

Reduced equations for models of laminated materials in thin domains. I.

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Abstract

In this paper, we consider a scalar wave equation on a thin, laminated, three-dimensional plate. We show that if the plate is sufficiently thin, then there is a hierarchy of two-dimensional equations whose dynamics model the dynamics of the full plate, each of which successively lengthens the time interval over which the approximation holds.

1 Introduction

Many problems involving the motion of elastic structures occur in domains where one or more of the dimensions of the body is significantly smaller than the others. One can think, for instance, of the vibrations of a long metal beam or rod, or the metal plates that cover the outside of an airplane or ship. In such situations, engineers have usually replaced the true equations of elasticity by simpler model equations – for instance, the Bernoulli or Timoshenko models of beams or the Kirchoff or Reisner-Mindlin models of plates. The reasons for such approximations were that the simplified models were more amenable to exact or asymptotic solution methods. Even with the advent of high speed computers, numerical solution of the full three dimensional elasticity problem is very time-consuming and these model equations are still extensively used because of the savings of time that they represent. In the last decade, through the work of numerous authors (see [9],[10], [11] [2], [1], [3], [26], [19] for a small sampling of this literature), a number of results have been established that justify using these model equations as approximations in problems of *static* elastic behavior

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– that is to say, when the beam or plate is at rest. Far less is known about the validity of these model equations at describing the dynamic behavior of these elastic media. We are aware only of the work of Raoult [23], [24], Durdević [14], and Xiao [27].

In this paper, we begin a study of the dynamics of thin elastic media and the approximation of their motion by “reduced equations” – that is, by equations on lower dimensional spatial domains. Our approach differs from those mentioned in the previous paragraph in that we are interested in developing a hierarchical family of reduced equations which allow us to approximate the dynamics over longer and longer time intervals – in particular for a time interval longer than any fixed inverse power of the thickness of the domain. In this respect, our work is similar in spirit to that of Schwab and Babuška [3] on the analogous static problem (although our methods are completely different) who point out that in real applications one is always faced with a domain of some fixed thickness and one needs to be able to construct an approximation to the desired accuracy for that given thickness.

We take as our starting point a model for thin laminated materials studied in the static case in [3]. We prove that if one fixes a tolerance for the error and a timescale on which one wishes to make the approximation, then for sufficiently thin beams or plates, one can construct a reduced 2D equation which will approximate the dynamics of the full problem to that error tolerance and on that timescale. As the length of time over which one wishes the approximation to be valid increases, the reduced equation becomes more complicated, but our method provides an algorithm for constructing refinements to the reduced equation inductively, and in a companion paper, [13], one of us shows that the coefficients in the reduced equations can be computed explicitly. We also hope that the theory generated in this work can eventually be generalized to methods which will derive similar reduced equations for the full equations of linear elasticity, just as in the static case the results of [3] were later extended to the static elasticity problem in [26].

The viewpoint we adopt in this paper is to think of the original three dimensional partial differential equation as an infinite dimensional Hamiltonian dynamical system. We identify in the (infinite dimensional) phase space of this dynamical system an (infinite dimensional) submanifold which is left invariant by the reduced PDE. We then show, by making appropriate canonical changes of variables in the Hamiltonian for our original PDE, that this invariant manifold is left approximately invariant by the flow of the full three-dimensional PDE. In fact, we make a sequence of canonical changes of variables which systematically: (1) change the submanifold slightly, leading to more and more refined approximating equations, and (2) show that, given initial conditions of our original PDE in some tubular neighborhood of this submanifold, the solution with these initial conditions is well approximated by the flow on the manifold (and hence by the reduced equation) for longer and longer time intervals.

The procedure is reminiscent of and inspired by the Nekhoroshev theory [20] of classical mechanics. That theory shows that for a nearly integrable Hamiltonian system, any solution behaves in a nearly quasi-periodic fashion for a very

long time. More geometrically, one may interpret it as saying that initial conditions which lie close to an invariant torus in the phase space can remain close to that torus for a very long time, and their motion is well approximated by the motion on the torus. Just as in the classical mechanics case, where the flow near the invariant torus both stays near the torus and is well-approximated by the flow on the torus, we will show that the flow near the invariant submanifold both stays near the submanifold and is well-approximated by the flow on the invariant submanifold. Note that in contrast to the classical Nekhoroshev theory, and also in contrast to the extensions of the Nekhoroshev theory for nearly integrable PDEs ([4], [21], [22], [7]), the manifold to which we remain close is itself infinite dimensional. In order to apply the classical Nekhoroshev theory, it is necessary that there be some small parameter in the problem. In our case, we take advantage of the fact that vibrations of our laminar material in its “thin” direction have much higher frequency than those in its “long” directions. The inverse of the ratio of these two frequencies (which is, in fact, a positive power of the thickness of the three-dimensional domain) is effectively the small parameter in our expansion. Nekhoroshev-type theories in which one exploits a large separation in frequencies have also been developed for models of one-dimensional gases [6] and for models of diatomic molecules [5].

One complication of this approach is that the approximating equations which arise most naturally in this procedure are sometimes ill-posed. We circumvent that problem here by showing that we can still approximate solutions of our original partial differential equation if we choose initial conditions for the reduced equation in a subspace in which it is well-posed. In a companion paper [13], a method of deriving reduced equations which are well-posed on the whole space is developed.

We conclude this introduction with a brief survey of the organization of the remainder of this section. In Section 1.1 we define the notation we will use and state the problem explicitly. In Section 1.2 we describe the problem with homogeneous boundary conditions and state Theorem 1, whose proof is the bulk of this paper. In Section 1.3 we consider the problem with inhomogeneous boundary conditions, and show how Theorem 1 and the results of [3] can be combined to prove Theorem 2. Finally, in Section 1.4, we outline the ideas behind our proof in some detail.

1.1 Notation and problem formulation

Let us consider $\omega \subset \mathbb{R}^2$, a bounded domain with \mathcal{C}^1 smooth boundary γ , for which we will use the coordinates $x = (x_1, x_2)$.

Given a two dimensional domain ω and a positive thickness parameter d we define the three-dimensional domain

$$\Omega = \omega \times (0, \pi d),$$

where we use coordinates (x_1, x_2, y) . The lateral boundary of Ω is

$$\Gamma = \gamma \times (0, \pi d),$$

and the top and bottom faces are

$$\begin{aligned} R_- &= \{(x_1, x_2, y) | (x_1, x_2) \in \omega, y = 0\}, \\ R_+ &= \{(x_1, x_2, y) | (x_1, x_2) \in \omega, y = \pi d\}. \end{aligned}$$

In Ω we consider the hyperbolic problem with prescribed forcing terms on the faces, i.e.

$$\begin{aligned} u_{tt} &= Lu \text{ in } \Omega \times \mathbb{R}, \\ u &= 0 \text{ on } \Gamma \times \mathbb{R}, \\ \partial_n u &= f^\pm \text{ on } R_\pm \times \mathbb{R}, \end{aligned} \tag{1}$$

where the operator L is given by

$$Lu = \frac{\partial}{\partial y} \left(a(y/d) \frac{\partial u}{\partial y} \right) + b(y/d) \nabla_x \cdot (C(x) \nabla_x u), \tag{2}$$

with $\nabla_x = (\partial/(\partial x_1), \partial/(\partial x_2))^T$, and $f_\pm \in L^2(\omega)$. We choose an initial condition $(u^0, u_t^0) \in H^s(\Omega) \times H^{s-1}(\Omega)$, and we show below that the solution to (1) will lie in $H^s(\Omega) \times H^{s-1}(\Omega)$, by which we mean that $u(x, y, t) \in H^s(\Omega)$ and $\frac{\partial}{\partial t} u(x, y, t) \in H^{s-1}(\Omega)$ for all t .

We further assume that $a(\cdot), b(\cdot) \in C^1(0, \pi)$ are independent of d and that there are numbers $\mathbf{a}, \mathbf{A}, \mathbf{b}$, and \mathbf{B} so that

$$0 < \mathbf{a} \leq a(z) \leq \mathbf{A}, \quad 0 < \mathbf{b} \leq b(z) \leq \mathbf{B},$$

and that the matrix $C(x)$ is symmetric and uniformly positive-definite, i.e. that there are constants $0 < \mathbf{c} \leq \mathbf{C} < \infty$ so that

$$\mathbf{c} |\xi|^2 \leq \xi^T C(x) \xi \leq \mathbf{C} |\xi|^2,$$

for all $\xi \in \mathbb{R}^2$, $x \in \omega$, and that C has C^∞ coefficients. The static version of (1) (i.e. the case in which u is independent of time) was introduced in [3] as a model for laminated materials. Although we are not considering a general second-order elliptic operator in (2), the form of L is natural in this context. First, we assume that all coefficient functions decompose into products in each direction, and second, we assume that the plate is laminated, namely that the first differential term does not depend on x . These are reasonable assumptions on the operator L in the dimensional reduction setting.

The goal of this paper is to show that we can approximate the solutions to (1) by the solutions to a 2-dimensional PDE in the variables (x_1, x_2) (or by a system of such equations). If we can find such a PDE which approximates the full 3-dimensional problem well, we will refer to it as a ‘‘reduced equation’’.

1.2 The homogeneous problem

We begin by studying an apparently simpler problem in which the inhomogeneous boundary conditions on the top and bottom face of the body are eliminated. After showing how we can approximate the motion of this problem by

a reduced problem, we will show in Section 1.3 that the same is true of (1) by combining our results with those of [3].

Letting L be as in (2), we consider:

$$\begin{aligned} u_{tt} &= Lu \text{ in } \Omega \times \mathbb{R}, \\ u &= 0 \text{ on } \Gamma \times \mathbb{R}, \\ \partial_n u &= 0 \text{ on } R_{\pm} \times \mathbb{R}, \end{aligned} \tag{3}$$

with the initial condition $(u^0, u_t^0) \in H^s(\Omega) \times H^{s-1}(\Omega)$. It will be useful in this problem to consider the operators

$$L_x u = \hat{b}(0) \nabla_x (C(x) \nabla_x u), \tag{4}$$

$$L_y u = (a(y/d) u_y)_y, \tag{5}$$

where $\hat{b}(0)$ denotes the average value of b , namely

$$\hat{b}(0) = \frac{1}{\pi d} \int_0^{\pi d} b(y/d) dy.$$

Note that L_x is a differential operator which depends only on x , and L_y is a differential operator which depends only on y . The motivation for choosing L_x and L_y in this manner is that it is the operator L defined in (2) and averaged it in the obvious way in both the x and y directions.

Our reduced equations will be equations defined on ω and thus it will be useful to have the projection operator $\Pi_1: H^s(\Omega) \rightarrow H^s(\omega)$ defined as

$$(\Pi_1 u)(x) = \frac{1}{\pi d} \int_0^{\pi d} u(x, y) dy. \tag{6}$$

As stated above, problem (1) is well-posed in $H^s(\Omega) \times H^{s-1}(\Omega)$. For technical reasons which will become clear, we will find it useful to consider the problem (1) projected into a finite dimensional subspace of $H^s(\Omega) \times H^{s-1}(\Omega)$, which we now describe. Let $\phi_k \in C^\infty(\omega)$ be the eigenfunctions of L_x , and fix an $\alpha \in (-1/2, 0)$. The functions ϕ_k will form an orthogonal basis for $H^s(\omega)$ (orthonormal if we scale them properly). We consider the finite dimensional subspace of $H^s(\omega)$ spanned by $\{\phi_k\}_{k < d^\alpha}$, and denote this $H_\alpha^s(\omega)$. We also define the projection $\Pi_2: H^s(\omega) \rightarrow H_\alpha^s(\omega)$ by

$$\Pi_2 \left(\sum_{k=0}^{\infty} a_k \phi_k \right) = \sum_{k < d^\alpha} a_k \phi_k. \tag{7}$$

If we have a solution u smooth enough that all the derivatives in (3) exist and are continuous, and (3) is satisfied for all $0 \leq t \leq T$, then we say that u is a strong solution of (3). We will typically study weak solutions of these equations, however. Define the time-dependent bilinear form

$$B[u, v; t] = \int_{\Omega} a(y/d) \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + b(y/d) (\nabla_x u)^T C(x) \nabla_x v dV. \tag{8}$$

We define the Hilbert space

$$\dot{H}(\Omega) = \{u \in H^2(\Omega) \mid \text{Trace}(u)=0 \text{ on } \Gamma, \text{Trace}(\partial_n u)=0 \text{ on } y = 0, y = \pi d\}.$$

Then [15] we say that u is a weak solution of (3) if

$$u \in L^2([0, T], \dot{H}(\Omega)), \dot{u} \in L^2([0, T], H^1(\Omega)), \ddot{u} \in L^2([0, T], L^2(\Omega)),$$

and u satisfies

$$\langle \ddot{u}, v \rangle + B[u, v; t] = 0 \tag{9}$$

for all $v \in \dot{H}(\Omega)$ and for almost every time $0 \leq t \leq T$, and

$$u(0) = u^0, \quad \dot{u}(0) = v^0.$$

The inner product in (9) denotes the standard pairing of H^{-m} with H^m . It is a standard exercise [15] to show that with the assumptions that we have made on the coefficient functions in the equation, (3) always has a unique weak solution. Furthermore, standard results also show that if u is a strong solution of (3), then it is also a weak solution, and any weak solution smooth enough to be defined as a strong solution is also a strong solution.

Remark 1. *It is more standard in problems of this type to impose Neumann boundary conditions by requiring that the solution take values in $H^1(\Omega)$, and in addition to satisfying the equation (9) one requires that*

$$\int_{R_{\pm}} a(y/d) \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} dx = 0.$$

However, since we will require that the solution lie in $H^2(\Omega)$ for some later estimates, we have chosen to enforce the boundary conditions in the definition of the function space $\dot{H}(\Omega)$.

This paper derives a series of approximating equations for (3) which are accurate approximations over longer and longer time intervals. More precisely we have:

Theorem 1. *Fix $n > 0$, $C_0 > 0$, $\epsilon > 0$, and $\alpha \in (-1/2, 0)$. Then there exist constants $C^{(2)}, C^{(3)}, \dots$, (which are computable in terms of the coefficient functions a, b , and C), a function $M(n)$ with $M(1) = \frac{5}{2}$, $M(n) = \frac{3}{2}n$ for $n > 1$, $d_0 = d_0(C_0, \epsilon, \alpha, n)$, and $C_T = C_T(C_0, \epsilon, \alpha, n)$, such that for all $d < d_0$, one has the following:*

Let $u(x, y, t)$ be the solution of (3) with initial conditions $(u^0, u_t^0) \in H^2(\Omega) \times H^1(\Omega)$ satisfying

$$\|u^0\|_{H^2(\Omega)} + \|u_t^0\|_{H^1(\Omega)} \leq C_0.$$

Let u_n^r be the solution of the equation

$$\partial_t^2 u_n^r = D_n u_n^r \tag{10}$$

where D_n is the differential operator defined by

$$D_n = \begin{cases} L_x, & n = 1, \\ L_x + \sum_{q=2}^n C^{(q)} d^{2(q-1)} L_x^q, & n > 1, \end{cases}$$

with initial conditions $(\Pi_2 \Pi_1 u^0, \Pi_2 \Pi_1 u_t^0) \in H_\alpha^2(\omega) \times H_\alpha^1(\omega)$. Then if we define $\mathbf{1}(\cdot)$ to be the constant function which is equal to one,

$$\|u(x, y, t) - u_n^{\mathbf{r}}(x, t) \mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} \leq \epsilon$$

for all $t \leq C_T d^{-M(n)}$.

We have several remarks concerning this theorem.

First, in terms of application, one can think of this theorem as follows. One fixes a tolerance for error ϵ and the order n of the reduced equation. The theorem then guarantees that for a sufficiently thin plate, there exists a reduced equation which will accurately approximate the solution of the three dimensional equation (up to the chosen error) for a time scale of $\mathcal{O}(d^{-M(n)})$. Conversely, if one wishes to approximate the solutions of the three dimensional problem for a predetermined timescale, one can choose n sufficiently large to find a reduced equation that provides a good approximation for at least this long. Not surprisingly, the longer one wishes to have accurate approximation, the more complicated (i.e. the larger n) the reduced equation must be. However, in a companion paper [13], explicit expressions for the coefficients $C^{(j)}$ in the reduced equation will be derived for all j so that in principle, the reduced equations can be computed to any order.

Second, the theorem as stated is not optimal. As can be seen from the statement of Theorem 3, it is sufficient to choose initial data in $H^s(\Omega) \times H^{s-1}(\Omega)$ with any $s > 3/2$. Moreover, the function $M(n)$ can be improved to be $(2n + 1) + (2n + 2)\alpha$ for $n > 1$ (see (93)). This also grows linearly, but for α chosen appropriately, this can grow faster than $\frac{3}{2}n$.

Third, one can also trade timescale for error in this theorem if this is desired. As stated, the error estimate is independent of d as this gives the maximal timescale over which the theorem is valid. However, as one can see from (105), we can trade up to $2 + 4\alpha$ powers of timescale and put it on the error term, i.e. we can get the estimate

$$\|u(x, y, t) - u_n^{\mathbf{r}}(x, t) \mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} \leq \epsilon d^\beta$$

for $t \sim d^{-M(n)+\beta}$, for any $\beta \in [0, 2 + 4\alpha]$.

Fourth and finally, the boundary conditions in equation (10) are those “inherited” from the definition of the operator L_x , i.e. we require

$$L_x^i u_n^{\mathbf{r}}|_{\partial\omega} = 0, \text{ for } i < n.$$

As mentioned above, when we speak of a solution of (10) we mean a weak solution. More precisely, since $-L_x$ is a positive definite, densely defined, symmetric

operator on $H_0^1(\omega)$, it has a positive-definite square root which we call \mathcal{L} . We define the bilinear form

$$B^{\mathbf{r}}[u, v; t] = \int_{\omega} (\mathcal{L}u)(\mathcal{L}v) + \sum_{q=2}^n C_q d^{2(q-1)} (\mathcal{L}^q u)(L^q v) dx. \quad (11)$$

A weak solution of (10) is any function

$$u \in L^2([0, T], H_0^m(\omega)), \dot{u} \in L^2([0, T], L^2(\omega)), \ddot{u} \in L^2([0, T], H^{-m}(\omega))$$

that satisfies

$$\langle \ddot{u}, v \rangle + B^{\mathbf{r}}[u, v; t] = 0,$$

for all $v \in H_0^m(\omega)$ and for almost every time $0 \leq t \leq T$, where

$$u(0) = \Pi_2 \Pi_1 u^0, \quad \dot{u}(0) = \Pi_2 \Pi_1 \dot{u}_t^0.$$

1.3 The inhomogeneous case

We now turn to the task of deriving reduced equations for our original equation (1) with nontrivial boundary conditions on the top and bottom faces. We will do this by combining Theorem 1 with the results of [3]. In [3], Babuška and Schwab considered the static version of (1), namely the elliptic equation:

$$\begin{aligned} 0 &= Lv \text{ in } \Omega, \\ v &= 0 \text{ on } \Gamma, \\ \partial_n v &= f^{\pm} \text{ on } R_{\pm}. \end{aligned} \quad (12)$$

They derived an efficient dimensional reduction method to approximate the solution $v(x, y)$. Given an s and an ϵ , it allows one to derive a system of (two dimensional) elliptic PDEs whose solution $\tilde{v}(x) \in H^s(\omega)$, and satisfies $\|v(x, y) - \tilde{v}(x)\mathbf{1}(y)\|_{H^1(\Omega)} < \epsilon$.

Given a solution $u(x, y, t)$ of (1), and a solution $v(x, y)$ of (12), the function $w(x, y, t) = u(x, y, t) - v(x, y)$ solves (3). We then use the results of [3] to derive a dimensionally reduced approximation to $v(x, y)$ and Theorem 1 to derive a dimensionally reduced approximation to $w(x, y, t)$. Their sum $u(x, y, t)$ will then give a dimensionally reduced approximation to the solution of (1). However, there will be one slight complication: since we do not know $v(x, y)$ exactly, we do not know the initial conditions of $w(x, y, 0) = u(x, y, 0) - v(x, y)$ to plug into (3). To circumvent this difficulty we need to make another approximation argument.

Fix $\beta > 0$. By [3] there exists a system of reduced equations for (12) whose solution $v^{\mathbf{r}} \in H^2(\omega)$ and which satisfies

$$\|v(x, y) - v^{\mathbf{r}}(x)\mathbf{1}(y)\|_{H^1(\Omega)} \leq \beta. \quad (13)$$

Now consider the equation

$$\begin{aligned}
w_{tt} &= Lw \text{ in } \Omega \times \mathbb{R}, \\
w &= 0 \text{ on } \Gamma \times \mathbb{R}, \\
\partial_n w &= 0 \text{ on } R_{\pm} \times \mathbb{R}, \\
w|_{t=0} &= u^0(x, y) - v^{\mathbf{r}}(x, y), \quad w_t|_{t=0} = u_t^0(x, y).
\end{aligned} \tag{14}$$

We would like to apply Theorem 1 to derive reduced equations for (14), but since $v^{\mathbf{r}}$ does not exactly satisfy the boundary conditions $\partial_n v = f^{\pm}$ on R_{\pm} on the top and bottom faces of the domain, $w|_{t=0} = u^0 - v^{\mathbf{r}}$ will not exactly satisfy $\partial_n w = 0$ on \mathbb{R}_{\pm} , giving a potential boundary layer. To circumvent this difficulty we appeal to the following lemma, which is proved in Appendix A.

Lemma 1. *For any $\beta > 0$, there exists a function $w^0(x, y) \in H^2(\Omega)$ such that $\partial_n w^0 = 0$ on R_{\pm} , $w^0|_{\Gamma} = 0$, and $\|w^0 - (u^0 - v^{\mathbf{r}})\|_{H^1(\Omega)} < \beta$.*

Now, replace the initial conditions given by Lemma 1 into (14):

$$\begin{aligned}
\tilde{w}_{tt} &= L\tilde{w} \text{ in } \Omega \times \mathbb{R}, \\
\tilde{w} &= 0 \text{ on } \Gamma \times \mathbb{R}, \\
\partial_n \tilde{w} &= 0 \text{ on } R_{\pm} \times \mathbb{R}, \\
\tilde{w}|_{t=0} &= w^0(x, y), \quad \tilde{w}_t|_{t=0} = u_t^0(x, y).
\end{aligned} \tag{15}$$

Given any $n > 0$, Theorem 1 guarantees that there exists a reduced equation whose solution $w_n^{\mathbf{r}}$ satisfies

$$\sup_{t \in [0, C_T d^{-M(n)}]} \|\tilde{w}(x, y, t) - w_n^{\mathbf{r}}(x, t)\mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} < \beta. \tag{16}$$

We will need another easy lemma, this one based on the fact that (14) and (15) are Hamiltonian equations and hence preserve energy.

Lemma 2. *Let $w(x, y, t)$ be the solution of the initial-boundary value problem (14), and let $\tilde{w}(x, y, t)$ be the solution of (15). Then there exists a constant $C > 0$ such that*

$$\|w(x, y, t) - \tilde{w}(x, y, t)\|_{H^1(\Omega) \times L^2(\Omega)} \leq C\beta \tag{17}$$

for all $t \geq 0$.

We postpone the proof of this lemma until Section 1.4 below, where the Hamiltonian nature of the problem is discussed. We are now in a position to state the second main theorem of this paper, which is in fact our most general result.

Theorem 2. *Fix any $n > 0$, $\epsilon > 0$, and $C_0 > 0$, and let $M(n)$ be defined as in Theorem 1. There exist constants $d_0 = d_0(C_0, \epsilon, n)$, and $C_T = C_T(C_0, \epsilon, n)$ such that for all $d \leq d_0$, if $u(x, y, t)$ is the solution of (1) with initial conditions*

$(u^0, u_t^0) \in H^2(\Omega) \times H^1(\Omega)$ satisfying $\|u^0\|_{H^2(\Omega)} + \|u_t^0\|_{H^1(\Omega)} \leq C_0$, then there exist reduced equations whose solutions, which we denote u_n^r , satisfy

$$\|u(x, y, t) - u_n^r(x, t)\mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} \leq \epsilon$$

for all $0 \leq t \leq C_T d^{-M(n)}$.

Proof. Define $u_n^r(x, t) = v^r(x) + w_n^r(x, t)$. We will use (13), (16), and (17) to bound

$$\begin{aligned} & \|u(x, y, t) - u^r(x, t)\mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} \\ & \leq \|w(x, y, t) - w_n^r(x, t)\mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} + \|v(x, y) - v^r(x)\|_{H^1(\Omega)} \\ & \leq \|w(x, y, t) - \tilde{w}(x, y, t)\|_{H^1(\Omega) \times L^2(\Omega)} + \|\tilde{w}(x, y, t) - w_n^r(x, t)\mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} \\ & \quad + \|v(x, y) - v^r(x)\|_{H^1(\Omega)} \\ & = C\beta + \beta + \beta. \end{aligned}$$

□

All that now remains is to prove Theorem 1. We describe the proof in the next section.

1.4 An outline of the proof

To prove Theorem 1, we will actually do most of our work in the spectral domain as opposed to the spatiotemporal. A common approach in applying KAM or Nekhoroshev theory to partial differential equations is to expand our solution in terms of a basis of the phase space (see [17]). We do that here and then rewrite the Hamiltonian in these new coordinates. We then describe a program which will allow us to change variables to get a Hamiltonian which will generate the type of flow we need to prove Theorem 1.

Our first step is to make the second-order system (1) into a first-order Hamiltonian system with the Hamiltonian function

$$\begin{aligned} \tilde{H}: H_1 \times L_2 & \rightarrow \mathbb{R} \\ (u, p) & \mapsto \frac{1}{2} \int_{\Omega} \left\{ p^2 + a(y/d)(u_y)^2 + b(y/d) (\nabla_x u)^T C(x) \nabla_x u \right\} dV. \end{aligned} \quad (18)$$

The canonical equations

$$\partial_t \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_u \tilde{H}(u, p) \\ \delta_p \tilde{H}(u, p) \end{pmatrix} = \begin{pmatrix} p \\ (a(y/d)u_y)_y + b(y/d) \nabla_x (C(x) \nabla_x u) \end{pmatrix}$$

are easily seen to be equivalent to (1). Furthermore, it is also easily seen that our assumptions on a , b , and C make the function \tilde{H} equivalent to the canonical norm on $H^1(\Omega) \times L^2(\Omega)$.

Let us define $X_s(\Omega) = H^s(\Omega) \times H^{s-1}(\Omega)$. Then \tilde{H} is a function defined on X_1 , and $\nabla\tilde{H}$ is anti-smoothing in that it maps X_s to X_{s-1} . Defining \tilde{J} to be the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ above, we also have that \tilde{J} maps X_s to X_{s-1} . The canonical equations above, stated in this language, say that our PDE can be written as

$$\dot{u} = \tilde{J}\nabla\tilde{H}(u) =: V_{\tilde{H}}(u)$$

where $V_{\tilde{H}}$ maps X_s to X_{s-2} .

To use the standard language of Hamiltonian PDE, we have that X_s is a Hilbert scale, $\nabla\tilde{H}$ and \tilde{J} are isomorphisms of the scale of order one, and $V_{\tilde{H}}$ is an isomorphism of the scale of order two.

Proof of Lemma 2. By standard results from the theory of Hamiltonian PDEs, the value of \tilde{H} is constant along solutions of (1). Furthermore, because of the assumptions on the coefficient functions, a , b , and C , there exist constants \underline{c} and \bar{c} such that

$$\underline{c}(\|u\|_{H^1} + \|p\|_{L^2}) \leq \tilde{H}(u, p) \leq \bar{c}(\|u\|_{H^1} + \|p\|_{L^2}). \quad (19)$$

Let $w(\cdot, t)$ be the solution of (14) and $\tilde{w}(\cdot, t)$ the solution of (15). Then the difference $w - \tilde{w}$ satisfies the same differential equation, with initial conditions which satisfy $\|(w - \tilde{w})|_{t=0}\|_{H^1} \leq \beta$, and $(w_t - \tilde{w}_t)|_{t=0} = 0$. But then the conservation of the Hamiltonian along solutions and the estimate in (19) immediately give the result. \square

We now choose a basis for this phase space, and expand the Hamiltonian with respect to that basis. We have defined the y -range of our domain to be $[0, \pi d]$, so we introduce the new variable $\eta = y/d$, which ranges over $[0, \pi]$. Consider the eigenproblems

$$\begin{aligned} L_\eta \psi &= \frac{\partial}{\partial \eta} \left(a(\eta) \frac{\partial \psi}{\partial \eta} \right) = -\lambda \psi, & \frac{\partial \psi}{\partial \eta}(0) &= \frac{\partial \psi}{\partial \eta}(\pi) = 0, \\ L_x \phi &= \hat{b}(0) \nabla_x (C(x) \nabla_x \phi) = -\mu \phi, & \phi|_{\partial \omega} &= 0. \end{aligned}$$

Classical results (see [16],[25],[8]) imply that the above non-singular Sturm-Liouville problems define complete L^2 -orthonormal sequences of eigenfunctions and associated eigenvalues:

$$L_\eta \psi_l = -\lambda_l \psi_l, \quad (20)$$

$$L_x \phi_k = -\mu_k \phi_k, \quad (21)$$

and we can index the eigenvalues so that

$$\begin{aligned} 0 &= \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots, \\ 0 &< \mu_1 \leq \mu_2 \leq \dots. \end{aligned}$$

Furthermore, the eigenvalues satisfy the asymptotic estimates $\lambda_l = \mathcal{O}(l^2)$ and $\mu_k = \mathcal{O}(k)$ for large l and k [8].

We expand u in a Fourier series as

$$u(x, y, t) = \sum_{k,l} \hat{u}_{k,l}(t) \phi_k(x) \psi_l(y/d), \quad (22)$$

where ϕ_k and ψ_l are the above eigenfunctions. Note that this expansion (when coupled with the transform of \dot{u}) is a canonical change of variables, by which we mean that the system of equations after the change of variables is Hamiltonian, and, furthermore, that the new Hamiltonian is simply the old Hamiltonian in the new coordinates (see §2.6 of [18]). Plugging (22) into (18) gives

$$\tilde{H} = \frac{d}{2} \sum_{k,l} \left(\dot{\hat{u}}_{k,l}(t) \right)^2 + \frac{1}{2d} \sum_{k,l} \lambda_l \left(\hat{u}_{k,l}(t) \right)^2 + \frac{d}{2} \sum_{k,l,l'} \mu_k \hat{u}_{k,l}(t) \hat{u}_{k,l'}(t) \frac{\hat{b}(l,l')}{\hat{b}(0)}, \quad (23)$$

where we define

$$\hat{b}(l,l') = \int_0^\pi b(\xi) \psi_l(\xi) \psi_{l'}(\xi) d\xi.$$

We now introduce complex coordinates for our Hamiltonian defining

$$z_{kl} = \frac{1}{\sqrt{2\omega_{kl}}} \dot{\hat{u}}_{k,l} + i \sqrt{\frac{\omega_{kl}}{2}} \hat{u}_{k,l}, \quad (24)$$

with

$$\omega_{kl}^2 = \frac{\lambda_l}{d^2} + \mu_k. \quad (25)$$

In terms of these new coordinates the Hamiltonian takes the form

$$\tilde{H} = d \sum_{k,l} \omega_{kl} |z_{kl}|^2 - \frac{d}{4} \sum_{k,l,l'} \frac{\mu_k}{\sqrt{\omega_{kl}\omega_{kl'}}} \beta_{l,l'} (z_{kl} - \bar{z}_{kl}) (z_{kl'} - \bar{z}_{kl'}),$$

where we define

$$\beta_{l,l'} = \frac{\hat{b}(l,l')}{\hat{b}(0)} - \delta_{l,l'}.$$

For ease of notation we define

$$\Gamma_{k,l,l'} = \frac{-\mu_k}{4\sqrt{\omega_{kl}\omega_{kl'}}} \beta_{l,l'}, \quad (26)$$

and thus our Hamiltonian finally takes the form

$$\tilde{H} = d \sum_{k,l} \omega_{kl} |z_{kl}|^2 + d \sum_{k,l,l'} \Gamma_{k,l,l'} (z_{kl} - \bar{z}_{kl}) (z_{kl'} - \bar{z}_{kl'}). \quad (27)$$

In these coordinates, (1) is equivalent to the Hamiltonian system

$$\dot{z} = J \nabla \tilde{H}(z), \quad (28)$$

where we write

$$J = i \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (29)$$

and we are using the vector z here as shorthand for the vector whose components are all of the z_{kl} in order, followed by all of the \bar{z}_{kl} in order. Thus by ∇ we mean here that

$$\nabla = \left(\dots, \frac{\partial \{\cdot\}}{\partial z_{kl}}, \dots, \frac{\partial \{\cdot\}}{\partial \bar{z}_{kl}}, \dots \right). \quad (30)$$

Note that, formally, $\tilde{H} = \mathcal{O}(d)$. We define a new Hamiltonian H so that $\tilde{H} = d \cdot H$. Note then that

$$\dot{z} = J \nabla \tilde{H}(z) = d J \nabla H(z). \quad (31)$$

If we define $\tau = d \cdot t$, then we have $\frac{dz}{d\tau} = J \nabla H(z)$. All we have done here is to rescale time from t to τ to remove a factor of d from the differential equation.

We are trying to find a dimensionally-reduced PDE, i.e. one which depends only on x_1 and x_2 , and not on y . One way to do this would be to consider only the modes with $l = 0$ and discard all $l > 0$ modes. For example, if we consider the reduced Hamiltonian $H^r = \sum_k \omega_{k0} |z_{k0}|^2 = \sum_k \frac{1}{2} (\hat{u}_{k,0}^2 + \mu_k \hat{u}_{k,0}^2)$, then reverting to continuous variables, we have

$$H^r = \frac{1}{2} \int_{\Omega} \left(\frac{\partial}{\partial t} (u^r) \psi_0(y/d) \right)^2 + \hat{b}(0) ((\nabla_x u^r \psi_0(y/d)))^T C(x) (\nabla_x u^r \psi_0(y/d)) dV,$$

whose canonical equations are easily seen to be equivalent to

$$u_{tt}^r = \hat{b}(0) \nabla_x \cdot (C(x) \nabla_x u^r) = L_x u^r,$$

an equation which depends only on the x_1, x_2 variables, and not y .

However, it is by no means clear that the resulting equations will provide an accurate approximation to the original dynamics. To make the replacement legitimate, we will have to show that the modes with $l = 0$ (the reduced solution) approximate all of the modes (the full solution) sufficiently well. In fact, as it stands it is not true, as chopping off these modes gives an error which is too large for our purposes. The way we get around this is to generate a certain change of variables after which this truncation introduces a smaller error. In fact we will iterate this process, and show that there is a sequence of changes of variables that make this truncation error as small as we would like.

Note that the second term in the Hamiltonian in (27) is the source of all of our difficulties. For example, if the second term in (27) did not couple any $l = 0$ terms with $l > 0$ terms, we would be done. Although the evolution of the $l > 0$ terms might be complicated under this Hamiltonian, it would not affect the $l = 0$ terms at all. Since, by the smoothness of the initial data, very little of the energy is in the $l > 0$ modes at $t = 0$, it would follow that the total energy in the $l > 0$ modes would stay small for all t . In this case, simply throwing away the $l > 0$ terms would give us a dimensionally-reduced PDE. Geometrically,

what we would have shown is the submanifold generated by the $l = 0$ modes is invariant under the flow generated by this Hamiltonian.

This is, of course, too much to hope for. This could only happen if $\Gamma_{k,l,0} = 0$ for all $l > 0$, and this would be true only if $\hat{b}(l, 0) = 0$ for all $l > 0$. But

$$\hat{b}(l, 0) = \int_0^\pi b(\eta) \psi_l(\eta) \psi_0(\eta) d\eta = \pi^{-1/2} \int_0^\pi b(\eta) \psi_l(\eta) d\eta$$

is (essentially) the l th Fourier coefficient of b , expanded with respect to the ψ_l . These will all vanish only in the case of constant b .

Thus, except in the most exceptional of cases, we are resigned to coupling between the $l = 0$ and the $l > 0$ terms. Our goal here is to make the coupling as small as possible. With this goal in mind, we write our Hamiltonian as

$$H = H_{\text{diag}}^0 + H_1^0 + H_2^0, \quad (32)$$

where

$$\begin{aligned} H_{\text{diag}}^0 &= \sum_{k,l} \omega_{kl} |z_{kl}|^2, \\ H_1 &= \sum_{\substack{k,l \\ l>0}} (\Gamma_{k,l,0} + \Gamma_{k,0,l}) (z_{kl} - \bar{z}_{kl}) (z_{k0} - \bar{z}_{k0}), \\ H_2 &= \sum_{\substack{k,l,l' \\ l,l'>0}} \Gamma_{k,l,l'} (z_{kl} - \bar{z}_{kl}) (z_{kl'} - \bar{z}_{kl'}). \end{aligned}$$

Each of the above is a quadratic function defined on an appropriate Hilbert space which we describe below. We will refer to quadratic functions on this Hilbert space as one of three types: A term is of Type 0 if it couples only $l = 0$ and $l = 0$ terms, of Type 1 if it couples $l = 0$ and $l > 0$ terms, and of Type 2 if it couples $l > 0$ and $l > 0$ terms. In (32), H_1^0 contains only Type 1 terms, and H_2^0 contains only Type 2 terms. However, H_{diag}^0 contains terms of Type 0 and of Type 2. There are no Type 0 terms in H except for those already in the diagonal piece H_{diag}^0 . We note in passing that this will not be true in general after we make our changes of variables, and we will generate extra Type 0 terms which must be accounted for.

If we had a Hamiltonian with no Type 1 terms, the system would decouple, meaning that the points

$$\{z_{kl} \mid z_{kl} = 0 \text{ if } l > 0\}$$

form an invariant submanifold. If we could find a change of coordinates in which our new Hamiltonian had no Type 1 terms, then the system would effectively decouple, or, in the new coordinates, the $l = 0$ terms would form an invariant manifold. We will find that we cannot do this, but we will be able to change coordinates so that the Hamiltonian in the new coordinates has a small Type 1 piece. In fact, we will find that there are successive changes of variables which

make this term as small as we would like. In other words, we will have found coordinates in which the submanifold generated by the $l = 0$ modes is nearly invariant and thus interacts with the rest of the space very weakly.

We implement this program as follows. In Section 2 we will formally describe the canonical changes of variables which ensure that the $l = 0$ part of the Hamiltonian provides a better and better approximation to the dynamics of the full system, and in Section 3 we will make these formal calculations rigorous. This will be enough to show that truncating the $l > 0$ terms introduces errors which are small enough for our purposes. Finally, in Section 4, we will show that after these changes, the $l = 0$ terms of the Hamiltonian correspond to a PDE of the type described in Theorem 1.

2 The transformed Hamiltonian

In this section we describe formally how the Hamiltonian (27) can be transformed so as to make the coupling between the modes with $l = 0$ and $l > 0$ smaller. In Section 2.1 we compute formally what the canonical transformation that reduces this coupling should be. Then in Section 2.2 we calculate the result of iterating these canonical transformations. The calculations in this section are mostly formal and will be made rigorous in Section 3.

2.1 The formal change of variables

In this section, we describe the type of canonical transformation we will use to reduce the size of the Type 1 terms, and prove a lemma which formally describes the effects of these canonical transformations.

If we have a Hamiltonian system with Hamiltonian χ , then the time- t map of the system, ϕ^t , is a symplectomorphism. Let us denote $\phi := \phi^1$, and we can calculate

$$H \circ \phi = H + \{\chi, H\} + \frac{1}{2} \{\chi, \{\chi, H\}\} + \dots$$

where $\{A, B\} = \langle J\nabla A, \nabla B \rangle$. (See (29) and (30) for the definitions of J and ∇ .)

In (32) we wrote $H = H_{\text{diag}}^0 + H_1 + H_2$, so that if we make a change of variables and expand, we get

$$\begin{aligned} H \circ \phi &= H_{\text{diag}}^0 + H_1 + H_2 \\ &+ \{\chi, H_{\text{diag}}^0\} + \{\chi, H_1\} + \{\chi, H_2\} \\ &+ \frac{1}{2} \{\chi, \{\chi, H_{\text{diag}}^0\}\} + \frac{1}{2} \{\chi, \{\chi, H_1\}\} + \frac{1}{2} \{\chi, \{\chi, H_2\}\} + \dots \end{aligned}$$

Now choose χ so that $\{\chi, H_{\text{diag}}^0\} = -H_1$, and then

$$\{\chi, \{\chi, H_{\text{diag}}^0\}\} = -\{\chi, H_1\},$$

so that

$$\begin{aligned} H \circ \phi &= H_{\text{diag}}^0 + H_2 + \{\chi, H_2\} + \{\chi, \{\chi, H_2\}\} + \dots \\ &+ \frac{1}{2} \{\chi, H_1\} + \left(\frac{1}{2} - \frac{1}{6}\right) \{\chi, \{\chi, H_1\}\} + \dots \end{aligned} \quad (33)$$

Although this is a complicated expression, we should note that the original Type 1 term has disappeared, and that everything in this expansion which may potentially be a Type 1 term is $\mathcal{O}(\chi)$. If χ is small, then this is an improved Hamiltonian from our standpoint, since the Type 1 terms are now smaller.

We could now repeat this process. If we identify the lowest-order Type 1 terms in $H \circ \phi$, then make a similar change of variables to cancel them, this could potentially improve our Hamiltonian further, in that the remaining Type 1 terms are yet smaller. Conceivably we might be able to iterate this process and continue to choose transformations which make the Type 1 term smaller at each step, until it is eventually as small as we require. In fact, we will see below that we can do this.

Let us first prove a lemma which helps us do these calculations:

Lemma 3. *Given a Type 1 term of the form*

$$A = \sum_{\substack{k,l \\ l \neq 0}} \hat{A}_{k,l} (z_{kl} - \bar{z}_{kl})(z_{k0} - \bar{z}_{k0}),$$

we can choose χ to solve

$$\{\chi, H_{\text{diag}}^0\} = -A.$$

This χ can be written in the expansion

$$\chi = \sum_{\substack{k,l \\ l > 0}} \chi_{kl} (z_{kl} z_{k0} - \bar{z}_{kl} \bar{z}_{k0}) + \chi_{k\bar{l}} (z_{kl} \bar{z}_{k0} - \bar{z}_{kl} z_{k0}),$$

with

$$\chi_{kl} = \frac{-i\hat{A}_{k,l}}{\omega_{k0} + \omega_{kl}}, \quad \chi_{k\bar{l}} = \frac{-i\hat{A}_{k,l}}{\omega_{k0} - \omega_{kl}}. \quad (34)$$

Furthermore, if B is a term of Type 0 or Type 2, then $\{\chi, B\}$ is of Type 1, and if B is of Type 1, then $\{\chi, B\}$ is of Type 0 + 2, by which we mean that $\{\chi, B\}$ has two pieces, one of which is Type 0, the other Type 2.

Proof. To prove the first part, we simply make the Ansatz that

$$\chi = \sum_{\substack{k,l \\ l > 0}} \chi_{kl} z_{kl} z_{k0} + \chi_{k\bar{l}} \bar{z}_{kl} z_{k0} + \chi_{k\bar{l}} z_{kl} \bar{z}_{k0} + \chi_{\bar{k}l} \bar{z}_{kl} \bar{z}_{k0}, \quad (35)$$

and solve the equation

$$\{\chi, H_{\text{diag}}^0\} = -A.$$

An easy calculation then shows that we can choose $\chi_{k\bar{l}} = -\chi_{k\bar{l}}$ and $\chi_{\bar{k}l} = -\chi_{kl}$, and that

$$\chi_{kl} = \frac{-i\hat{A}_{k,l}}{\omega_{k0} + \omega_{kl}}, \quad \chi_{k\bar{l}} = \frac{-i\hat{A}_{k,l}}{\omega_{k0} - \omega_{kl}}. \quad (36)$$

We write χ as

$$\chi = \sum_{\substack{k,l \\ l>0}} \chi_{kl}(z_{kl}z_{k0} - \bar{z}_{kl}\bar{z}_{k0}) + \chi_{k\bar{l}}(z_{kl}\bar{z}_{k0} - \bar{z}_{kl}z_{k0}),$$

and consider the general quadratic function $H = H_{k,l,l'}(z_{kl} - \bar{z}_{kl})(z_{kl'} - \bar{z}_{kl'})$. Using the definition of the Poisson bracket, we have

$$\begin{aligned} \{\chi, H\} &= \sum_{\substack{k,l,l' \\ l>0}} (H_{k,0,l'} + H_{k,l',0}) S_{kl} (z_{kl} - \bar{z}_{kl})(z_{kl'} - \bar{z}_{kl'}) \\ &\quad + \sum_{k,l} \left(\sum_{l'>0} (H_{k,l,l'} + H_{k,l',l}) D_{kl'} \right) (z_{kl} - \bar{z}_{kl})(z_{k0} - \bar{z}_{k0}), \end{aligned} \quad (37)$$

where

$$S_{kl} = i(\chi_{kl} + \chi_{k\bar{l}}), \quad (38)$$

$$D_{kl} = i(\chi_{kl} - \chi_{k\bar{l}}). \quad (39)$$

If H is of Type 0, then the second sum disappears entirely, and the first sum only has terms with $l' = 0$, so that $\{\chi, H\}$ is of Type 1. If H is of Type 2, then the first sum disappears, and the second sum has terms with $l > 0$, so $\{\chi, H\}$ is again of Type 1. Finally, if H is of Type 1, then the first sum has terms with $l' > 0$, making it Type 2, and the second sum has terms with $l = 0$, and is thus of Type 0. \square

Remarks. First, if we iterate this lemma twice, we see that if H is of Type 1, then so is $\{\chi, \{\chi, H\}\}$, and if H is of Type 0+2, then so is $\{\chi, \{\chi, H\}\}$.

Second, it will be useful later to have formulas for S_{kl} and D_{kl} , and we can calculate directly that

$$S_{kl} = i(\chi_{kl} + \chi_{k\bar{l}}) = \frac{\hat{A}_{kl}}{\omega_{k0} + \omega_{kl}} + \frac{\hat{A}_{kl}}{\omega_{k0} - \omega_{kl}} = -\frac{2d^2\omega_{k0}\hat{A}_{kl}}{\lambda_l}, \quad (40)$$

$$D_{kl} = i(\chi_{kl} - \chi_{k\bar{l}}) = \frac{\hat{A}_{kl}}{\omega_{k0} + \omega_{kl}} - \frac{\hat{A}_{kl}}{\omega_{k0} - \omega_{kl}} = \frac{2d^2\omega_{kl}\hat{A}_{kl}}{\lambda_l}. \quad (41)$$

Finally, we point out a potentially confusing abuse of notation. Recall that we have defined H_1^0 as the Type 1 piece of the Hamiltonian H^0 , and that we have

$$H_1^0 = \sum_{\substack{k \\ l>0}} (\Gamma_{k,l,0} + \Gamma_{k,0,l})(z_{kl} - \bar{z}_{kl})(z_{k0} - \bar{z}_{k0}).$$

Since we do have that $\Gamma_{k,l,0} = \Gamma_{k,0,l}$, we can write

$$H_1^0 = \sum_{\substack{k \\ l>0}} 2\Gamma_{k,l,0} (z_{kl} - \bar{z}_{kl}) (z_{k0} - \bar{z}_{k0}).$$

Throughout the paper, we will solve equations of the form

$$\{\chi, H_{\text{diag}}^0\} = G_1,$$

where G is some Hamiltonian, and G_1 its Type 1 piece. We will frequently choose coefficients for symmetry, i.e. so that $G_{1,kl0} = G_{1,k0l}$. In this case, if we apply Lemma 3, the term \hat{A}_{kl} will be $2G_{1,kl0}$ and not $G_{1,kl0}$. Also, we calculate that

$$S_{kl} = -\frac{4d^2\omega_{k0}G_{kl0}}{\lambda_l}, \quad D_{kl} = \frac{4d^2\omega_{kl}G_{kl0}}{\lambda_l}.$$

We now extend the previous lemma and compute the form of the terms in $\{\chi, H\}$:

Lemma 4. *If H is symmetric in the sense that*

$$H_{k,l,l'} = H_{k,l',l} \text{ for all } l, l',$$

then we can choose coefficients so that $\{\chi, H\}$ is also symmetric. Furthermore, the coefficients of $\{\chi, H\}$ can be found in the following chart:

H	$\{\chi, H\}$
$\sum H_{k,0,0} (z_{k0} - \bar{z}_{k0}) (z_{k0} - \bar{z}_{k0})$	$\{\chi, H\}_{k10} = H_{k,0,0} S_{kl}$
$\sum H_{k,l,0} (z_{kl} - \bar{z}_{kl}) (z_{k0} - \bar{z}_{k0})$	$\{\chi, H\}_{0,k00} = 2 \sum_{l>0} H_{k,l,0} D_{kl}$ $\{\chi, H\}_{2,kl'l'} = H_{k,l',0} S_{kl} + H_{k,l,0} S_{kl'}$
$\sum H_{k,l,l'} (z_{kl} - \bar{z}_{kl}) (z_{kl'} - \bar{z}_{kl'})$	$\{\chi, H\}_{k10} = \sum_{l'>0} H_{k,l,l'} D_{kl'}$

where we define

$$S_{kl} = i (\chi_{kl} + \chi_{k\bar{l}}),$$

$$D_{kl} = i (\chi_{kl} - \chi_{k\bar{l}}).$$

Recall that if H is Type 1, then $\{\chi, H\}$ is Type 0 + 2. We denote this in the second line of the table by saying that $\{\chi, H\}$ has two pieces, one of Type 0, $\{\chi, H\}_0$, and one of Type 2, $\{\chi, H\}_2$.

The proof is a straightforward computation applying (37).

2.2 Iterating the canonical transformations

In this subsection we study (still largely on a formal level) what happens if we iterate the sort of canonical transformations constructed in Section 2.1. Let us define the Hilbert space \mathcal{H}^s as

$$\mathcal{H}^s = \left\{ \{z_{kl}\}_{k,l=0}^\infty \mid \sum_{k,l} \omega_{kl}^{2s} |z_{kl}|^2 < \infty \right\},$$

with inner product

$$\langle z, w \rangle_{\mathcal{H}^s} = \sum_{k,l} \omega_{kl}^{2s} z_{kl} \bar{w}_{kl}.$$

Remark 2. Note that $z \in \mathcal{H}^{1/2}$ if and only if $H_{\text{diag}}^0(z) = \sum_{k,l} \omega_{kl} |z_{kl}|^2 < \infty$. Thus, H_{diag}^0 is equivalent to the $\mathcal{H}^{1/2}$ -norm, and thus $\mathcal{H}^{1/2}$ is a natural space in which to work. Furthermore, we will also show below in Lemma 17 that the full Hamiltonian H is also equivalent to this norm.

We say that H is a quadratic functional on \mathcal{H}^s if $H: \mathcal{H}^s \rightarrow \mathbb{R}$, and $H(z) = \langle z, Az \rangle_{\mathcal{H}^0}$ where A is a linear map on \mathcal{H}^s . We denote the set of all \mathcal{C}^1 quadratic functionals on \mathcal{H}^s as $\mathcal{QF}(\mathcal{H}^s)$. Thus $H \in \mathcal{QF}(\mathcal{H}^s)$ if H is quadratic, and H_* , called the derivative of H , exists. For each x , $H_*(x)$ is by definition a linear map. We should note that since H is quadratic, $H_*(x)$ is also linear in the argument x .

Since $H_*(x): \mathcal{H}^s \rightarrow \mathbb{R}$ is a linear map, we can (by the Riesz Representation Theorem) identify it with a member of $(\mathcal{H}^s)^*$. We will define our dual spaces with respect to the \mathcal{H}^0 inner product, i.e.

$$\|z\|_{(\mathcal{H}^s)^*} := \sup_{\substack{w \in \mathcal{H}^s \cap \mathcal{H}^0 \\ \|w\|_{\mathcal{H}^s} = 1}} \langle z, w \rangle_{\mathcal{H}^0}. \quad (42)$$

There is a canonical identification of $(\mathcal{H}^s)^*$ and \mathcal{H}^{-s} (see Section 1.2 of [17]) so, if $H_*: \mathcal{H}^s \rightarrow \mathbb{R}$, we define ∇H to be the corresponding element of \mathcal{H}^{-s} , i.e.

$$\langle \nabla H(x), y \rangle_{\mathcal{H}^0} = H_*(x)y.$$

By the above remarks, if we have an $H \in \mathcal{QF}(\mathcal{H}^{1/2})$, then we know that $\nabla H(z)$ exists, and a priori, $\nabla H(z) \in \mathcal{H}^{-1/2}$, for $z \in \mathcal{H}^{1/2}$. Since ∇H happens to be linear in its argument (since H is quadratic), we can think of ∇H as a linear map from $\mathcal{H}^{1/2}$ to $\mathcal{H}^{-1/2}$. This makes ∇H an anti-smoothing linear map, i.e. the range of ∇H is a Sobolev space which contains functions less smooth than those in the domain. Let us consider $\mathcal{A} \subset \mathcal{QF}(\mathcal{H}^{1/2})$, where $H \in \mathcal{A}$ if and only if ∇H maps into $\mathcal{H}^{1/2}$, and $\|\nabla H\|_{\mathcal{H}^{1/2}, \mathcal{H}^{1/2}} < \infty$, and define $\|H\|_{\mathcal{A}} = \|\nabla H\|_{\mathcal{H}^{1/2}, \mathcal{H}^{1/2}}$. Roughly, our goal in the following will be to show that the terms H_1^0 and H_2^0 are in \mathcal{A} , and that our changes of variables produce

terms also in \mathcal{A} . This will prevent us from losing smoothness as the inductive argument proceeds.

If we have a Hamiltonian flow under χ with an associated flow map ϕ^t , then it is again a standard result [17] that the canonical transformation $\phi = \phi^{t=1}$ gives

$$H \circ \phi = \sum_{i=0}^{\infty} \frac{L^i(H)}{i!}, \quad (43)$$

provided the sum converges. Here $L^i(H)$ is defined by

$$L(H) = \{\chi, H\}, \quad L^{i+1}(H) = \{\chi, L^i(H)\}.$$

Furthermore, one has the ‘‘partial expansion’’

$$H \circ \phi = \sum_{i=0}^{K-1} \frac{L^i(H)}{i!} + \int_0^1 \int_0^{t_1} \dots \int_0^{t_{K-1}} L^K(H) \circ \phi^{t_K} dt_K \dots dt_1. \quad (44)$$

Let’s assume that we start with a Hamiltonian of the form $H_{\text{diag}}^0 + H_{0+2} + H_1$ as above, where

$$H_{\text{diag}}^0(z) = \sum \omega_{kl} |z_{kl}|^2,$$

and H_{0+2} combines the remaining terms of Type 0+2, and H_1 consists of terms of Type 1. The separation of Type 1 terms from the others is natural in light of Lemma 3 and our goal of eliminating the Type 1 terms. Assume further that we chose χ so that $\{\chi, H_{\text{diag}}^0\} = -H_1$. Then, following (43),

$$\begin{aligned} H \circ \phi &= \sum_{i=0}^{\infty} \frac{L^i(H_{\text{diag}}^0)}{i!} + \sum_{i=0}^{\infty} \frac{L^i(H_1)}{i!} + \sum_{i=0}^{\infty} \frac{L^i(H_{0+2})}{i!} \\ &= H_{\text{diag}}^0 + \sum_{i=0}^{\infty} \frac{L^i(H_{0+2})}{i!} + \sum_{i=1}^{\infty} \left(\frac{1}{i!} - \frac{1}{(i+1)!} \right) L^i(H_1), \end{aligned}$$

where the second line uses the fact that $L(H_{\text{diag}}^0) = -H_1$ implies $L^i(H_{\text{diag}}^0) = -L^{i-1}(H_1)$, and hence a cancellation occurs and we only need consider $L^i(H_1)$ for $i > 0$.

Denoting $\xi(i) = (i+1)!/i!$, we can write the above as

$$H \circ \phi = H_{\text{diag}}^0 + \sum_{i=0}^{\infty} \frac{L^i(H_{0+2})}{i!} + \sum_{i=1}^{\infty} \frac{L^i(H_1)}{\xi(i)}. \quad (45)$$

From Lemma 3, we know that the Poisson brackets respect a certain kind of parity, by which we mean if we apply an even number of Poisson brackets to a term, then the result has the same parity as the input, namely if we apply an even number of Poisson brackets to a term of Type 1, our result is a term of Type 1, and if we apply an even number of Poisson brackets to a term of Type 0+2, then the result is a term of Type 0+2. Moreover, applying an odd number of Poisson brackets to a term reverses this parity.

Accordingly, if we want to calculate, for example, the Type 0 + 2 term of $H \circ \phi$, it is

$$(H \circ \phi)_{0+2} = \sum_{i \text{ odd}} \frac{L^i(H_1)}{\xi(i)} + \sum_{i \text{ even}} \frac{L^i(H_{0+2})}{i!}, \quad (46)$$

and similarly, the Type 1 term of $H \circ \phi$ is

$$(H \circ \phi)_1 = \sum_{i \text{ odd}} \frac{L^i(H_{0+2})}{i!} + \sum_{\substack{i \text{ even} \\ i > 0}} \frac{L^i(H_1)}{\xi(i)}. \quad (47)$$

We point out that the second term of (46) includes a summand for $i = 0$, i.e. the term H_{0+2} itself, whereas the second term of (47) does not have a term with $i = 0$, because of the cancellation.

In our particular case, we start with our original Hamiltonian given by (27), which we denote as H^0 . Separating by type as we did above, we write $H^0 = H_{\text{diag}}^0 + H_1^0 + H_2^0$. In this case the last term happens to be purely of Type 2 (see Section 1.4).

We choose χ_0 so that $\{\chi_0, H_{\text{diag}}^0\} = -H_1^0$, and we define ϕ_0 to be the time-1 map of the flow under χ_0 . Defining $L_0(H) = \{\chi_0, H\}$, we are left with

$$H^1 = H^0 \circ \phi_0 = \sum_{i=0}^{\infty} \frac{L_0^i(H^0)}{i!} = H_{\text{diag}}^0 + \sum_{i=0}^{\infty} \frac{L_0^i(H_2^0)}{i!} + \sum_{i=1}^{\infty} \frac{L_0^i(H_1^0)}{\xi(i)}.$$

If we then choose to write

$$H^1 = H_{\text{diag}}^0 + H_1^1 + H_{0+2}^1,$$

then we know from (47) and (46) that

$$H_{0+2}^1 = \sum_{i \text{ odd}} \frac{L_0^i(H_1^0)}{\xi(i)} + \sum_{i \text{ even}} \frac{L_0^i(H_2^0)}{i!}, \quad (48)$$

$$H_1^1 = \sum_{i \text{ odd}} \frac{L_0^i(H_2^0)}{i!} + \sum_{\substack{i \text{ even} \\ i > 0}} \frac{L_0^i(H_1^0)}{\xi(i)}. \quad (49)$$

We mention again that in (48), the second sum includes a term with $i = 0$, i.e. that one of the contributions to H_{0+2}^1 is H_2^0 itself. Assume that we have shown that L_0 is an operator of small norm, so that the above power series make sense. If we think of H_1^0 and H_2^0 as terms of about the same size, then (formally) the leading order term in H_1^1 is $L_0(H_2^0)$. If we then ignore all but the leading order term, then we expect that H_1^1 can be bounded as

$$\|H_1^1\|_{\mathcal{A}} \leq \|L_0\|_{\mathcal{A}, \mathcal{A}} \|H_2^0\|_{\mathcal{A}}. \quad (50)$$

We will show below that all of these power series do in fact converge. But if everything in (50) makes sense, and $\|L_0\|_{\mathcal{A}, \mathcal{A}}$ really is small, then we have

improved our Hamiltonian. This is true because the efficacy of our dimensional reduction is limited exactly by the size of the Type 1 term, and the Type 1 term of our new Hamiltonian H^1 really is smaller than the Type 1 term of the original Hamiltonian H^0 .

We iterate this procedure and the induction will work as follows. Assume that we have done n steps of this process, so that we have the Hamiltonian

$$H^n = H^0 \circ \phi_0 \circ \cdots \circ \phi_{n-1},$$

which we write as

$$H^n = H_{\text{diag}}^0 + H_1^n + H_{0+2}^n. \quad (51)$$

We separate out the lowest-order (in d) terms of H_1^n to form $H_{1,\text{low}}^n$. Define

$$\widehat{H}_1^n = H_1^n - H_{1,\text{low}}^n,$$

and choose χ_n to solve $\{\chi_n, H_{\text{diag}}^0\} = -H_{1,\text{low}}^n$. Note by Lemma 3 this means that

$$\chi_{n,kl} = \frac{-i2H_{1,\text{low},kl}^n}{\omega_{k0} + \omega_{kl}}, \quad \chi_{n,k\bar{l}} = \frac{-i2H_{1,\text{low},kl}^n}{\omega_{k0} - \omega_{kl}}. \quad (52)$$

Doing the coordinate change as above gives us

$$\begin{aligned} H^{n+1} &= H^n \circ \phi_n = \sum_{i=0}^{\infty} \frac{L_n^i(H^n)}{i!} \\ &= H_{\text{diag}}^0 + \sum_{i=1}^{\infty} \frac{L_n^i(H_{1,\text{low}}^n)}{\xi(i)} + \sum_{i=0}^{\infty} \frac{L_n^i(\widehat{H}_1^n + H_{0+2}^n)}{i!}. \end{aligned}$$

Applying Lemma 3 once again, we have

$$H_{0+2}^{n+1} = \sum_{i \text{ odd}} \frac{L_n^i(H_{1,\text{low}}^n)}{\xi(i)} + \sum_{i \text{ odd}} \frac{L_n^i(\widehat{H}_1^n)}{i!} + \sum_{i \text{ even}} \frac{L_n^i(H_{0+2}^n)}{i!}, \quad (53)$$

$$H_1^{n+1} = \sum_{i \text{ odd}} \frac{L_n^i(H_{0+2}^n)}{i!} + \sum_{\substack{i \text{ even} \\ i > 0}} \frac{L_n^i(H_{1,\text{low}}^n)}{\xi(i)} + \sum_{i \text{ even}} \frac{L_n^i(\widehat{H}_1^n)}{i!}. \quad (54)$$

By inspection, we see that the lowest-order terms in (54) will be $L_n(H_{0+2}^n)$ and \widehat{H}_1^n . The lowest-order term of H_{0+2}^n will be H_2^0 , so we expect

$$\|L_n(H_{0+2}^n)\|_{\mathcal{A}} \leq \|L_n\|_{\mathcal{A},\mathcal{A}} \|H_2^0\|_{\mathcal{A}}.$$

Since \widehat{H}_1^n was chosen to be the terms *not* of lowest order in H_1^n , we expect \widehat{H}_1^n to be of higher order in d than H_1^n . So, if $\|L_n\|_{\mathcal{A},\mathcal{A}}$ is of higher order in d than $\|L_{n-1}\|_{\mathcal{A},\mathcal{A}}$, then we expect $\|H_1^{n+1}\|_{\mathcal{A}}$ to be of higher order than $\|H_1^n\|_{\mathcal{A}}$, and our approximation has improved.

Unfortunately, there is one difficulty with the above argument, and that is that $\|H_2^0\|_{\mathcal{A}}$ may not be finite. For example, we calculate:

$$\begin{aligned}
\|\nabla H_2^0\|_{\mathcal{H}^{1/2}}^2 &= \sum_{\substack{k \\ l>0}} \omega_{kl} |\nabla H_{2,kl}^0|^2 = \sum_{\substack{k \\ l>0}} \omega_{kl} \left| \sum_{l' \neq l} \Gamma_{k,l,l'}(z_{kl'} - \bar{z}_{kl'}) \right|^2 \\
&\leq \sum_{\substack{k \\ l>0}} \omega_{kl} \sum_{\substack{l' \neq l \\ l'>0}} \frac{\omega_{k0}^4}{\omega_{kl}\omega_{kl'}} |\beta_{l,l'}|^2 |z_{kl'} - \bar{z}_{kl'}|^2 \\
&\leq \mathcal{B}_{2,\infty} \sum_{\substack{k \\ l'>0}} \omega_{k0}^4 \omega_{kl'}^{-1} |z_{kl'} - \bar{z}_{kl'}|^2,
\end{aligned} \tag{55}$$

where we have defined $\mathcal{B}_{2,\infty} = \sup_{l'>0} \sum_{l>0} |\beta_{l,l'}|^2$, which is finite for $b(\cdot) \in C^1$, by the Plancherel theorem and the definition of $\beta_{l,l'}$.

Looking at the final line in (55), we see that since $\omega_{kl} > \omega_{k0}$ for $l > 0$,

$$\|\nabla H_2^0(z)\|_{\mathcal{H}^{1/2}} \leq \mathcal{B}_{2,\infty} \|z\|_{\mathcal{H}^{3/2}}.$$

Since we're trying to calculate $\|H_2^0\|_{\mathcal{A}}$, this would require the right-hand side to involve $\|z\|_{\mathcal{H}^{1/2}}$ instead. This suggests that the $\|H_2^0\|_{\mathcal{A}}$ is not finite, and in fact one can see that there are choices of $z \in \mathcal{H}^{1/2}$ for which (55) is not finite.

Looking more carefully at the bounds in (55), we see the obstruction to an estimate of this type is not the behavior of our sum for large l , but for large k . For example, if we choose an $\alpha < 0$, and only consider terms in the above series for $\omega_{k0} < d^\alpha$, then, using (55), we have that

$$\|H_2^0\|_{\mathcal{A}} \leq \mathcal{B}_{2,\infty} d^{2\alpha}. \tag{56}$$

This suggests that to make this scheme work, we should chop off the terms for which $k > d^\alpha$. We will show that this will not affect our scheme much, since all of our Hamiltonians above have the nice property that they do not couple terms with different k values. Furthermore, it follows from Lemma 3 that none of the changes of variables give rise to a term which couples different k , i.e. there is no quadratic term in the sum of the form $z_{kl}z_{k'l}$ with $k \neq k'$.

Lemma 5. *Consider a Hamiltonian H of the form $H = \sum_K H_K$ with*

$$H_K = \sum_{l,l'>0} \hat{H}_{K,l,l'}(z_{Kl} - \bar{z}_{Kl})(z_{Kl'} - \bar{z}_{Kl'}). \tag{57}$$

If $K \neq K'$, then

$$\{z_{Kl}, H_{K'}\} = 0 \text{ for all } l. \tag{58}$$

In short, for $K \neq K'$, the evolution of z_{Kl} does not depend at all on $H_{K'}$. So if we replace our Hamiltonian H with

$$H_\alpha = \sum_{K < d^\alpha} H_K,$$

then we will not affect the evolution of any z_{kl} for $k < d^\alpha$.

Proof. The proof is an easy calculation using $\frac{\partial H_K}{\partial z_{K'l}} = 0$ for $K \neq K'$. Since

$$\frac{\partial z_{kl}}{\partial z_{k'l}} = \delta_{k,k'},$$

(58) follows. Furthermore, writing $H = H_\alpha + (H - H_\alpha)$, we see that for $k < d^\alpha$,

$$\{z_{kl}, H\} = \{z_{kl}, H_\alpha\}.$$

Thus the evolution of z_{kl} is the same whether we use H or H_α as our Hamiltonian. \square

We define \mathcal{H}_α^s as

$$\mathcal{H}_\alpha^s = \left\{ \{z_{kl}\}_{k,l=0}^\infty \mid \sum_{\substack{k < d^\alpha \\ l > 0}} \omega_{kl}^{2s} |z_{kl}|^2 < \infty \right\}.$$

It is a consequence of Lemma 5 that if we choose $z \in \mathcal{H}_\alpha^s$, i.e. a z with all modes zero for $k > d^\alpha$, and define a flow of the form

$$\dot{z} = J\nabla H(z),$$

where H satisfies the assumptions of Lemma 5, then the modes of z for $k > d^\alpha$ will stay zero for all time, and \mathcal{H}_α^s is invariant. Thus it makes sense to measure not $\|H\|_{\mathcal{A}} = \|\nabla H\|_{\mathcal{H}^{1/2}, \mathcal{H}^{1/2}}$, but $\|H\|_{\mathcal{A}_\alpha} = \|\nabla H\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}}$.

If we can determine that we have a small Type 1 term, in the sense that $\|H_1^p\|_{\mathcal{A}_\alpha}$ is small, and we choose initial conditions in $\mathcal{H}_\alpha^{1/2}$, then the formal arguments earlier in this section show that the flow nearly decouples, and the $l = 0$ modes are a good approximation for the whole system. Also, if $H \in \mathcal{A}_\alpha \subset \mathcal{QF}(\mathcal{H}_\alpha^{1/2})$, we can again think of ∇H as a linear map from $\mathcal{H}_\alpha^{1/2}$ to itself. Finally, the functions H_1^0 and H_2^0 are bounded functions on $\mathcal{H}_\alpha^{1/2}$.

Lemma 6. *There exist constants C_1 and C_2 , independent of d , such that*

$$\|H_1^0\|_{\mathcal{A}_\alpha} \leq C_1 d^{1+\alpha}, \quad \|H_2^0\|_{\mathcal{A}_\alpha} \leq C_2 d^{1+\alpha}. \quad (59)$$

Proof. If we repeat the estimates of (55) we arrive this time at

$$\|\nabla H_2^0\|_{\mathcal{A}_\alpha}^2 \leq \mathcal{B}_{2,\infty} \sum_{\substack{k < d^\alpha \\ l' > 0}} \omega_{k0}^4 \omega_{kl'}^{-2} \omega_{kl'} |(z_{kl'} - \bar{z}_{kl'})|^2, \quad (60)$$

But $\omega_{k0}^4 = \mu_k^2 \leq Ck^2 < Cd^{2\alpha}$ using the asymptotics of the eigenvalues μ_k and the fact that $k < d^\alpha$. Similarly, $\omega_{kl'}^{-2} \leq \frac{Cd^2}{(l')^2}$, and we see that

$$\|\nabla H_2^0\|_{\mathcal{A}_\alpha}^2 \leq \mathcal{B}_{2,\infty} d^{2+2\alpha} \sum_{\substack{k < d^\alpha \\ l' > 0}} l'^{-2} \omega_{kl'} |(z_{kl'} - \bar{z}_{kl'})|^2 \leq 4\mathcal{B}_{2,\infty} d^{2+2\alpha} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2. \quad (61)$$

The calculation of $\|\nabla H_1^0\|_{\mathcal{A}_\alpha}^2$ is very similar and we omit the details. \square

Remark 3. *By very similar estimates one can show that*

$$|H_1^0(z)| \leq C_1 d^{1+\alpha} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2, \quad |H_2^0(z)| \leq C_2 d^{1+\alpha} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2. \quad (62)$$

Once again, the estimates are so similar to those above that we omit the details.

3 Estimates on the transformed Hamiltonian

In this section we prove a number of estimates which justify the formal arguments of the previous section concerning the size of terms in the transformed Hamiltonian. In particular, in Section 3.1, we estimate the action of arbitrary compositions of the Poisson brackets of our Hamiltonian with the functions χ_n defined in (52). We then use these estimates to bound the size of the interaction terms in the Hamiltonian after making n canonical transformations. In Section 3.2, we collect a number of technical estimates which we will need in Section 4 to prove our approximation theorems.

3.1 Bounds on the interaction terms in the Hamiltonian

The main goal of this section is to calculate the size of $\|H_1^n\|_{\mathcal{A}_\alpha}$, in particular its order in powers of d . We define

$$M_n = \{I \in \mathbb{Z}^n \mid I_1 \geq I_2 \geq \cdots \geq I_n \geq 0\}, \quad \text{with } M_0 = \{\emptyset\},$$

and

$$M = \bigcup_{n \geq 0} M_n, \quad \text{and } M^+ = \bigcup_{n > 0} M_n.$$

In this context, we will refer to I as a multiindex. We also define

$$\mathcal{M}_n = \{I \in M \mid I_1 \leq n\}.$$

Note that if $I \in \mathcal{M}_n$, then $I_k \leq n$ for all k . For any $I \in M_n$ for $n \geq 1$, we define

$$|I| = n, \quad \text{and } p(I) = \sum_{k=1}^n 2(I_k + 1).$$

We further define $|\emptyset| = p(\emptyset) = 0$.

Note that $I \in \mathcal{M}_n$ is a sequence of some number of n 's, followed by some number of $n-1$'s, etc. all the way to some number of 0's. We will denote these numbers by a sequence $J(I) = \{J(I)_m\}_{m=n, \dots, 0}$, where $J(I)_m$ gives the number of times m appears in I . Clearly we can also reconstruct I from $J(I)$. Given a function H and $I \in M_n$, we define

$$L_I(H) = L_{I_1} \circ \cdots \circ L_{I_n}(H),$$

and we define L_\emptyset to be the identity.

We will sometimes write multiindices out in multiplicative notation, with multiplication meaning concatenation. In particular, $n^\alpha m^\beta$ will refer to the multiindex which is α copies of the number n followed by β copies of the number m , and we will write

$$L_{n^\alpha m^\beta} = L_n^\alpha \circ L_m^\beta.$$

For example, this means that $L_{n^\alpha} = L_n^\alpha$, in either case being the map L_n iterated α times. We note that $I = n^\alpha m^\beta$ could also be specified as the multiindex with $J(I)_n = \alpha$, $J(I)_m = \beta$, and $J(I)_k = 0$ if $k \neq n, m$.

For a given quadratic function G , we want to know the type of $L_I(G)$. First, choose $G = H_1^0$. Then $L_I(H_1^0)$ is of Type 1 if $|I|$ is even, and of Type 0 + 2 if $|I|$ is odd. On the other hand, if we choose $G = H_2^0$, then the reverse is true: $L_I(H_2^0)$ is of Type 0 + 2 if $|I|$ is even, and of Type 1 if $|I|$ is odd.

For the rest of this paper, G will always be either H_1^0 or H_2^0 . We define $L_I(G)_0$, $L_I(G)_1$, and $L_I(G)_2$ as follows. First, let $|I|$ be even. Then, if we want $L_I(G)$ to be a term of Type 0 + 2, we need to choose $G = H_2^0$. We then define $L_I(G)_0$ and $L_I(G)_2$ to be the Type 0 and Type 2 pieces of $L_I(H_2^0)$. If we want $L_I(G)$ to be Type 1, we have to choose $G = H_1^0$, and so we define $L_I(G)_1$ to be $L_I(H_1^0)$. Now, if $|I|$ is odd, we have to make the opposite choices at each step.

Thus, to define $L_I(G)_q$ for $q = 0, 1$, or 2 , we choose G to be either H_1^0 or H_2^0 (whichever one will give us the correct type), compute $L_I(G)$, and take the part of Type q . Whenever we write $L_I(G)_q$ below, we know that G is determined by I and q .

Lemma 7. *Every term in $H_{0+2}^n + H_1^n$ is of the form $L_I(H_1^0)$ or $L_I(H_2^0)$, for some $I \in \mathcal{M}_{n-1}$. Moreover, if we write*

$$\begin{aligned} H_1^n &= \sum_{I \in \mathcal{M}_{n-1}} \Theta_1(I) L_I(G)_1, \\ H_{0+2}^n &= \sum_{I \in \mathcal{M}_{n-1}} \Theta_{0+2}(I) L_I(G)_{0+2}, \end{aligned}$$

then $\Theta_1(I)$ and $\Theta_{0+2}(I)$ have the property that

$$\max(|\Theta_1(I)|, |\Theta_{0+2}(I)|) \leq \frac{3^{p(I)}}{\prod_{j=0}^{n-1} J(I)_j!}. \quad (63)$$

We emphasize again that G on the right hand side of these formulas is chosen to be either H_1^0 or H_2^0 , whichever gives a term of the correct type.

Remark 4. *In [13] an explicit formula for the coefficients $\Theta_j(I)$ will be derived, but to prove the results of this paper, we need only know that they satisfy (63).*

Proof. We prove the lemma inductively. Equations (49) and (48) show that it holds for $n = 1$. We assume that it holds up to some $n \geq 1$ and prove it holds for $n + 1$. We provide the details for H_1^{n+1} , as the proof is the same for

H_2^{n+1} . Recalling (54) we have

$$H_1^{n+1} = \sum_{i \text{ odd}} \frac{L_n^i(H_{0+2}^n)}{i!} + \sum_{\substack{i \text{ even} \\ i > 0}} \frac{L_n^i(H_{1,\text{low}}^n)}{\xi(i)} + \sum_{i \text{ even}} \frac{L_n^i(\widehat{H}_1^n)}{i!}.$$

Since $H_{1,\text{low}}^n$ is the sum of the lowest order terms in d of H_1^n , there is a subset $\mathcal{I} \subsetneq \mathcal{M}_{n-1}$ so that

$$H_{1,\text{low}}^n = \sum_{I \in \mathcal{I}} \Theta_1(I) L_I(G)_1, \quad \text{and} \quad \widehat{H}_1^n = \sum_{I \in \mathcal{M}_{n-1} \setminus \mathcal{I}} \Theta_1(I) L_I(G)_1.$$

Putting this together, we can write

$$\begin{aligned} H_1^{n+1} &= \sum_{i \text{ odd}} \frac{L_n^i \left(\sum_{I \in \mathcal{M}_{n-1}} \Theta_{0+2}(I) L_I(G)_{0+2} \right)}{i!} \\ &\quad + \sum_{\substack{i \text{ even} \\ i > 0}} \frac{L_n^i \left(\sum_{I \in \mathcal{I}} \Theta_1(I) L_I(G)_1 \right)}{\xi(i)} \\ &\quad + \sum_{i \text{ even}} \frac{L_n^i \left(\sum_{I \in \mathcal{M}_{n-1} \setminus \mathcal{I}} \Theta_1(I) L_I(G)_1 \right)}{i!}. \end{aligned}$$

Let us consider the first sum:

$$\sum_{\substack{i \text{ odd} \\ I \in \mathcal{M}_{n-1}}} \frac{\Theta_{0+2}(I)}{i!} L_n^i(L_I(G)_{0+2}).$$

Since i is odd, $L_n^i(L_I(G)_{0+2})$ is of Type 1, so we can write it as $L_{\tilde{I}}(G)_1$, where $\tilde{I} = n^i I$. Clearly, $\tilde{I} \in \mathcal{M}_n$. We define $\Theta_1(\tilde{I}) = \Theta_{0+2}(I)/i!$. If $|\Theta_{0+2}(I)| < 3^{p(I)}/(\prod_j J(I)_j!)$, then $|\Theta_1(\tilde{I})| < 3^{p(\tilde{I})}/(\prod_j J(I)_j!)$ also. In passing, note that if we consider $\tilde{I} = n^0 I$, for some $I \in \mathcal{M}_{n-1}$, i.e. if $\tilde{I} = I$, then $\Theta_1(\tilde{I}) = \Theta_{0+2}(I)$, so that the functions Θ_j do not depend on n , but only on I .

We will get similar results if we consider the other two sums. Since we are applying some power of L_n , the multiindices that will arise are all in \mathcal{M}_n . \square

The next two lemmas give rigorous estimates for the size of the expressions that appear in H^n . Their proofs are long and somewhat technical and are deferred until Appendix A.

Lemma 8. *There exists a constant C_I and functions $Q_I(l), f_I(l), g_I(l)$ such that for all $I \in M^+$, we can write*

$$L_I(G)_{0,k00} = C_I d^{p(I)} \omega_{k0}^{p(I)+1}, \quad (64)$$

$$L_I(G)_{1,kl0} = Q_I(l) d^{p(I)} \omega_{k0}^{p(I)+1} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}}, \quad (65)$$

$$L_I(G)_{2,kl_1 l_2} = d^{p(I)} \omega_{k0}^{p(I)+2} \left(\frac{f_I(l_1) g_I(l_2)}{\omega_{kl_1}^{1/2} \omega_{kl_2}^{1/2}} + \frac{f_I(l_2) g_I(l_1)}{\omega_{kl_1}^{1/2} \omega_{kl_2}^{1/2}} \right), \quad (66)$$

where Q_I , f_I , and g_I are all in ℓ^2 (with respect to l). Furthermore, for every n ,

$$H_{1,\text{low}}^n = \sum_{\substack{I \in \mathcal{M}_{n-1} \\ p(I)=2n}} \Theta_1(I) L_I(G)_1.$$

Lemma 9. *We have the following estimates for $I \in M^+$ and d sufficiently small:*

$$\begin{aligned} \|L_I(G)_0\|_{\mathcal{A}_\alpha} &\leq 8C_I d^{p(I)(1+\alpha)+\alpha}, \\ \|L_I(G)_1\|_{\mathcal{A}_\alpha} &\leq 4 \|Q_I\|_{\ell^2} d^{p(I)(1+\alpha)+2\alpha}, \\ \|L_I(G)_2\|_{\mathcal{A}_\alpha} &\leq 4 \|f_I\|_{\ell^2} \|g_I\|_{\ell^2} d^{p(I)(1+\alpha)+2\alpha}. \end{aligned}$$

These lemmas justify the formal arguments above. Recall that we chose $H_{1,\text{low}}^n$, at each step, to be the terms of Type 1 of lowest order in d . These lemmas specifically tell us that, when we chose the term with the formally lowest power of d , we were choosing a term which actually was smallest in d in the sense of the operator norm on \mathcal{A}_α .

Combining Lemmas 7, 8 and 9 gives us

Lemma 10. *For all n and for sufficiently small d , there exists $C_n > 0$ such that*

$$\|H_{1,\text{low}}^n\|_{\mathcal{A}_\alpha} \leq C_n d^{2n(1+\alpha)+2\alpha}.$$

Proof. We know from Lemma 8 that $H_{1,\text{low}}^n = \sum_{\substack{I \in \mathcal{M}_{n-1} \\ p(I)=2n}} \Theta_1(I) L_I(G)_1$.

Using Lemma 9, we have that

$$\begin{aligned} \|H_{1,\text{low}}^n\|_{\mathcal{A}_\alpha} &\leq 4d^{(2n)(1+\alpha)+2\alpha} \sum_{\substack{I \in \mathcal{M}_{n-1} \\ p(I)=2n}} |\Theta_1(I)| \|Q_I\|_{\ell^2} \\ &\leq 4C_{q,n} d^{(2n)(1+\alpha)+2\alpha} \sum_{\substack{I \in \mathcal{M}_{n-1} \\ p(I)=2n}} \frac{3^{p(I)}}{\prod J(I)_j!}, \end{aligned} \quad (67)$$

where the last inequality used Lemma 7 to bound the factor of Θ_1 and set $C_{q,n} = \max_{\substack{I \in \mathcal{M}_{n-1} \\ p(I)=2n}} \|Q_I\|_{\ell^2}$. The sum over I can be bounded by $3^{p(I)} \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!}\right)^{(2n)} \leq (3e)^{(2n)}$ and the estimate of the Lemma follows. \square

3.2 Some technical details

In this subsection, we collect some estimates related to those of the previous subsection that we will need to prove Theorem 3 in Section 4. In particular, we calculate the size of $\|\chi_n\|_{\mathcal{A}_\alpha}$ for any n and we show that our original Hamiltonian H^0 and any subsequent Hamiltonian H^n define equivalent norms on $\mathcal{H}_\alpha^{1/2}$. Finally, we prove some lemmas about how a Hamiltonian flow affects the size of various norms.

Lemma 11. *If $k \leq d^\alpha$, then for d sufficiently small,*

$$|\omega_{kl} - \omega_{k0}| \geq \frac{|\omega_{kl}|}{2}, \text{ for } l > 0.$$

The proof is an immediate consequence of the asymptotics of the eigenvalues of (20) and (21).

Lemma 12. *For any n and for d sufficiently small,*

$$\|\chi_n\|_{\mathcal{A}_\alpha} \leq Cd^{2n(1+\alpha)+1+2\alpha}.$$

Proof. By (52), we know that

$$\chi_{n,kl} = \frac{-i2H_{1,\text{low},kl}^n}{\omega_{k0} + \omega_{kl}}, \quad \chi_{n,k\bar{l}} = \frac{i2H_{1,\text{low},kl}^n}{\omega_{k0} - \omega_{kl}}.$$

Using Lemma 11, we see that we have the two estimates

$$|\chi_{n,kl}| \leq CdH_{1,\text{low},kl}^n, \text{ and } |\chi_{n,k\bar{l}}| \leq CdH_{1,\text{low},kl}^n.$$

From this it is clear that $\|\chi_n\|_{\mathcal{A}_\alpha} \leq Cd\|H_{1,\text{low}}^n\|_{\mathcal{A}_\alpha}$, and this combined with Lemma 10 completes the proof. \square

Remark 5. *Recall that $L_n(G) = \{\chi_n, G\}$. Thus, the operator norm of L_n , considered as a linear operator on \mathcal{A}_α , is bounded by $\|\chi_n\|_{\mathcal{A}_\alpha}$.*

Corollary 13. *Under the hypotheses of Lemma 12, there exists $C_n > 0$ such that the canonical transformation ϕ_n defined as the time-one map of the Hamiltonian system*

$$\dot{z} = \{\chi_n, z\}, \tag{68}$$

is a bounded linear map on $\mathcal{H}_\alpha^{1/2}$ satisfying

$$\begin{aligned} \|\phi_n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} &\leq \exp(C_n d^{2n(1+\alpha)+1+2\alpha}), \\ \|\phi_n - \mathbf{1}\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} &\leq C_n d^{2n(1+\alpha)+1+2\alpha}. \end{aligned}$$

Proof. These estimates follow immediately by applying Gronwall's inequality to the (linear) system of differential equations (68), and using the result of Lemma 12. \square

Remark 6. *We can also consider ϕ_n as a bounded, linear, canonical transformation on $\mathcal{H}^{1/2}$ by extending it as the identity on $\mathcal{H}^{1/2} \setminus \mathcal{H}_\alpha^{1/2}$.*

By combining Lemma 12 with (44) we obtain a representation of the Hamiltonian after n canonical transformations as a *finite* sum of terms of the form $L_I(H^0)$, plus a remainder that can be made arbitrarily small. More precisely, we have:

Proposition 14. Fix $N > 0$ and $n \geq 1$. There exist positive constants d_0 and $C_{n,N}$, a finite subset $\mathcal{I}_{n,N} \subset \mathcal{M}_{n-1}$, and functions $\tilde{\Theta}(I)$ satisfying

$$|\tilde{\Theta}(I)| \leq 1 / \left(\prod_j J(I)_j! \right), \quad (69)$$

such that if $0 < d < d_0$,

$$H^n(z) = H^0 \circ \phi^0 \circ \dots \circ \phi^n(z) = \sum_{I \in \mathcal{I}_{n,N}} \tilde{\Theta}(I) L_I(H^0) + \mathcal{R}_{n,N}(z), \quad (70)$$

where

$$\|\mathcal{R}_{n,N}\|_{\mathcal{A}_\alpha} \leq C_{n,N} d^{(1+2\alpha)N}. \quad (71)$$

Proof. The proof is inductive. For $n = 1$, (44) implies that

$$H^1(z) = H^0 \circ \phi^0(z) = \sum_{j=0}^{N-1} \frac{1}{j!} L_0^j(H^0) + \int_0^1 \dots \int_0^{t_{N-1}} L_0^N(H^0) \circ (\phi_0)^{t_N} dt_N \dots dt_1. \quad (72)$$

From Lemma 12 and Corollary 13, there exists $C > 0$ such that

$$\|L_0^N(H^0) \circ (\phi_0)^t\|_{\mathcal{A}_\alpha} \leq C d^{(1+2\alpha)(N-1)} \|L_0(H^0)\|_{\mathcal{A}_\alpha}. \quad (73)$$

But $L_0(H^0) = \{\chi_0, H_{\text{diag}}^0\} + \{\chi_0, H_1^0\} + \{\chi_0, H_2^0\} = -H_1^0 + \{\chi_0, H_1^0\} + \{\chi_0, H_2^0\}$, and from Lemma 6 we know that $\|H_1^0\|_{\mathcal{A}_\alpha} \leq C d^{1+\alpha}$ while χ_0 is bounded with the aid of Lemma 12. Thus, the integral term in (72) can be bounded by

$$C d^{(1+2\alpha)N} \int_0^1 \dots \int_0^{t_{N-1}} dt_N \dots dt_1 = \frac{C}{(N-1)!} d^{(1+2\alpha)N},$$

and the lemma follows for $n = 1$ if we take $\mathcal{I}_{1,N} = \{\emptyset, 0^1, 0^2, \dots, 0^{N-1}\}$, and $\tilde{\Theta}(I = 0^j) = 1/j!$.

We now assume that the lemma holds for all $n = 1, \dots, M-1$ and prove that it holds for $n = M$. We have

$$\begin{aligned} H^M(z) &= H^{M-1} \circ \phi_{M-1}(z) = \sum_{j=0}^{N-1} \frac{1}{j!} L_{M-1}^j(H^{M-1}) \\ &\quad + \int_0^1 \dots \int_0^{t_{N-1}} L_{M-1}^N(H^{M-1}) \circ (\phi_{M-1})^{t_N} dt_N \dots dt_1 \end{aligned} \quad (74)$$

by (44). By the induction hypothesis we have

$$L_{M-1}^j(H^{M-1}) = \sum_{I \in \mathcal{I}_{M-1,N}} \frac{1}{j!} \tilde{\Theta}(I) L_{M-1}^j L_I(H^0) + L_{M-1}^j(\mathcal{R}_{M-1,N}). \quad (75)$$

From Lemma 12 and the inductive estimate on $\mathcal{R}_{M-1,N}$, we have

$$\|L_{M-1}^j(\mathcal{R}_{M-1,N})\|_{\mathcal{A}_\alpha} \leq C d^{j[2(M-1)(1+\alpha)+1+2\alpha]} d^{(1+2\alpha)N}. \quad (76)$$

On the other hand $L_{M-1}^j L_I(H^0) = L_{\tilde{I}}(H^0)$, where $\tilde{I} = M - 1^j I \in \mathcal{M}_{M-1}$, so we can define $\mathcal{I}_{M,N}$ to be those $\tilde{I} \in \mathcal{M}_{M-1}$ for which $\tilde{I} = (M - 1)^j I$ for some $I \in \mathcal{I}_{M-1,N}$, and for some $j = 0, 1, \dots, N - 1$. In particular, this is a finite set. Defining $\tilde{\Theta}(\tilde{I}) = \frac{1}{j!} \tilde{\Theta}(I)$, we see that $\tilde{\Theta}(\tilde{I})$ satisfies (69).

By estimates very similar to those above, we can bound the integral term on the right hand side of (74) by

$$\begin{aligned} & \int_0^1 \dots \int_0^{t_{N-1}} C d^{[2(M-1)(1+\alpha)+1+2\alpha](N-1)} dt_N \dots dt_1 \\ & \leq \frac{C}{(N-1)!} d^{[2(M-1)(1+\alpha)+1+2\alpha](N-1)}. \end{aligned}$$

Thus, if we define

$$\begin{aligned} \mathcal{R}_{M,N} &= \sum_{j=0}^{N-1} L_{M-1}^j (\mathcal{R}_{M-1,N}) + \\ & \int_0^1 \dots \int_0^{t_{N-1}} L_{M-1}^N (H^{M-1}) \circ (\phi_{M-1})^{t_N} dt_N \dots dt_1, \end{aligned}$$

we see that there exists a constant $C_{M,N}$ such that $\|\mathcal{R}_{M,N}\|_{\mathcal{A}_\alpha} \leq C_{M,N} d^{(1+2\alpha)N}$, and the lemma follows. \square

Remark 7. *If we compare the representation of $H^n(z)$ derived in the proof of Proposition 14 with that derived in Subsection 2.2 and Lemma 7, we see that in Proposition 14 we have just restricted the expansion to term $L_I(H^0)$ with $I \in \mathcal{M}_{n-1}$, and $J(I)_l \leq N$ for all l . Thus, there exists a finite subset $\tilde{\mathcal{I}}_{n,N} \subset \mathcal{M}_{n-1}$ such that*

$$\sum_{I \in \tilde{\mathcal{I}}_{n,N}} \tilde{\Theta}(I) L_I(H^0) = H_{\text{diag}}^0 + \sum_{\tilde{I} \in \tilde{\mathcal{I}}_{n,N}} \Theta_1(I) L_I(G)_1 + \sum_{\tilde{I} \in \tilde{\mathcal{I}}_{n,N}} \Theta_{0+2}(I) L_I(G)_{0+2} \quad (77)$$

(Note: The H_{diag}^0 term comes from the term on the left hand side with $I = \emptyset$.)

We now derive from this representation of the transformed Hamiltonian several corollaries concerning the size of the transformed Hamiltonian.

Lemma 15. *For all $n \geq 1$, there exists $d_0 > 0$ and $C_n > 0$ such that for $0 < d < d_0$,*

$$\|H_1^n\|_{\mathcal{A}_\alpha} = \|\nabla H_1^n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \leq C_n d^{2n(1+\alpha)+2\alpha}, \quad (78)$$

$$\|H_0^n\|_{\mathcal{A}_\alpha} = \|\nabla H_0^n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \leq C_n d^{2+3\alpha}, \quad (79)$$

$$\|H_2^n\|_{\mathcal{A}_\alpha} = \|\nabla H_2^n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \leq C_n d^{1+\alpha}. \quad (80)$$

Proof. Choosing N such that $(1+2\alpha)N > 2n(1+\alpha) + 2\alpha$, we can apply Proposition 14 to write

$$H^n(z) = \sum_{I \in \tilde{\mathcal{I}}_{n,N}} \tilde{\Theta}(I) L_I(H^0) + R_{n,N}(z), \quad (81)$$

where $\|\mathcal{R}_{n,N}\|_{\mathcal{A}_\alpha} \leq C_n d^{2n(1+\alpha)+2\alpha}$. By Remark 7, there exists a finite subset $\tilde{\mathcal{I}}_{n,N} \subset \mathcal{M}_{n-1}$ such that

$$\sum_{I \in \mathcal{I}_{n,N}} \tilde{\Theta}(I) L_I(H^0) = H_{\text{diag}}^0 + \sum_{\tilde{I} \in \tilde{\mathcal{I}}_{n,N}} \Theta_1(\tilde{I}) L_I(G)_1 + \sum_{\tilde{I} \in \tilde{\mathcal{I}}_{n,N}} \Theta_{0+2}(\tilde{I}) L_I(G)_{0+2}. \quad (82)$$

Thus

$$H_1^n(z) = \sum_{\tilde{I} \in \tilde{\mathcal{I}}_{n,N}} \Theta_1(I) L_I(G)_1 + \mathcal{R}_{n,N}^{(1)}(z), \quad (83)$$

where $\mathcal{R}_{n,N}^{(1)}$ is the Type 1 part of $\mathcal{R}_{n,N}$ and hence satisfies the estimate (71). Since $\tilde{\mathcal{I}}_{n,N}$ is finite, there exists some integer P_{\max} such that $p(\tilde{I}) \leq P_{\max}$ for all $\tilde{I} \in \tilde{\mathcal{I}}_{n,N}$. On the other hand, the canonical transformations $\phi^0, \dots, \phi^{n-1}$ were constructed so that all Type 1 terms in the Hamiltonian with $p(\tilde{I}) < 2n$ were canceled. Hence we can write

$$H_1^n(z) = \sum_{\substack{p=2n \\ p \text{ even}}}^{P_{\max}} \sum_{\substack{\tilde{I} \in \tilde{\mathcal{I}}_{n,N} \\ p(\tilde{I})=p}} \Theta_1(I) L_I(G)_1 + \mathcal{R}_{n,N}^{(1)}(z). \quad (84)$$

Just as in the proof of Lemma 10, we can bound

$$\left\| \sum_{\substack{\tilde{I} \in \tilde{\mathcal{I}}_{n,N} \\ p(\tilde{I})=p}} \Theta_1(I) L_I(G)_1 \right\|_{\mathcal{A}_\alpha} \leq C_p d^{p(1+\alpha)+2\alpha} \quad (85)$$

since there are only finitely many terms in the sum. The sum over p is also finite and hence easily bounded and $\|\mathcal{R}_{n,N}^{(1)}(z)\|_{\mathcal{A}_\alpha}$ is bounded by (71), yielding the estimate of (78).

To estimate $\|H_0^n(z)\|_{\mathcal{A}_\alpha}$ we proceed as above but since the construction of the canonical transformations does not eliminate the low order (in d) terms in H_0^n , we arrive at an expression for the Type 0 part of the Hamiltonian of the form

$$H_0^n(z) = \sum_{\substack{p=2 \\ p \text{ even}}}^{P_{\max}} \sum_{\substack{\tilde{I} \in \tilde{\mathcal{I}}_{n,N} \\ p(\tilde{I})=p}} \Theta_{0+2}(I) L_I(G)_0 + \mathcal{R}_{n,N}^{(0)}(z), \quad (86)$$

where $\mathcal{R}_{n,N}^{(0)}$ is the Type 0 part of $\mathcal{R}_{n,N}$. With the aid of Lemma 9 we can again bound

$$\left\| \sum_{\substack{\tilde{I} \in \tilde{\mathcal{I}}_{n,N} \\ p(\tilde{I})=p}} \Theta_1(I) L_I(G)_0 \right\|_{\mathcal{A}_\alpha} \leq C_p d^{p(1+\alpha)+\alpha}. \quad (87)$$

This is a geometric series in p , and combined with the estimate of (71), this implies that (79) holds.

The estimate (80) follows in exactly the same fashion except that since the original Hamiltonian contains a term H_2^0 , but no H_0^0 term, that additional term

must be added to the right hand side of the analogue of (86), and it is estimated by Lemma 6. \square

Remark 8. *If we proceed as in the previous lemma, using the expressions in Lemma 8 rather than the bounds in Lemma 9, we can bound the values of the Hamiltonian on $\mathcal{H}_\alpha^{1/2}$ by*

$$|H_1^n(z)| \leq Cd^{(2n)(1+\alpha)} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2, \quad (88)$$

$$|H_0^n(z)| \leq Cd^{2+\alpha} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2, \quad (89)$$

$$|H_2^n(z)| \leq Cd^{1+\alpha} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2, \quad (90)$$

just as we did in the proof of Lemma 15.

We define

$$\mathcal{O}_R(\mathcal{H}_\alpha^s) = \left\{ z \in \mathcal{H}_\alpha^s \mid \|z\|_{\mathcal{H}_\alpha^s} \leq R \right\},$$

and have:

Lemma 16. *Let $B \subseteq \mathcal{H}_\alpha^{1/2}$ be bounded, i.e. there is an R with $B \subset \mathcal{O}_R(\mathcal{H}_\alpha^{1/2})$. Then there is a constant C such that for sufficiently small d ,*

$$|H^n(z) - H^0(z)| \leq Cd^{1+\alpha}, \quad \text{for all } z \in B.$$

Proof. This follows immediately from (88)–(90) and Remark 3. \square

Lemma 17. *If we define H_0^n as the Type 0 piece of H^n , then $\langle z, z \rangle_{H_{\text{diag}}^0 + H_0^n} = H_{\text{diag}}^0(z) + H_0^n(z)$, defines an inner product on $\mathcal{H}^{1/2}$ with the property that for any bounded set $B \in \mathcal{H}_\alpha^{1/2}$ there is a C with*

$$\frac{1}{C} \|z\|_{\mathcal{H}_\alpha^{1/2}} \leq \|z\|_{H_{\text{diag}}^0 + H_0^n} \leq C \|z\|_{\mathcal{H}_\alpha^{1/2}},$$

for all $z \in B$.

Proof. A positive-definite symmetric linear operator $A: \mathcal{H}^{1/2} \rightarrow \mathcal{H}^{-1/2}$ defines an inner product via

$$\langle x, y \rangle_A = \langle Ax, y \rangle = \left\langle A^{1/2}x, A^{1/2}y \right\rangle, \quad (91)$$

where $A^{1/2}$ is the positive-definite square root of A . Since the gradient of $H = H_{\text{diag}}^0 + H_0^n$ is positive-definite and symmetric, we can define the inner product

$$H(z) = \langle \nabla H(z), z \rangle.$$

Thus, to complete the proof of the lemma, we need only show that

$$H_{\text{diag}}^0(z) + H_0^n(z) \sim H_{\text{diag}}^0(z),$$

for all $z \in B \subset \mathcal{H}_\alpha^{1/2}$, by which we mean that there exists a constant C with

$$\frac{1}{C} (H_{\text{diag}}^0(z) + H_0^n(z)) \leq H_{\text{diag}}^0(z) \leq C (H_{\text{diag}}^0(z) + H_0^n(z)),$$

for all $z \in B$.

Now, using Lemma 16, it is easy to see that for $z \in B$,

$$H^n(z) \sim H^0(z).$$

Inequalities (88) and (90) imply that there is a choice of d_0 such that for $d < d_0$, we have

$$|H_1^n(z) + H_2^n(z)| \leq \frac{1}{2} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2 = \frac{1}{2} H_{\text{diag}}^0(z) \text{ if } z \in \mathcal{H}_\alpha^{1/2}. \quad (92)$$

(The larger n is, the easier it is to satisfy (92), and this is why the estimate can be made uniformly in n .) Using this for $n = 0$ gives us that since $H_{\text{diag}}^0 = H^0 - H_1^0 - H_2^0$, then

$$\frac{1}{2} |H^0(z)| \leq |H_{\text{diag}}^0(z)| \leq \frac{3}{2} |H^0(z)|,$$

or that $H^0 \sim H_{\text{diag}}^0$. The same argument gives us that $H^n \sim H_{\text{diag}}^0 + H_0^n$, and we are done. \square

The utility of the norms defined in the previous lemma comes from the fact that the Hamiltonian flow with Hamiltonian H preserves the value of H and hence the norm defined by H . More precisely we have:

Corollary 18. *Assume that we have a Hamiltonian H which generates an inner product, $\langle \cdot, \cdot \rangle_H$, as in the previous lemma. Then if we define a flow by*

$$\dot{z} = J\nabla H(z),$$

then this flow preserves the norm $\|\cdot\|_H$, i.e. $\frac{d}{dt} \|z\|_H = 0$, where we define $\|z\|_H^2 = \langle z, z \rangle_H$.

As a consequence of this estimate, and the fact that the norm defined by H_{diag}^0 is the $\mathcal{H}_\alpha^{1/2}$ norm, we have

Corollary 19. *We define, on $\mathcal{H}_\alpha^{1/2}$, the flow $\dot{z} = J\nabla H(z)$, with $H = H_{\text{diag}}^0 + H_{>}$, where $|H_{>}(z)| \leq \frac{1}{2} |H_{\text{diag}}^0(z)|$, for all $z \in \mathcal{H}_\alpha^{1/2}$. Then*

$$\|z(t)\|_{\mathcal{H}_\alpha^{1/2}}^2 \leq 3 \|z(0)\|_{\mathcal{H}_\alpha^{1/2}}^2, \quad \text{for all } t.$$

Proof. Note that

$$\frac{1}{2} |H_{\text{diag}}^0(z(t))| \leq |H(z(t))| \leq \frac{3}{2} |H_{\text{diag}}^0(z(0))|.$$

Since $H(z(t)) = H(z(0))$ for all t , and $H_{\text{diag}}^0(z) = \|z\|_{\mathcal{H}_\alpha^{1/2}}^2$, the estimate follows. \square

Finally, we estimate the way in which modes with a fixed value of k evolve.

Lemma 20. *Let $z \in \mathcal{H}^s$, and define $P^K(z)$ as*

$$(P^K(z))_{kl} = \delta_{k,K} z_{kl}.$$

We define $\|z\|_{\mathcal{H}^{1/2,K}} := \|P^K(z)\|_{\mathcal{H}^{1/2}}$. Then if we have the flow $\dot{z} = J\nabla H^0(z)$, there is a C (independent of K), such that

$$\|z(t)\|_{\mathcal{H}^{1/2,K}} \leq C \|z(0)\|_{\mathcal{H}^{1/2,K}}.$$

Proof. The proof is an easy exercise, using Lemma 5 which shows that the flow defined by H does not couple modes with different values of k , and the type of energy estimates used in Corollary 19. \square

4 The Approximation Theorems

In Section 4.1, we prove that the flow of the original Hamiltonian, H^0 , can be approximated in $\mathcal{H}_\alpha^{1/2}$ for a very long time by the flow given by $H_{\text{diag}}^0 + H_0^n$. In Section 4.2 we then revert to the continuous variables and show how this discrete approximation theorem implies the existence and approximation properties of the reduced equations defined in Theorem 1.

4.1 The approximation theorem in spectral space

In this section, we want to state and prove the theorem in the discrete variables which will imply Theorem 1 in the continuous variables. (We will establish this implication in Section 4.2.)

To state our theorem, we need to define one more Hilbert space. Recall that our standard Hilbert space $\mathcal{H}^{1/2}$ corresponds (roughly) to functions which are once differentiable in x and y . We want to consider functions which may have different degrees of differentiability in the x and y directions. So we define

$$\|z\|_{\mathcal{H}^{s,t}} = \sum_k \omega_{k0}^{2s} |z_{k0}|^2 + \sum_{\substack{k \\ l>0}} \omega_{kl}^{2t} |z_{kl}|^2,$$

which corresponds (if $s > t$) to functions with s derivatives in the long (x) direction, and t derivatives in the thin (y) direction.

Theorem 3. *Given H^0 as defined in (27), we make the coordinate changes defined in the iterative scheme of Section 2.1. Define*

$$N(n) = 2n + 1 + (2n + 2)\alpha. \tag{93}$$

Choose $z(0) \in \mathcal{H}^{s,r}$, with $s > 1, r > 1/2$, and define the flows

$$\begin{aligned} \dot{z} &= J\nabla H^0(z), \\ \dot{\gamma} &= J\nabla(H_{\text{diag}}^0 + H_0^n)(\gamma), \end{aligned} \tag{94}$$

with $\gamma(0)$ is chosen as

$$\gamma_{kl}(0) = \begin{cases} z_{kl}(0), & k < d^\alpha, l = 0, \\ 0, & \text{else.} \end{cases}$$

Then, for any $\epsilon > 0$, there exists $C_n = C_n(a, b, C, \|z(0)\|_{\mathcal{H}^{s,t}})$ and $d_0 = d_0(a, b, C, s)$ such that for $d < d_0$, we have the estimate

$$\|z(t) - \gamma(t)\|_{\mathcal{H}^{1/2}}^2 \leq C_n d^{N(n)} t + 3\epsilon,$$

for $t \leq d^{-N(n)}$.

Proof. Choose $\epsilon > 0$. We define $z^{\mathbf{r}}$ and $z^>$ by

$$z_{kl}^{\mathbf{r}} = \delta_{l,0} z_{kl}, \quad z_{kl}^> = z - z^{\mathbf{r}}.$$

We want to make both $\|z(t)\|_{\mathcal{H}^{1/2, k \geq d^\alpha}} < \epsilon$ and $\|z^>(t)\|_{\mathcal{H}^{1/2}} < \epsilon$. We know from Lemma 19 that if $\|z(0)\|_{\mathcal{H}^{s,t}} = R$, then $z(t) \in \mathcal{O}_{3R}(\mathcal{H}_\alpha^{1/2})$ for all t . We assume implicitly in all that follows that z is in this bounded set.

According to Lemma 20, we know that there is a κ independent of K with

$$\|z(t)\|_{\mathcal{H}^{1/2, K}} \leq \kappa \|z(0)\|_{\mathcal{H}^{1/2, K}}. \quad (95)$$

and if we define

$$\|z\|_{\mathcal{H}^{1/2, K \geq d^\alpha}} = \sum_{K \geq d^\alpha} \|z\|_{\mathcal{H}^{1/2, K}},$$

then

$$\|z(t)\|_{\mathcal{H}^{1/2, K \geq d^\alpha}} \leq \kappa \|z(0)\|_{\mathcal{H}^{1/2, K \geq d^\alpha}}.$$

So we simply need to make sure that

$$\|z(0)\|_{\mathcal{H}^{1/2, K \geq d^\alpha}} < \frac{\epsilon}{\kappa}.$$

We chose $z(0) \in \mathcal{H}^{s,r}$, with $s > 1$, $r > 1/2$. From this

$$\sum_k \omega_{k0}^{2s} |z_{k0}(0)|^2 \leq \|z(0)\|_{\mathcal{H}^{s,r}}^2.$$

The asymptotics of the eigenvalues in (21) imply that $\omega_{k0}^{2s} = \mu_k^s = M(b, C)k^s$ where $M(b, C)$ is a constant which depends only on $b(\eta), C(x)$. Thus

$$|z_{k0}(0)| \leq \frac{\|z(0)\|_{\mathcal{H}^{s,r}}}{M |k|^s},$$

and

$$\begin{aligned} \|z(0)\|_{\mathcal{H}^{1/2, k > d^\alpha}}^2 &= \sum_{k > d^\alpha} \omega_{k0} |z_{k0}(0)|^2 \leq \|z(0)\|_{\mathcal{H}^{s,r}}^2 \sup_{k > d^\alpha} \omega_{k0}^{1-2s} \\ &\leq M(b, C) \|z(0)\|_{\mathcal{H}^{s,r}}^2 d^{\frac{s}{2}(1-2s)}. \end{aligned}$$

In summary,

$$\|z(0)\|_{\mathcal{H}^{1/2}, k > d^\alpha}^2 \leq \kappa d^{\frac{\alpha}{2}(1-2s)},$$

where κ is a constant which depends only upon $b(\eta)$, $C(x)$, and $\|z(0)\|_{s,r}$. This and (95) give

$$\|z(t)\|_{\mathcal{H}^{1/2}, k > d^\alpha}^2 = \tilde{\kappa} d^{\frac{\alpha}{2}(1-2s)}, \quad (96)$$

where $\tilde{\kappa}$ is a constant which depends only upon $b(\eta)$, $C(x)$, and $\|z(0)\|_{s,r}$.

So for $s > 1$, there is a d_0 such that for all $d < d_0$, we can make (96) smaller than ϵ . This d_0 depends on the coefficient functions $a(\eta)$, $b(\eta)$, and $C(x)$ defined in (1), and also on s , since for larger s , the right hand side of (96) will shrink more quickly as d shrinks. Note further that if we choose our initial condition smoother, the thickness of our domain can be larger.

Now let us consider $z^>(t)$. We have chosen $z(0) \in \mathcal{H}^{s,r}$ for some $s > 1, r > 1/2$. Now, according to (25), we have

$$\omega_{kl} > \frac{\kappa(a)}{d} \text{ for } l > 0,$$

where $\kappa(a)$ depends only on the function $a(\eta)$. But

$$\begin{aligned} \|z^>(0)\|_{\mathcal{H}^{s,r}}^2 &= \sum_{l > 0} \omega_{kl}^{2r} |z_{kl}(0)|^2 = \sum_{l > 0} \omega_{kl} \omega_{kl}^{2(r-1/2)} |z_{kl}(0)|^2 \\ &> \frac{\kappa}{d^{2(r-1/2)}} \|z^>(0)\|_{\mathcal{H}^{1/2}}^2, \end{aligned}$$

so that $\|z^>(0)\|_{\mathcal{H}^{1/2}}^2 \leq \kappa d^{2r-1} \|z^>(0)\|_{\mathcal{H}^{s,r}}^2$. Since $r > 1/2$, we can choose d_0 such that for $d < d_0$, we have

$$\|z^>(0)\|_{\mathcal{H}^{1/2}} < \epsilon \quad (97)$$

for any $\epsilon > 0$. Again, note that if we choose r larger (and thus our initial condition more smooth), then the thickness of our domain can be larger.

Now we show that we can approximate the solutions of $\dot{z} = J\nabla H^0(z)$ by those of $\dot{\gamma} = J\nabla H_0^n(\gamma)$. Recall that ϕ_0, \dots, ϕ_n are defined so that $H^n = H^0 \circ \phi_0 \circ \dots \circ \phi_n$. Denote $\phi = \phi_0 \circ \dots \circ \phi_n$. If we define $\zeta = \phi^{-1}(z)$, then we have $\dot{\zeta} = J\nabla(H^0 \circ \phi)(\zeta) = J\nabla H^n(\zeta)$, with $\zeta(0) = \phi^{-1}(0)$. The $z(t)$ and $\zeta(t)$ equations express the same dynamics in different coordinates. However, now note that the transformation ϕ is a near identity transformation. More precisely, for any $R > 0$, Corollary 13 implies that there exists $C = C(R)$ such that on the ball of radius R in $\mathcal{H}_\alpha^{1/2}$,

$$\|\phi - \mathbf{1}\|_{\mathcal{H}_\alpha^{1/2}} \leq Cd^{1+2\alpha}. \quad (98)$$

Thus, we see that we so long as $z(t)$ remains within a ball of radius R ,

$$\|z(t) - \zeta(t)\|_{\mathcal{H}_\alpha^{1/2}} \leq Cd^{1+2\alpha}R,$$

and hence if $Cd^{1+2\alpha}R \ll \epsilon$, we can approximate either the z -dynamics or the ζ -dynamics, whichever is more convenient, to within the error we allow. It is easier to approximate the ζ dynamics because in the Hamiltonian H^n the coupling between the $l = 0$ and $l > 0$ modes is very weak.

Recall that $H_{\text{diag}}^0(z) = \sum_{k,l} \omega_{kl} |z_{kl}|^2$, and we now write

$$H_{\text{diag}}^{0\mathbf{r}}(z) = \sum_k \omega_{k0} |z_{k0}|^2, \quad H_{\text{diag}}^{0>} = \sum_{\substack{k \\ l>0}} \omega_{kl} |z_{kl}|^2.$$

We remind the reader that throughout the remainder of this proof we will always be working in the Hilbert space $\mathcal{H}_\alpha^{1/2}$.

We need to verify that if we define

$$\dot{\zeta} = J\nabla H^n(\zeta), \quad (99)$$

$$\dot{\gamma} = J\nabla(H_{\text{diag}}^0 + H_0^n)(\gamma), \quad (100)$$

with $\gamma(0)$ the projection of $z^{\mathbf{r}}(0)$ onto $\mathcal{H}_\alpha^{1/2}$, then γ tracks ζ sufficiently well. Since $\gamma(0)$ lies in the submanifold generated by the $\{z_{k0}\}$, and this submanifold is preserved by the Type 0 Hamiltonian, we can write the system of (99) and (100) as

$$\begin{aligned} \dot{\zeta} &= J\nabla H^n(\zeta), \\ \dot{\gamma} &= J\nabla(H_{\text{diag}}^{0\mathbf{r}} + H_0^n)(\gamma). \end{aligned}$$

If we write $\zeta = z^{\mathbf{r}} + z^>$, then the equation for ζ becomes

$$\dot{z}^{\mathbf{r}} = J\nabla H_{\text{diag}}^{0\mathbf{r}}(z^{\mathbf{r}}) + J\nabla H_0^n(z^{\mathbf{r}}) + J\nabla H_1^n(z^>), \quad (101)$$

$$\dot{z}^> = J\nabla H_{\text{diag}}^{0>}(z^>) + J\nabla H_1^n(z^{\mathbf{r}}) + J\nabla H_2^n(z^>). \quad (102)$$

We need to bound (101) and (102). First, we concentrate on (101). We define $\xi = z^{\mathbf{r}} - \gamma$, and this gives

$$\begin{aligned} \dot{\xi} &= J\nabla H_{\text{diag}}^{0\mathbf{r}}(\xi) + J\nabla H_0^n(\xi) + J\nabla H_1^n(z^>), \\ &= J(\nabla(H_{\text{diag}}^{0\mathbf{r}} + H_0^n))(\xi) + J\nabla H_1^n(z^>). \end{aligned}$$

where $\|\xi(0)\|_{\mathcal{H}_\alpha^{1/2}} = \mathcal{O}(d^{1+2\alpha})$ (from the fact that $\|\zeta(0) - z(0)\|_{\mathcal{H}_\alpha^{1/2}} \leq Cd^{1+2\alpha}$). We want to control $\|\xi(t)\|_{\mathcal{H}_\alpha^{1/2}}$. In Lemma 17, we proved that $\|\cdot\|_{\mathcal{H}_\alpha^{1/2}} \sim \|\cdot\|_{H_{\text{diag}}^0 + H_0^n}$ for $z \in \mathcal{O}_{3R}(\mathcal{H}_\alpha^{1/2})$. So it suffices to bound $\|\xi(t)\|_{H_{\text{diag}}^0 + H_0^n}$ instead.

Then we calculate

$$\begin{aligned} \frac{d}{d\tau} \|\xi(\tau)\|_{H_{\text{diag}}^0 + H_0^n}^2 &= \langle \xi, J\nabla H_1^n(z^>) \rangle_{H_{\text{diag}}^0 + H_0^n} + \langle J\nabla H_1^n(z^>), \xi \rangle_{H_{\text{diag}}^0 + H_0^n} \\ &\leq 2 \|\xi(\tau)\|_{H_{\text{diag}}^0 + H_0^n} \|J\nabla H_1^n(z^>(\tau))\|_{H_{\text{diag}}^0 + H_0^n} \\ &\leq \kappa \|\xi(\tau)\|_{\mathcal{H}_\alpha^{1/2}} \|J\nabla H_1^n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \|z^>(\tau)\|_{\mathcal{H}_\alpha^{1/2}} \\ &\leq \kappa_n d^{N(n)-1} (1 + \|\xi\|_{\mathcal{H}_\alpha^{1/2}}^2). \end{aligned} \quad (103)$$

where the last inequality used the fact since $\|z(t)\|_{\mathcal{H}_\alpha^{1/2}} < 3R$ for all t , so is $\|z^>(\tau)\|_{\mathcal{H}_\alpha^{1/2}}$ and the result of Lemma 15 that $\|J\nabla H_1^n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \leq \kappa_n d^{N(n)-1}$.

Recall from (31) that $\tau = d \cdot t$, so that

$$\frac{d}{dt} \|\xi(\tau)\|_{H_{\text{diag}}^0 + H_0^n}^2 = d \cdot \frac{d}{d\tau} \|\xi(\tau)\|_{H_{\text{diag}}^0 + H_0^n}^2 \leq \kappa_n d^{N(n)} (1 + \|\xi\|_{\mathcal{H}^{1/2}}^2). \quad (104)$$

Since $\|\xi(0)\|_{\mathcal{H}^{1/2}}^2 < C' d^{2+4\alpha}$ from (98), another application of Gronwall's Inequality gives

$$\|\xi(t)\|_{\mathcal{H}_\alpha^{1/2}}^2 \leq e^{\kappa_n C} \left(C' d^{2+4\alpha} + C_n d^{N(n)} t \right), \quad (105)$$

so long as $t < C d^{-N(n)}$. For any $\epsilon > 0$, we can certainly choose d sufficiently small so that $C' d^{2+4\alpha} < \epsilon$, and thus

$$\|\xi(t)\|_{\mathcal{H}_\alpha^{1/2}}^2 \leq C'_n d^{N(n)} t + \epsilon. \quad (106)$$

Now we consider (102), the $z^>$ equation. An analysis similar to that in (103) gives us that

$$\frac{d}{dt} \|z^>\|_{\mathcal{H}_\alpha^{1/2}}^2 \leq \kappa \|J\nabla H_1^n\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \|z^>\|_{\mathcal{H}_\alpha^{1/2}}.$$

A Gronwall argument combined with the estimate in (97) gives us that, for some constant C''_n ,

$$\|z^>\|_{\mathcal{H}_\alpha^{1/2}} \leq C''_n d^{N(n)} t + \epsilon. \quad (107)$$

Combining this estimate with Equation (96) to control the modes with large k gives us that

$$\|\zeta - z^{\mathbf{r}}\|_{\mathcal{H}^{1/2}} \leq \epsilon.$$

This combined with (106) and (107) gives us

$$\|\zeta(t) - \gamma(t)\|_{\mathcal{H}^{1/2}} \leq (C'_n + C''_n) d^{N(n)} t + 3\epsilon.$$

□

4.2 Proof of Theorem 1

In this section we show that Theorem 3 implies Theorem 1. We begin by noting the map embodied by transformations (22) and (24) (the map from continuous u variables to discrete z variables) is a continuous linear map with continuous inverse. We know that z and γ stay close in Theorem 3. This together with Lemma 21 will show that the continuous analogs of z and γ also stay close.

Lemma 21. *We define the map Ξ , written as $\Xi_2 \circ \Xi_1$, where*

$$\begin{aligned} \Xi_1(u(x, y, t), u_t(x, y, t)) &= \{\hat{u}_{k,l}, \dot{\hat{u}}_{k,l}\}, \\ \Xi_2(\{\hat{u}_{k,l}, \dot{\hat{u}}_{k,l}\}) &= \{z_{kl}, \bar{z}_{kl}\}, \end{aligned}$$

with

$$u(x, y, t) = \sum_{kl} \hat{u}_{k,l}(t) \phi_k(x) \psi_l(y/d),$$

$$z_{kl} = \frac{1}{\sqrt{2\omega_{kl}}} \dot{\hat{u}}_{k,l} + i \sqrt{\frac{\omega_{kl}}{2}} \hat{u}_{k,l}.$$

Then given s a half-integer, the map $\Xi: H^{s+1/2}(\Omega) \times H^{s-1/2}(\Omega) \rightarrow \mathcal{H}^s$ is a bounded invertible linear transformation with bounded inverse. Also, Ξ is a canonical transformation with multiplier i .

Furthermore, if we restrict the domain of Ξ to $H_\alpha^{s+1/2} \times H_\alpha^{s-1/2}$, then it remains a canonical transformation from this set onto its image.

Proof. This is a straightforward calculation using the asymptotics of the eigenvalues μ_k and λ_l , analogous to Section 1.2 in Kuksin [17]. \square

In Theorem 3, we assumed that the initial condition $z(0) \in \mathcal{H}^{s,r}$ for $s > 1$, $r > 1/2$. One possible choice would be to choose $s = r > 1$. We should note that in this case, the norm $\|z\|_{\mathcal{H}^{s,s}} = \|z\|_{\mathcal{H}^s}$. Then, the previous lemma tells us that choosing an initial condition $z(0) \in \mathcal{H}^s$ with $s > 1$ corresponds to choosing an initial pair $(u^0, u_t^0) \in H^{s+1/2}(\Omega) \times H^{s-1/2}(\Omega)$. For example, choosing $(u^0, u_t^0) \in H^2(\Omega) \times H^1(\Omega)$ will do.

We want to say that the equation

$$\dot{\gamma} = J\nabla H_0^n(\gamma)$$

corresponds to an equation of the form $u_{tt} = D_n u$, so let us make this notion clear. Let $(u(0), u_t(0)) \in H^2(\Omega) \times H^1(\Omega)$ be the initial conditions for the equation $u_{tt} = D_n u$, where D_n is some operator for which we have existence and uniqueness. Then for any time $\tau > 0$, the pair $(u(\tau), u_t(\tau))$ is uniquely determined. On the other hand, we can consider $z(0) = \Xi(u(0), u_t(0))$, and assume that it satisfies $\dot{z} = J\nabla H_0^n(z)$, thus specifying $z(\tau)$ for any $\tau > 0$. Then if $z(\tau) = \Xi(u(\tau), u_t(\tau))$, we say that the PDE $u_{tt} = D_n u$ corresponds to the equation $\dot{\gamma} = J\nabla H_0^n(\gamma)$.

To finish our proof of Theorem 1, we need to calculate explicitly what the H_0^n term is after n changes of variables, which will allow us to calculate explicitly its equivalent PDE.

From the proof of Theorem 3 we know that it suffices to consider the evolution in $\mathcal{H}_\alpha^{1/2}$, (since the modes with $k > d^\alpha$ are initially very small, and never grow beyond the size of our allowed error) and thus in the remaining discussion we restrict our attention to this space. Note that by combining (86) and Lemma 8 we see that

$$\begin{aligned} H_0^n(z) &= \sum_{\substack{p=2 \\ p \text{ even}}}^{P_{max}} \sum_{\substack{\tilde{I} \in \tilde{\mathcal{I}}_{n,N} \\ p(\tilde{I})=p}} \Theta_1(I) C_I d^p \sum_{k < d^\alpha} \omega_{k0}^{p+1} (z_{k0} - \bar{z}_{k0})^2 + \mathcal{R}_{n,N}^{(0)}(z) \\ &= -\frac{1}{2} \sum_{q=2}^{Q_{max}} \tilde{C}^{(q)} d^{2(q-1)} \sum_{k < d^\alpha} \omega_{k0}^{2q-1} (z_{k0} - \bar{z}_{k0})^2 + \mathcal{R}_{n,N}^{(0)}(z). \end{aligned} \quad (108)$$

where in the last line we set $p = 2(q-1)$, $\tilde{C}^{(\ell)} = \sum_{\substack{\tilde{I} \in \tilde{\mathcal{I}}_{n,N} \\ p(\tilde{I})=\ell}} (-2C_I)\Theta_{0+2}(I)$. Note that if N is finite, so are P_{max} and Q_{max} .

Remark 9. *From the formula given above it appears that the coefficients $C^{(\ell)}$ could depend on n or N . In [13], a closed form expression for these coefficients is derived from which it follows that $C^{(\ell)}$ is independent of both n and N , but this will not affect the conclusions of this paper.*

We are not completely done. In order to derive our reduced PDE we will replace H_0^n by an approximation with finitely many terms and we must show that this can be done without affecting the dynamics up to our order of approximation. For this we use the following lemma:

Lemma 22. *Fix $R > 0$. There exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the following is true: Consider two quadratic Hamiltonians $H = H_{\text{diag}}^{\text{Or}} + h_1$ and $\tilde{H} = H_{\text{diag}}^{\text{Or}} + h_2$, both of which define norms equivalent to the norm defined by $H_{\text{diag}}^{\text{Or}}$, and assume that $\|h_1 - h_2\|_{\mathcal{A}_\alpha} < \epsilon$. If we consider the two flows*

$$\dot{u} = J\nabla(H_{\text{diag}}^{\text{Or}} + h_1), \text{ and } \dot{v} = J\nabla(H_{\text{diag}}^{\text{Or}} + h_2),$$

with $u(0) = v(0)$ and $\|u(0)\|_{\mathcal{H}_\alpha^{1/2}} < R$, then there exists $C(R) > 0$ (independent of $u(0)$), so long as $\|u(0)\|_{\mathcal{H}_\alpha^{1/2}} < R$ such that

$$\|u(t) - v(t)\|_{\mathcal{H}_\alpha^{1/2}} \leq C\epsilon dt .$$

Proof. Noting that $u(t)$ and $v(t)$ are solutions of linear systems of Hamiltonian, ordinary differential equations, and that the difference of the corresponding vector fields has Lipschitz constant bounded by ϵ , the estimate then follows from a straightforward application of Gronwall's inequality using the fact that the Hamiltonians are conserved along their respective orbits. The extra factor of d on the right hand side of this inequality comes from the rescaling of time as in inequality (104). \square

We continue for the moment to assume that R is fixed and that we consider sequences $\{z_{k0}\}$ with $\|z\|_{\mathcal{H}_\alpha^{1/2}} < R$. In Section 4.1, we have already specified an order in d to which our reduced equation is accurate, so we will simply cut off the expression for H_0^n at exactly the same order in d . As stated in Theorem 3, after we have done n changes of variables, our approximation is accurate to $\mathcal{O}(d^{N(n)})$ where $N(n) = 2n + 1 + (2n + 2)\alpha$. Thus, we first choose N such that $(1 + 2\alpha)N > N(n)$, which guarantees that

$$\|R_{n,N}^{(0)}\|_{\mathcal{A}_\alpha} \leq Cd^{(1+2\alpha)N} < Cd^{N(n)}.$$

If we define

$$h_q = \sum_{k < d^\alpha} \omega_{k0}^{2q-1} (z_{k0} - \bar{z}_{k0})^2,$$

an easy estimate like that leading to (59) implies

$$\|h_q\|_{\mathcal{A}_\alpha} \leq 4d^{(2q-1)\alpha}.$$

Defining $\bar{C} = \max_{q=2, \dots, Q_{max}} |\tilde{C}^{(q)}|$, we have

$$\left\| \sum_{q=Q+1}^{Q_{max}} \tilde{C}^{(q)} d^{2(q-1)} \sum_{k < d^\alpha} \omega_{k0}^{2q-1} (z_{k0} - \bar{z}_{k0})^2 \right\|_{\mathcal{A}_\alpha} \leq 4Q_{max} \bar{C} d^{2Q(1+\alpha)+\alpha}, \quad (109)$$

for d sufficiently small. Thus, if we choose $2Q(1+\alpha) + \alpha > N(n) - 1$, the expression on the left hand side of (109) is less than or equal to $Cd^{N(n)-1}$, and so if we define

$$\widetilde{H}_0^n = \sum_{q=2}^Q \tilde{C}^q d^{2(q-1)} \sum_{k < d^\alpha} \omega_{k0}^{2q-1} (z_{k0} - \bar{z}_{k0})^2, \quad (110)$$

we have

$$\|H_0^n - \widetilde{H}_0^n\|_{\mathcal{A}_\alpha} < Cd^{N(n)-1}. \quad (111)$$

Note also that the inequality $2Q(1+\alpha) + \alpha > N(n) - 1$ can be satisfied by choosing $Q = n$.

As an immediate corollary of (111) and Lemma 22 we have

Lemma 23. *If we consider the two flows*

$$\begin{aligned} \dot{u} &= J\nabla(H_{\text{diag}}^{0\mathbf{r}} + H_0^n)(u), \\ \dot{v} &= J\nabla(H_{\text{diag}}^{0\mathbf{r}} + \widetilde{H}_0^n)(v), \end{aligned}$$

with $u(0) = v(0) \in \mathcal{H}_\alpha^{1/2}$, then

$$\|u(t) - v(t)\|_{\mathcal{H}_\alpha^{1/2}}^2 \leq Cd^{N(n)}t,$$

where we choose Q as in (110).

Finally, we examine the form of \widetilde{H}_0^n in continuous coordinates. Since by (24), $(z_{k0} - \bar{z}_{k0}) = i\sqrt{2\omega_{k0}}\hat{u}_{k0}$, we have

$$\widetilde{H}_0^n = \sum_{q=2}^Q \tilde{C}^{(q)} d^{2(q-1)} \sum_{k < d^\alpha} \omega_{k0}^{2q} \hat{u}_{k0}^2 = \sum_{q=2}^Q \tilde{C}^{(q)} d^{2(q-1)} \sum_{k < d^\alpha} \mu_k^q \hat{u}_{k0}^2 \quad (112)$$

With this form of the Hamiltonian, we are finally in a position to prove:

Lemma 24. *Consider the form of \widetilde{H}_0^n given in (112). The differential equation (on $\mathcal{H}_\alpha^{1/2}$) given by*

$$\dot{z} = J\nabla \left(H_{\text{diag}}^{0\mathbf{r}} + \widetilde{H}_0^n \right) (z),$$

if converted into continuous coordinates by Ξ^{-1} as defined in Lemma 21, gives a PDE of the form

$$u_{tt}^{\mathbf{r}} = L_x u^{\mathbf{r}} + \sum_{q=2}^Q C^q d^{2(q-1)} L_x^q u^{\mathbf{r}} \text{ on } \omega, \quad (113)$$

with the boundary conditions

$$L_x^i u|_{\partial\omega} = 0 \text{ for all } i < Q, \quad (114)$$

where $C^q = (-1)^{q-1} 2\tilde{C}^q$.

Proof. Reexpressing $H_{\text{diag}}^{0\text{r}}$ in continuous coordinates as we did \widetilde{H}_n^0 in (112) we see that $H_{\text{diag}}^{0\text{r}} = \frac{1}{2} \sum_k (|\hat{u}_{k0}|^2 + \mu_k |\hat{u}_{k0}|^2)$.

Now consider the subspace of $H^{2Q}(\omega)$ defined by

$$W^Q(\omega) = \{u \in H^{2Q}(\omega) \mid L_x^j u|_{\partial\omega} = 0, j = 0, 1, \dots, Q-1\}.$$

(Note that if $z \in \mathcal{H}_\alpha^{1/2}$, the corresponding u reconstructed by undoing the various changes of variables will be in $H^s(\omega)$ for any s .) Then an easy inductive argument using the definition of \mathcal{L} shows that $\int_\omega (\mathcal{L}^j u)^2 dx = (-1)^Q \int_\omega u L_x^j u dx$, for $j = 0, \dots, Q$.

From this observation it easily follows that if $u \in W^Q$ and $q \leq Q$, and we write $u = \sum \hat{u}_{k0} \phi_k(x)$ we have $\sum_k \mu^q \hat{u}_{k0}^2 = \int (u L_x^q u) dx = (-1)^q \int (\mathcal{L}^q u)^2 dx$. So

$$H_{\text{diag}}^{0\text{r}} + \widetilde{H}_n^0 = \int \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} (\mathcal{L}u)^2 + \sum_{q=2}^Q (-1)^q \tilde{C}^{(q)} d^{2(q-1)} (\mathcal{L}^q u)^2 \right\} dx. \quad (115)$$

Hamilton's equations for this Hamiltonian are readily seen to be the partial differential equation (113), with boundary conditions (114), and we are done. \square

Proof of Theorem 1. Assume that we have $(u^0(x, y), u_t^0(x, y)) \in H^2(\Omega) \times H^1(\Omega)$, and compute $(\Pi_2 \Pi_1 u^0, \Pi_2 \Pi_1 u_t^0) \in H_\alpha^2(\omega) \times H_\alpha^1(\omega)$ (see (6) and (7) to recall the definitions of $\Pi_{1,2}$).

Define $z(0) = \Xi(u^0, u_t^0)$ and $\gamma(0) = \Xi(\Pi_2 \Pi_1 u^0, \Pi_2 \Pi_1 u_t^0)$, and let $z(t), \gamma(t)$ flow under (94). Then $z, \gamma \in \mathcal{H}^s$ with $s > 1$, and Theorem 3 tells us that for any $\delta > 0$,

$$\|z(t) - \gamma(t)\|_{\mathcal{H}^{1/2}}^2 \leq C d^{N(n)} t + 3\delta.$$

From Lemma 21 and the linearity and continuity of Ξ^{-1} we have that

$$\|\Xi^{-1}(z(t)) - \Xi^{-1}(\gamma(t))\|_{H^1(\Omega) \times L^2(\Omega)} \leq C(d^{N(n)} t + 3\delta). \quad (116)$$

By Lemma 24, $\Xi^{-1}(\gamma(t))$ is the flow of (113). Therefore, given an ϵ , we can find a C such that for all $t \leq C d^{-N(n)}$, the right hand side of (116) is less than ϵ . Since

$$\|u(x, y, t) - u_n^{\text{r}}(x, t) \mathbf{1}(y)\|_{H^1(\Omega) \times L^2(\Omega)} = \|\Xi^{-1}(z(t)) - \Xi^{-1}(\gamma(t))\|_{H^1(\Omega) \times L^2(\Omega)}$$

where u_n^{r} is the solution of (113), this proves the theorem. \square

A Proofs of Lemmas 1, 8 and 9

Proof of Lemma 1. The proof is a straightforward approximation argument. Define $\mathcal{U}(x, y) = (u^0(x, y) - v^{\mathbf{r}}(x)\mathbf{1}(y))$. Choose $\mathcal{U}^\infty \in C^\infty(\Omega) \cap H^2(\Omega)$ such that $\|\mathcal{U}^\infty - \mathcal{U}\|_{H^2(\Omega)} < \beta/4$. Furthermore since $\mathcal{U}|_\Gamma = 0$, we can assume the same of \mathcal{U}^∞ . Since \mathcal{U}^∞ is smooth, we have

$$\mathcal{U}^\infty(x, y) = \mathcal{U}^\infty(x^*, y^*) + \int_{(x^*, y^*)}^{(x, y)} D\mathcal{U}^\infty(\xi, \eta) \cdot d\ell ,$$

for any points (x, y) and (x^*, y^*) in Ω . Now replace $D\mathcal{U}^\infty$ by $\Delta \in C^\infty(\Omega)$ where $\Delta(x, y) = D\mathcal{U}^\infty(x, y)$ if $x \in \omega$, $y \in (\nu d, (1 - \nu)\pi d)$, while $\Delta(x, y) = 0$ if $y \in (0, \frac{\nu}{2}d) \cup ((1 - \frac{\nu}{2})d\pi, \pi d)$, and $\|D\mathcal{U}^\infty - \Delta\|_{L^2(\Omega)} < \beta/4$. Setting

$$w^0(x, y) = \mathcal{U}^\infty(x^*, y^*) + \int_{(x^*, y^*)}^{(x, y)} \Delta(\xi, \eta) \cdot d\ell ,$$

we find that $\partial_n w^0|_{R_\pm} = 0$ and $w^0|_\Gamma = 0$ by construction while

$$\|w^0 - (u^0 - v^{\mathbf{r}})\|_{H^1(\Omega)} \leq \|w^0 - (u^0 - v^{\mathbf{r}})\|_{L^2(\Omega)} + \|Dw^0 - D(u^0 - v^{\mathbf{r}})\|_{L^2(\Omega)}.$$

The second term on the right hand side of this inequality is bounded by

$$\|\Delta - D\mathcal{U}^\infty\|_{L^2(\Omega)} + \|D\mathcal{U}^\infty - D(u^0 - v^{\mathbf{r}})\|_{L^2(\Omega)} < \frac{\beta}{4} + \frac{\beta}{4}.$$

On the other hand, $w^0(x, y) - (u^0 - v^{\mathbf{r}})(x, y) = 0$ if $x \in \omega$ and $y \in (\nu\delta, (1 - \nu)\pi\delta)$, so the first term can be made less than $\frac{\beta}{2}$ by making ν sufficiently small. \square

Proof of lemma 8. We prove this lemma by induction. We begin by considering the case where $I \in \mathcal{M}_0 \cap \mathcal{M}^+$, i.e. $I = 0^n$ for some $n > 0$. (Recall that if $I = 0^n$, and χ_0 is the Hamiltonian whose time 1 map gives the first canonical transformation, then $L_I(G) = \{\chi_0, \{\chi_0 \dots \{\chi_0, G\} \dots\}\}$, where there are a total of n Poisson brackets.)

Let us first calculate the Type 0 term when $n = 1$, i.e. $L_0(G)_{0, k00}$. By part 2 of Lemma 3, G must be H_1^0 , and from the table in Lemma 4, we see that

$$\begin{aligned} L_0(G)_{0, k00} &= 2 \sum_{l>0} H_{1, kl0}^0 D_{0, kl} \\ &= 2 \sum_{l>0} \frac{\omega_{k0}^{3/2}}{4\omega_{kl}^{1/2}} \beta_{l,0} \times d^2 \omega_{k0}^1 (\omega_{k0} \omega_{kl})^{1/2} \frac{\beta_{l,0}}{\lambda_l} \\ &= \frac{1}{2} d^2 \omega_{k0}^3 \sum_{l>0} \frac{(\beta_{l,0})^2}{\lambda_l}. \end{aligned}$$

This verifies the formula for $L_I(G)_{0, k00}$ in Lemma 8 in the case $I = 0$ if we take $C_{0^1} = \frac{1}{2} \sum_{l>0} \frac{(\beta_{l,0})^2}{\lambda_l}$. (The convergence of the sum follows from the fact that the $\beta_{l,0}$ are the Fourier coefficients of an L^2 function.)

The derivation of the formulas for the Type 1 and 2 terms are very similar and are left as an exercise. (The details are also presented in Appendix H of [12].)

We now assume that Lemma 8 holds for $I = 0, 0^2, \dots, 0^n$ and prove that it holds for $I = 0^{n+1}$. Once again, we will perform the detailed calculations for the Type 0 terms and leave the analogous computations of the Type 1 and 2 terms as an exercise. We know that $L_{0^{n+1}}(G)_0 = L_0(L_{0^n}(G)_1)$. So we calculate

$$\begin{aligned} L_0(L_{0^n}(G)_1) &= 2 \sum_{l>0} \left(Q_{0^n}(l) d^{2n} \omega_{k0}^{2n+1} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} \right) \left(-d^2 \omega_{k0}^2 \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} \frac{\beta_{l,0}}{\lambda_l} \right) \\ &= -2d^{2n+2} \omega_{k0}^{2n+3} \sum_{l>0} \frac{Q_{0^n}(l) \beta_{l,0}}{\lambda_l}. \end{aligned}$$

Thus

$$C_{0^{n+1}} = -2 \sum_{l>0} \frac{Q_{0^n}(l) \beta_{l,0}}{\lambda_l}.$$

Since $Q_{0^n}, \beta_{l,0} \in \ell^2$, this sum converges. We now know that the expressions for $L_I(G)_{j,kl}$ in Lemma 8 holds for all $I \in \mathcal{M}_0 \cap \mathcal{M}^+$.

We next check that $H_{1,\text{low}}^1$ has the stated form. Recall that by definition, $H_{1,\text{low}}^1$ is the sum of the lowest order terms in d in H_1^1 . But given the expression for $L_I(G)_{1,kl}$ established above, these will be exactly those terms with $p(I) = 2$ – i.e. those with $I_1 = 0$. Thus, if $n = 1$, these are precisely the terms in \mathcal{M}_0 with $p(I) = 2$ and the formula for $H_{1,\text{low}}^1$ follows.

We now complete the proof of Lemma 8 by showing that if it holds for $I \in \mathcal{M}_{n-1} \cap \mathcal{M}^+$, then it also holds for $I \in \mathcal{M}_n \cap \mathcal{M}^+$.

By the induction hypothesis,

$$\begin{aligned} H_{1,\text{low},kl}^{n-1} &= \sum_{\substack{I \in \mathcal{M}_{n-1} \cap \mathcal{M}^+ \\ p(I)=2n}} \Theta_1(I) Q_I(l) d^{p(I)} \omega_{k0}^{p(I)+1} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} \\ &= d^{2n} \omega_{k0}^{2n+1} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} \sum_{\substack{I \in \mathcal{M}_{n-1} \cap \mathcal{M}^+ \\ p(I)=2n}} \Theta_1(I) Q_I(l). \end{aligned} \quad (117)$$

If we define

$$K^n(l) = 4 \sum_{\substack{I \in \mathcal{M}_{n-1} \cap \mathcal{M}^+ \\ p(I)=2n}} \Theta_1(I) Q_I(l), \quad (118)$$

then

$$\|K^n(\cdot)\|_{\ell^2} \leq 4 \sum_{\substack{I \in \mathcal{M}_{n-1} \cap \mathcal{M}^+ \\ p(I)=2n}} |\Theta_1(I)| \|Q_I(l)\|_{\ell^2} < \infty,$$

since $\|Q_I(\cdot)\|_{\ell^2}$ is finite for all I and there are finitely many terms in the sum. Finally,

$$H_{1,\text{low},kl}^n = \frac{1}{4} d^{2n} \omega_{k0}^{2n+1} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} K^n(l).$$

Then we have (see Lemma 3)

$$D_{n,kl} = -d^{2n+2} \omega_{k0}^{2n+1} (\omega_{k0} \omega_{kl})^{1/2} \frac{K^n(l)}{\lambda_l}, \quad (119)$$

$$S_{n,kl} = d^{2n+2} \omega_{k0}^{2n+2} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} \frac{K^n(l)}{\lambda_l}. \quad (120)$$

To establish the formulas for $L_I(G)$ for $I \in \mathcal{M}_n$, note that if $I \in \mathcal{M}_n$, then either $I = n^q$ for some integer q or $I = n^q \tilde{I}$ for some $\tilde{I} \in \mathcal{M}_{n-1}$. We will first show that the formulas in Lemma 8 hold if $I = n$. This is again a straightforward calculation. For the Type 0 terms we have

$$\begin{aligned} L_n(G)_{0,k00} &= 2 \sum_{l>0} H_{1,kl0}^0 D_{n,kl} \\ &= 2 \sum_{l>0} \left(\frac{\omega_{k0}^{3/2}}{-4\omega_{kl}^{1/2}} \beta_{l,0} \right) \left(-d^{2n+2} \omega_{k0}^{2n+1} (\omega_{k0} \omega_{kl})^{1/2} \frac{K^n(l)}{\lambda_l} \right) \\ &= \frac{1}{2} d^{2n+2} \omega_{k0}^{2n+3} \sum_{l>0} \frac{K^n(l) \beta_{l,0}}{\lambda_l}, \end{aligned}$$

and then (64) follows with $C_{I=n} = \sum_{l>0} \frac{K^n(l) \beta_{l,0}}{\lambda_l}$. Since both $K^n(l)$ and $\beta_{l,0}$ are in ℓ^2 the sum is bounded. Again, we leave the analogous computations of $L_n(G)_{1,kl0}$ and $L_n(G)_{2,klv}$ as exercises.

Now suppose that we know that (64)–(66) hold for some I . We prove that they also hold for $I' = nI$. Since we know that they hold for $I = n$ or for $I \in \mathcal{M}_{n-1}$, the observation above about the form of $I \in \mathcal{M}_n$ and an inductive argument completes the proof. As above we compute in detail the formula for the Type 0 terms and leave the Type 1 and 2 terms as exercises. According to Lemma 4, we have that

$$\begin{aligned} L_{nI}(G)_{0,k00} &= 2 \sum_{l>0} \left(d^{p(I)} \omega_{k0}^{p(I)+1} \frac{\omega_{k0}^{1/2}}{\omega_{kl}^{1/2}} Q_I(l) \right) \left(-d^{2n+2} \omega_{k0}^{2n+1} \omega_{k0}^{1/2} \omega_{kl}^{1/2} \frac{K^n(l)}{\lambda_l} \right) \\ &= -2 d^{p(I)+2n+2} \omega_{k0}^{p(I)+2n+3} \sum_{l>0} \frac{Q_I(l) K^n(l)}{\lambda_l} \\ &= d^{p(nI)} \omega_{k0}^{p(nI)+1} C_{nI}, \end{aligned}$$

with

$$C_{nI} = -2 \sum_{l>0} \frac{Q_I(l) K^n(l)}{\lambda_l}. \quad (121)$$

Since $Q_I, K^n \in \ell^2$, this sum converges and, $L_{nI}(G)_{0,k00}$ satisfies the estimate in (64). Now that we have proven (64)—(66), the formula for $H_{1,\text{low}}^n$ follows immediately from the definition. \square

Proof of Lemma 9. As in the proof of Lemma 8 we will provide all the details of the estimate of $L_I(G)_0$, and leave as exercises the very similar estimates of $L_I(G)_1$ and $L_I(G)_2$.

We use the fact that $L_I(G)_0 = C_I d^{p(I)} \omega_{k0}^{p(I)+1}$, and

$$\partial_{z_{k0}} L_I(G)_0(z) = 2C_I d^{p(I)} \omega_{k0}^{p(I)+1} (z_{k0} - \bar{z}_{k0}).$$

By definition,

$$\|\nabla L_I(G)_0(z)\|_{\mathcal{H}_\alpha^{1/2}} = \sum_{k < d^\alpha} \omega_{k0} |\partial_{z_{k0}} L_I(G)_0|^2.$$

Then we calculate

$$\begin{aligned} \|\nabla L_I(G)_0(z)\|_{\mathcal{H}_\alpha^{1/2}}^2 &= \sum_{k < d^\alpha} \omega_{k0} |\partial_{z_{k0}} L_I(G)_0|^2 \\ &\leq \sum_{k < d^\alpha} \omega_{k0} 8C_I d^{2p(I)} \omega_{k0}^{2p(I)+2} |z_{k0}|^2 \\ &\leq 8C_I d^{2p(I)+\alpha(2p(I)+2)} \sum_{k < d^\alpha} \omega_{k0} |z_{k0}|^2 \\ &= 8C_I d^{2p(I)+\alpha(2p(I)+2)} \|z\|_{\mathcal{H}_\alpha^{1/2}}^2. \end{aligned}$$

Thus

$$\|\nabla L_I(G)_0\|_{\mathcal{H}_\alpha^{1/2}, \mathcal{H}_\alpha^{1/2}} \leq 8C_I d^{p(I)(1+\alpha)+\alpha}.$$

\square

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