Introduction to Differential Equations – Math 285 G1
Spring 2009
Quiz 2

1. Write down the general solution to $y'' - 4y' + 3y = 0$.

   Solution. Making the Ansatz $y(x) = e^{rx}$, we obtain the characteristic equation

   $$r^2 - 4r + 3 = 0.$$

   We see that this has two roots, $r = 1, 3$. Thus two independent solutions to the ODE are

   $$y(x) = e^x, \quad y(x) = e^{3x},$$

   and the general solution is

   $$y(x) = C_1 e^x + C_2 e^{3x}.$$

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2. Write down *any* solution to $y'' - 4y' + 3y = e^{2x}$.

   Solution. We can see from the previous problem that the right-hand side is not a solution to the homogeneous problem, so we should guess

   $$y_p(x) = Ae^{2x}.$$

   We compute:

   $$y_p'(x) = 2Ae^{2x},$$
   $$y_p''(x) = 4Ae^{2x},$$

   and plugging in gives

   $$4Ae^{2x} - 8Ae^{2x} + 3Ae^{2x} = -Ae^{2x}.$$

   This means that $A = -1$, so we have

   $$y_p(x) = -e^{2x}.$$

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3. Consider an object modeled by the equation $x''(t) + 4x(t) = A\cos(\omega t)$.

   How do we choose A, ω in such a way as to break the object as fast as possible?

   Solution. The short answer to the question is "make the forcing resonant". To do this, we need the forcing frequency to be the same as the natural frequency.

   To get the natural frequency, consider the homogeneous equation

   $$x'' + 4x = 0.$$

   The characterstic equation is $r^2 + 4$, which has roots $\pm 2i$, which means that the general solution to the homogeneous problem is

   $$C_1 \cos(2t) + C_2 \sin(2t).$$

   Thus if we choose $\omega = 2$ (and A anything not zero), then the forcing is resonant, and the particular solution which comes from this forcing will be of the form

   $$y_p(t) = C_1 t \sin(2t) + C_2 t \cos(2t),$$

   meaning the amplitude of the solution grows without bound for most initial conditions.
