Math 285 G1 — Midterm 2 Solutions

1. (**20 points.**) Find the solution to

$$y'' + 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

Solution. This is a linear homogeneous equation with constant coefficients. So we make the Ansatz

$$y(x) = e^{rx},$$

and this gives the characteristic equation

$$r^2 + 3r + 2 = (r+1)(r+2) = 0,$$

which has the solutions $r = -1, -2$. So two solutions to the equation are

$$y_1(x) = e^{-x}, \quad y_2(x) = e^{-2x}.$$

The general solution is then

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

Plugging in to check the initial conditions, we obtain

$$\begin{aligned} C_1 + C_2 &= 2, \\ -C_1 - 2C_2 &= 1, \end{aligned}$$

and this has solution $C_1 = 5, C_2 = -3$, giving the solution

$$y(x) = 5e^{-x} - 3e^{-2x}.$$

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2. (**20 points.**) Write down the general solution of

$$y''' - 4y'' + 3y' = e^{2x}.$$

Solution. We first solve the homogeneous equation. Since it is constant coefficient, we can use the Ansatz as in the previous problem, and get the characteristic equation

$$r^3 - 4r^2 + 3r = r(r-1)(r-3) = 0,$$

so the general solution to the homogeneous problem is

$$C_1 + C_2 e^x + C_3 e^{3x}.$$

The inhomogeneous term is not a solution to the homogeneous problem, so we should guess

$$y_p(x) = Ae^{2x}.$$

Plugging this in gives

$$8Ae^{2x} - 16Ae^{2x} + 6Ae^{2x} = e^{2x},$$

or

$$-2A = 1,$$

or $A = -1/2$. So then

$$y_p(x) = -\frac{1}{2}e^{2x},$$

and the general solution is

$$y(x) = C_1 + C_2 e^x + C_3 e^{3x} - \frac{1}{2}e^{2x}.$$

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3. (**20 points.**) Find the solution to

$$y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. This is another constant coefficient problem; the characteristic equation is

$$r^2 + 9 = 0,$$

or $r = \pm 3i$. This means that the general solution is

$$y(x) = C_1 \cos(3x) + C_2 \sin(3x).$$

Plugging in initial conditions gives

$$C_1 = 1,$$
$$3C_2 = 0,$$

so the solution is

$$y(x) = \cos(3x).$$

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4. (**20 points.**) Assume we have a mass-spring system where the mass is 4 kg and the spring constant is 16 kg/s^2. Write down the equation of motion for this system, assuming that there is no friction.

Now assume that we force the system with a periodic force of the form $f(t) = A\cos(\omega t)$. How should we choose A, ω to make the system break? How should we choose A, ω if we want the system to *not* break?

Solution. The equation for the mass spring system without friction is

$$mx'' + kx = 0,$$

or, in this case,

$$4x'' + 16x = 0.$$

To find the natural frequency of this problem, we solve this equation to obtain

$$x(t) = C_1\cos(2t) + C_2\sin(2t),$$

so the natural frequency is $\omega_0 = 2$.

If we want to break the system, then the easiest way to do this is to choose the frequency of the forcing to be near the natural frequency, i.e. $\omega \approx 2$. This will set up a near-resonance solution with large response. Even better, choose $\omega = 2$ and obtain resonance. Of course, you must choose $A \neq 0$ in this case. Another way to break the system is to just choose A really large.

To make the system not break, we want the response to be not large, so we need to be away from resonance, so we choose ω far away from 2.

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5. (**20 points.**) We are given the equation

$$t^2x''(t) - 2tx'(t) + 2x(t) = 0,$$

and we are told that some (but not all) of the following functions are solutions:

$$\sin(t), t^2, 1, t, e^t.$$

From this information, write down the general solution of this equation.

Solution. This is a linear equation, *but not a constant-coefficient one!*, so we cannot use the characteristic equation. However, we have been given potential solutions, so we just need to plug these into the equation to check.

(a) Plugging in $\sin(t)$ gives

$$-t^2 \sin(t) - 2t\cos(t) + 2\sin(t) \neq 0,$$

so not a solution.

(b) Plugging in t^2 gives

$$2t^2 - 4t^2 + 2t^2 = 0,$$

so it is a solution.

(c) Plugging in 1 gives

$$0 + 0 + 2 \neq 0,$$

so not a solution.

(d) Plugging in t gives

$$0 - 2t + 2t = 0,$$

so it is a solution.

(e) Plugging in e^t gives

$$t^2 e^t - 2te^t + 2e^t \neq 0,$$

so not a solution.

Thus we have two solutions to the problem, $y_1(t) = t, y_2(t) = t^2$. We need that these are independent, and for two solutions we can either make sure their quotient is not constant, or check the Wronskian. Using the first technique, we see that $y_2(t)/y_1(t) = t$ which is not a constant. Using the Wronskian gives

$$\det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t^2 - t^2 = t^2,$$

which is not zero. Either way, the solutions are independent. So, if we have two independent solutions to a second-order equation, the general solution is a general linear combination of the two, and thus the general solution is

$$x(t) = C_1 t + C_2 t^2.$$
