

Math 285 G1 — Midterm 1 SOLUTIONS

Question 1, Version 1. Classify each of the following equations. In each case, identify the dependent and independent variables, determine the order of the equation, tell whether it is linear or nonlinear, separable or not, etc.

1. $\frac{dy}{dx} + y = 4x,$

2. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0,$

3. $y' + y^2 = 4,$

4. $y''' + 4x^2y'' - 14\sin(x)y' = -8.$

Solution.

1. independent variable: x , dependent variable: y , 1st order, linear, non-separable.

2. IV: x, t DV: u , 1st order, linear, non-separable.

3. IV: not specified, could be x or t , DV: y , 1st order, nonlinear, separable.

4. IV: x , DV: y , 3rd order, linear, non-separable.

Question 1, Version 2. Classify each of the following equations. In each case, identify the dependent and independent variables, determine the order of the equation, tell whether it is linear or nonlinear, separable or not, etc.

1. $\frac{dy}{dx} + \sin(y) = x,$

2. $\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$

3. $y' + y = 4y^2,$

4. $y'''' + y'' - y' = -8e^x.$

Solution.

1. IV: x , DV: y , 1st order, nonlinear, non-separable.

2. IV: x, y , DV: u , 2nd order, linear, non-separable.

3. IV: unspecified, could be x or t , DV: y , 1st order, nonlinear, separable.

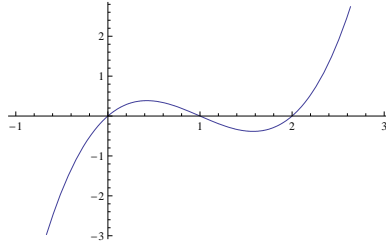
4. IV: x , DV y , 4th order, linear, non-separable.

Question 2, Version 1. Let $P(t)$ be a solution to

$$\frac{dP}{dt} = P(P - 1)(P - 2)e^{P^2 \sin(P^6)}.$$

If we consider the solution with $P(0) = 1/2$, what is $\lim_{t \rightarrow \infty} P(t)$? Same question, but if $P(0) = 3/2$? Same question, but $P(0) = 5/2$?

Solution. There are several ways to solve this, but the most efficient is to think of the “phase line” approach; basically, we need to know when the right hand side is positive or negative, and this will determine the long-time behavior of the solution. Notice that the exponential term at the end, while complicated, is always positive, and thus if we replace it with any constant, does not change the sign of the right-hand side. So we need to determine only where $P(P - 1)(P - 2)$ is zero, positive, and negative. Graphing this function gives us:



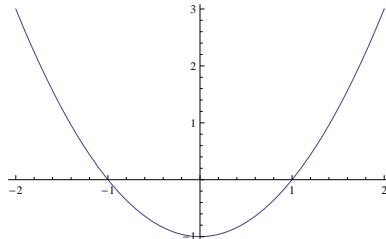
From this we can see that any initial condition between 0 and 1 will increase and approach 1, so the answer to the first question is 1. Any initial condition between 1 and 2 will decrease and approach 1, so the answer to the second question is also 1. Finally, any initial condition larger than 2 will increase forever, so the answer to the last question is $+\infty$.

Question 2, Version 2. Let $P(t)$ be a solution to

$$\frac{dP}{dt} = (P^2 - 1)(\sin(P^2) + 17)^2.$$

If we consider the solution with $P(0) = 0$, what is $\lim_{t \rightarrow \infty} P(t)$? Same question, but if $P(0) = 3$? Same question, but $P(0) = -17$?

Solution. There are several ways to solve this, but the most efficient is to think of the “phase line” approach; basically, we need to know when the right hand side is positive or negative, and this will determine the long-time behavior of the solution. Notice that the squared term at the end, while complicated, is always positive ($\sin > -1$), and thus if we replace it with any constant, does not change the sign of the right-hand side. So we need to determine only where $P^2 - 1$ is zero, positive, and negative. Graphing this function gives us:



From this we can see that any initial condition between -1 and 1 will decrease and approach -1 , so the answer to the first question is -1 . Any initial condition greater than 1

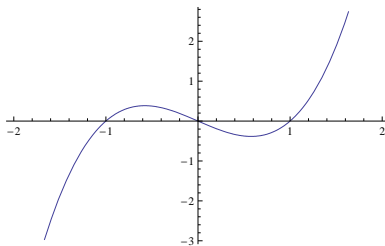
will increase without bound, so the answer to the second question is $+\infty$. Finally, any initial condition less than -1 will increase and approach -1 , so the answer to the last question is -1 .

Question 2, Version 3. Let $P(t)$ be a solution to

$$\frac{dP}{dt} = P(P^2 - 1).$$

Identify all of the equilibrium solutions and determine whether they are stable or unstable.

Solution. We graph the right-hand side and obtain



From this we see that there are three equilibria, at $P = -1, 0, 1$. We can also see from this graph that $P = 0$ is stable, since points approach it from both angles (if P is slightly negative, it increases, and if P is slightly positive, it decreases.) By the same argument, the other two equilibria $P = \pm 1$ are unstable: points slightly less decrease, and points slightly more increase.

Question 3, Version 1. Solve

$$y' + 2xy = e^{-x^2}, \quad y(0) = 2.$$

Solution. This is a first-order linear ODE, so we should use the integrating factor. Using the notation of the book, we have

$$P(x) = 2x, \quad Q(x) = e^{-x^2},$$

and so the integrating factor is

$$\mu(x) = e^{\int P(x)} = e^{x^2}.$$

Multiplying the equation by e^{x^2} gives

$$e^{x^2} y' + 2xe^{x^2} y = 1.$$

Now, noticing that the left-hand side is a derivative from the product rule, we write

$$\begin{aligned} \frac{d}{dx}(e^{x^2} y) &= 1, \\ e^{x^2} y &= x + C, \\ y(x) &= xe^{-x^2} + Ce^{-x^2}. \end{aligned}$$

Plugging in the initial condition gives us $C = 2$, and so the solution we want is

$$y(x) = xe^{-x^2} + 2e^{-x^2} = (x + 2)e^{-x^2}.$$

Question 3, Version 2. Solve

$$xy' + 2y = 3x, \quad y(1) = 5.$$

Solution. This is a first-order linear equation, so we should use the integrating factor. We first divide through by x to make the coefficient of y' equal to 1, and we get

$$y' + \frac{2}{x}y = 3.$$

We now have

$$P(x) = \frac{2}{x}, \quad Q(x) = 3,$$

and we multiply by the integrating factor

$$\mu(x) = e^{\int P(x)} = e^{2 \ln x} = e^{\ln(x^2)} = x^2.$$

Multiplying through by x^2 gives

$$x^2y' + 2xy = 3x^2.$$

Noting that the left-hand side is a derivative from the product rule, we obtain

$$\begin{aligned} \frac{d}{dx}(x^2y) &= 3x^2, \\ x^2y &= x^3 + C, \\ y(x) &= x + \frac{C}{x^2}. \end{aligned}$$

Plugging in the initial condition gives

$$5 = 1 + C,$$

or $C = 4$, and thus the solution is

$$y(x) = x + \frac{4}{x^2}.$$

Question 3, Version 3. Solve

$$y' + y = e^x, \quad y(1) = 2.$$

Solution. This is a first-order linear ODE, so we should use the integrating factor. Using the notation of the book, we have

$$P(x) = 1, \quad Q(x) = e^x,$$

and so the integrating factor is

$$\mu(x) = e^{\int P(x)} = e^x.$$

Multiplying the equation by e^x gives

$$e^x y + e^x y = e^{2x}.$$

Now, noticing that the left-hand side is a derivative from the product rule, we write

$$\begin{aligned} \frac{d}{dx}(e^x y) &= e^{2x}, \\ e^x y &= \frac{1}{2}e^{2x} + C, \\ y(x) &= \frac{1}{2}e^x + Ce^{-x}. \end{aligned}$$

Plugging in the initial condition gives

$$\frac{e}{2} + \frac{C}{e} = 2,$$

or $C = 2e - e^2/2$, and so the solution we want is

$$y(x) = \frac{1}{2}e^x + \left(2e - \frac{e^2}{2}\right)e^{-x}.$$

Question 4, Version 1. Solve

$$\frac{dy}{dx} = x^2 y, \quad y(0) = 3.$$

Solution. This is a separable equation, so we write

$$\frac{dy}{y} = x^2 dx.$$

Integrating both sides gives

$$\begin{aligned} \ln y &= \frac{x^3}{3} + C, \\ y(x) &= \exp\left(\frac{x^3}{3} + C\right) = Ce^{x^3/3}. \end{aligned}$$

Plugging in the initial condition gives $C = 3$, so our solution is

$$y(x) = 3e^{x^3/3}.$$

Question 4, Version 2. Solve

$$\frac{dy}{dx} = xy^2, \quad y(1) = 4.$$

Solution. This is a separable equation, so we write

$$\frac{dy}{y^2} = x dx.$$

Integrating both sides gives

$$-\frac{1}{y} = \frac{x^2}{2} + C,$$
$$y(x) = -\frac{1}{x^2/2 + C}.$$

Plugging in the initial condition to both sides gives

$$4 = -\frac{1}{C + 1/2},$$

or

$$C + \frac{1}{2} = -\frac{1}{4},$$

or $C = -3/4$. Thus our solution is

$$y(x) = -\frac{1}{x^2/2 - 3/4} = -\frac{4}{2x^2 - 3}.$$

Question 4, Version 3. Solve

$$\frac{dy}{dt} = \cos(t)y, \quad y(0) = -1.$$

Solution. This is a separable equation, so we write

$$\frac{dy}{y} = \cos(t) dt.$$

Integrating both sides gives

$$\ln y = \sin(t) + C,$$
$$y(t) = \exp(\sin(t) + C) = Ce^{\sin(t)}.$$

Plugging in the initial condition gives $C = -1$, so our solution is

$$y(x) = -e^{\sin(t)}.$$

Question 5, Version 1. The top of the bell tower of Altgeld Hall is 40m above the ground. How fast do you need to throw an object from ground level so that it goes this high? (Assume you're throwing from exactly 0m height, and that the acceleration due to gravity is -10m/s^2 .)

Solution. After we've thrown the ball, the only force is due to gravity and thus the ball is under the constant acceleration -10m/s^2 . Integrating twice, we obtain

$$\begin{aligned}v(t) &= -10t + v_0, \\x(t) &= -5t^2 + v_0t + x_0.\end{aligned}$$

Since we've started at ground level, $x_0 = 0$, but of course v_0 is still undetermined. The first question we could ask is when the ball reaches its maximum height, i.e. at what time t^* is $v(t^*) = 0$? Using the first equation gives us $t^* = v_0/10$. Plugging this into the second equation gives us

$$x(t^*) = -5(t^*)^2 + v_0t^* = -\frac{1}{20}v_0^2 + \frac{1}{10}v_0^2 = \frac{1}{20}v_0^2.$$

Since this should be equal to 40m , we have $v_0^2 = 800$ or $v_0 = \sqrt{800}$.

Question 5, Version 2. Imagine that we have two separate populations of bacteria whose population size is well-modeled by an exponential-growth model. In the first population, we start with 1000 bacteria and see that after one day, we now have 3000 bacteria. In the second population, we start with 2000 bacteria and after one day have 18,000 bacteria. First, which population has a larger growth rate? Second, if we let k_1, k_2 be the growth rates of the two populations, what is k_2/k_1 ? Finally, will the first population ever be larger than the second?

Solution. We use $P_1(t)$ to be the first population at time t , and $P_2(t)$ to be the second population. We're assuming an exponential growth model, so we know

$$P_1(t) = P_1(0)e^{k_1t}, \quad P_2(t) = P_2(0)e^{k_2t}.$$

We know from the problem that

$$\begin{aligned}P_1(0) &= 1000, & P_1(1 \text{ day}) &= 3000, \\P_2(0) &= 2000, & P_2(1 \text{ day}) &= 18000.\end{aligned}$$

From this we have

$$\begin{aligned}e^{k_1(1 \text{ day})} &= 3, \text{ or } k_1 = \ln 3 \text{ day}^{-1}, \\e^{k_2(1 \text{ day})} &= 9, \text{ or } k_2 = \ln 9 \text{ day}^{-1},\end{aligned}$$

Thus $k_2 > k_1$, and, moreover, we have

$$\frac{k_2}{k_1} = \frac{\ln 9}{\ln 3} = \frac{\ln 3^2}{\ln 3} = \frac{2 \ln 3}{\ln 3} = 2.$$

Finally, to answer the last, can it ever be true that $P_2(t) < P_1(t)$? Consider the quotient

$$\frac{P_2(t)}{P_1(t)} = \frac{P_2(0)}{P_1(0)} e^{(k_2 - k_1)t}.$$

Since $k_2 > k_1$, $e^{(k_2 - k_1)t} > 1$ for all positive t . Since $P_2(0) > P_1(0)$ as well, this means the product is always greater than one, so that $P_2(t) > P_1(t)$ for all t . Thus the answer is no.

Question 5, Version 3. Assume that a car's acceleration is proportional to the difference between 300 km/hr and its velocity, and that the car can go from rest to 100 km/hr in 5 seconds. How fast will it be going in 10s? What is the maximum velocity it will ever attain?

Solution. According to the problem, the acceleration $a(t)$ satisfies

$$a(t) = k(300 - v(t)).$$

Since $v'(t) = a(t)$, this gives us the differential equation

$$\frac{dv}{dt} = k(300 - v(t)),$$

which is both a linear first-order equation, and a separable equation. We can solve this in any of a number of ways, e.g.

$$\begin{aligned} \frac{dv}{300 - v} &= k dt \\ -\ln |300 - v| &= kt + C, \\ 300 - v &= Ce^{-kt}. \end{aligned}$$

Since $v(0) = 0$, this gives $C = 300$, so that

$$v(t) = 300(1 - e^{-kt}).$$

We also know that $v(5) = 100$, which we can write as

$$\frac{1}{3} = 1 - e^{-5k},$$

or

$$k = -\frac{1}{5} \ln \frac{2}{3}.$$

Plugging in $t = 10$ gives

$$\begin{aligned} v(10) &= 300(1 - e^{-k(10)}) = 300(1 - e^{2 \ln 2/3}) \\ &= 300(1 - e^{\ln 4/9}) = 300(1 - 4/9) = 300 * 5/9 = 500/3. \end{aligned}$$

Finally, we compute

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} 300(1 - e^{-kt}) = 300,$$

since $k > 0$.