly, but we can do another approximation. If we expand  $\sin\psi$  in a Taylor series around  $\psi^*$ , namely

$$\sin(\psi) = \sin(\psi^*) + \cos(\psi^*)(\psi - \psi^*) + \frac{-\sin(\psi^*)}{2}(\psi - \psi^*)^2 + \dots,$$

then

$$\begin{aligned} \int_{-\infty}^{\infty} \sin(\psi) p^\psi(\psi) d\psi &\approx \sin\psi^* \int_{-\infty}^{\infty} p^\psi(\psi) d\psi - \frac{\sin\psi^*}{2} \int_{-\infty}^{\infty} (\psi - \psi^*)^2 p^\psi(\psi) d\psi \\ &= \sqrt{\frac{2\pi\varepsilon}{a}} \sin\psi^* - \sqrt{2\pi} \left(\frac{\varepsilon}{a}\right)^{3/2} \frac{\sin\psi^*}{2} \\ &= \sqrt{\frac{2\pi\varepsilon}{a}} \sin(\psi^*) \left(1 - \frac{\varepsilon}{2a}\right), \end{aligned}$$

and

$$Z_\psi = \sqrt{\frac{2\pi\varepsilon}{a}}.$$

This gives

$$(35) \quad I_3 = \sqrt{a^2 + b^2} \frac{I_2}{-2a} \sin(\psi^*) \left(1 - \frac{\varepsilon}{2a}\right) = -\frac{1}{2} \left(-2\mu - \frac{\varepsilon\mu}{2a}\right) \left(1 - \frac{\varepsilon}{2a}\right) = \mu - \varepsilon \frac{1}{4} \frac{\mu}{a}$$

Putting (34), (35) together gives

$$\lambda = \mu - 2\mu - \varepsilon \frac{\mu}{2a} + \mu - \varepsilon \frac{\mu}{4a} + O(\varepsilon^2) = -\varepsilon \frac{3\mu}{4a} + O(\varepsilon^2),$$

which is negative for  $\varepsilon$  sufficiently small.

One thing which might seem paradoxical is that the correction to  $I_3$  is always negative, whereas we would expect it to be positive in some cases. For example, as  $\varepsilon \rightarrow 0$ , we converge to the delta function at  $\psi^*$ , but if  $\sin(\psi)$  is concave up at that point, then the correction when we let  $\varepsilon > 0$  would be positive on both sides, i.e.  $\sin$  is below its local average. However, it is always the case that  $\sin(\psi^*) > 0$ , since it has to be an *attracting* fixed point of the ODE  $\psi' = b + \sqrt{a^2 + b^2} \cos\psi$ . Thus the fixed point only exists in a region where the curve  $\sin$  is concave down, and thus the correction is always negative. Similarly for  $I_2$ , since the curve is concave up, broadening the exponential makes the integrand increase, but then we are multiplying by a negative coefficient. Thus in the small noise limit, we always approach zero from below.

**4.3. Shear Instability.** We plot the results of simulating  $\lambda$  in Figure 1. We can observe that increasing  $b$  seems to uniformly increase  $\lambda$ , but larger  $\mu$  require larger  $b$  to make  $\lambda > 0$ . We show below that these observations are consistent with certain asymptotic scalings which we present below.

We have

$$\lambda = \mu - 2a\mathbf{E}[r^2] + \sqrt{a^2 + b^2}\mathbf{E}[r^2 \sin(\psi)],$$

where  $\mathbf{E}[\cdot]$  is with respect to the invariant measure of (16). As we see from (25),  $\mathbf{E}[r^2] \approx \mu/a$ , so we will have

$$\lambda \approx -\mu + \sqrt{a^2 + b^2}\mathbf{E}[r^2 \sin(\psi)]$$

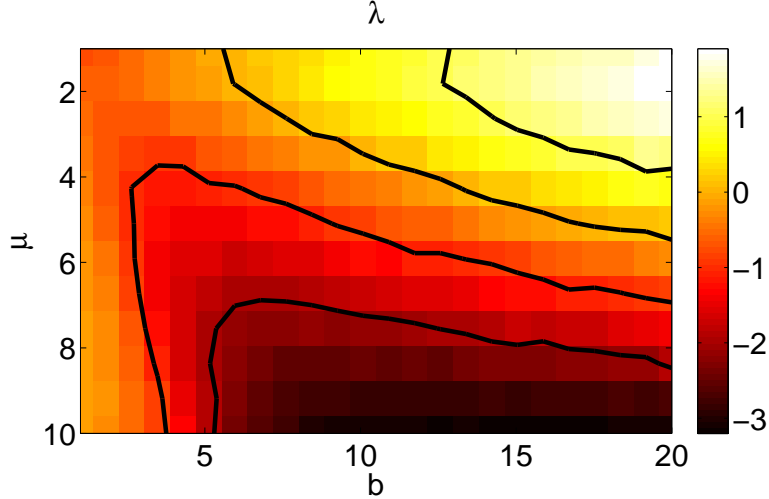


FIGURE 1. Numerically-computed values of  $\lambda$  versus  $\mu, b$  for  $a = \sigma = 1$ . The solid black lines are the integer-valued level sets of  $\lambda$ . To compute  $\lambda$  for a given  $\mu, b$ , we simulated (16) directly with  $dt = 10^{-3}$  until the final time  $T = 10^4$ , and computed the time average of  $Q$  over this simulation. We then ran simulations for 2,376 different parameter values in the square above, binned them into a  $12 \times 24$  grid, then plotted the ensemble average of  $\lambda$  for all the parameter values which lie inside each of the grid squares.

We expect  $\lim_{b \rightarrow 0} \mathbf{E} [r^2 \sin(\psi)] \rightarrow 0$ : it is easy to see that we expect that  $w(\psi) = p(\psi|r)$  should be concentrated around the unique attracting fixed point, but this fixed point approaches  $\pi$  from below. However, if it turns out that

$$\lim_{b \rightarrow 0} \mathbf{E} [r^2 \sin(\psi)] \rightarrow 0 \text{ slower than } b^{-1},$$

then the third term in  $\lambda$  will actually grow.

First, consider the limit where  $b \rightarrow 0$ . Then we have

$$d\psi_t = 2a \cos \psi - \frac{2\sigma}{r_t} dW$$

In the absence of noise, this system has a globally-attracting fixed point near  $\psi = \pi/2$ . As long as  $\sigma$  is not too large, we have that

$$\mathbf{E} [\sin(\psi)] \approx \sin(\pi/2) = 1$$

(of course, the correction to this will be  $r$ -dependent). If we again assume that  $r \approx \sqrt{\mu/a}$ , then

$$\lambda \approx -\mu + \mu = 0,$$

meaning that we expect  $\lambda$  to be negative, but get close to zero  $b \rightarrow 0$  (and it will get closer to zero as  $\sigma \rightarrow 0$  or  $\mu/a \rightarrow \infty$ ). This is consistent with the data plotted on the left edge of Figure 1.

Now take  $b$  large. Consider the  $\psi$  SDE with a time rescaling:

$$(36) \quad d\psi_t = 1 + \sqrt{1 + a^2/b^2} \cos(\psi) dt - \frac{\sqrt{2}\sigma}{\sqrt{b}r_t^2} dW.$$

As  $b \rightarrow \infty$ , we would expect the invariant measure to approach a delta function, since the noise is going to zero. However, as  $b \rightarrow \infty$ , we are also approaching a saddle-node bifurcation, which means that small noise can easily push the system away from the attracting fixed point. So, in short, what we have is that as  $b \rightarrow \infty$ , we expect to get a distribution localized around the fixed point  $\psi = \cos^{-1}(-1/\sqrt{1 + a^2/b^2}) \rightarrow \pi$ , and thus  $\mathbf{E} [\sin(\psi)]$ , to first approximation, is zero. The problem is, in the formula for  $Q$ , we are multiplying this number by a number which is  $O(b)$ , and therefore even if the distribution has zero mean in the limit, the product could blow up. And in fact, this is what we see numerically: in the limit as  $b \rightarrow \infty$ , the quantity  $\mathbf{E} [\sin(\psi)]$  seems to scale like  $b^{-1/3}$ ; see Figure 2.

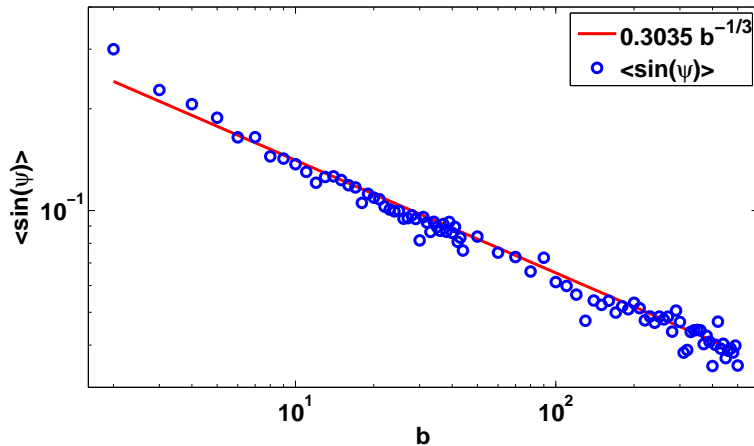


FIGURE 2. Numerically-computed quantity  $\mathbf{E}[\sin(\psi)]$  for the invariant measure coming from (36), where we have chosen  $a = \mu = \sigma = 1$  and various  $b$ .

## 5. CONCLUSIONS

We have considered a stochastically-perturbed system near Hopf bifurcation and studied the top Lyapunov exponent. We have shown it can be of either sign depending on parameters, but that it is always negative in the small-noise limit — in direct contrast to the analogous two-dimensional Hamiltonian system. Moreover, we have shown the existence of shear leads to a positive Lyapunov exponent, i.e. the system has a shear instability.

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