Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 4 Solutions

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1. Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 2,$$

and now solve

$$y'' - y = 0, \quad y(0) = 0, \quad y'(0) = -1.$$

Solution: We make the exponential Ansatz $y(x) = e^{rx}$, which leads to the equation

$$r^2 - 1 = 0,$$

which has roots $r = \pm 1$. Therefore, two solutions to this system are $y_1(x) = e^x$ and $y_2(x) = e^{-x}$, and so the general solution to this system is

$$y(x) = C_1 e^x + C_2 e^{-x}.$$

We also have

$$y'(x) = C_1 e^x - C_2 e^{-x},$$

so

$$y(0) = C_1 + C_2, \quad y'(0) = C_1 - C_2.$$

To solve the first problem we obtain

$$\begin{aligned} C_1 + C_2 &= 1, \\ C_1 - C_2 &= 2. \end{aligned}$$

which we solve as $C_1 = 3/2, C_2 = -1/2$, so the solution is

$$y(x) = \frac{3}{2} e^x - \frac{1}{2} e^{-x}.$$

For the second problem, we obtain

$$\begin{aligned} C_1 + C_2 &= 0, \\ C_1 - C_2 &= -1. \end{aligned}$$

which we solve as $C_1 = -1/2, C_2 = 1/2$, so the solution is

$$y(x) = -\frac{1}{2} e^x + \frac{1}{2} e^{-x}.$$

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2. Solve the initial value problem

$$y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 2,$$

and now solve

$$y'' + y = 0, \quad y(0) = -1, \quad y'(0) = 0.$$

Solution: Making the Ansatz $y(x) = e^{rx}$ and plugging in gives

$$r^2 + 1 = 0,$$

which has roots $\pm i$. Thus our two solutions are $\cos(x)$ and $\sin(x)$, and the general solution is

$$y(x) = C_1 \cos(x) + C_2 \sin(x).$$

We also have

$$y'(x) = -C_1 \sin(x) + C_2 \cos(x),$$

so

$$y(0) = C_1, \quad y'(0) = C_2.$$

Thus the first problem has $C_1 = 1, C_2 = 2$, or

$$y(x) = \cos(x) + 2\sin(x).$$

The second problem has $C_1 = -1, C_2 = 0$, or

$$y(x) = -\cos(x).$$

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3. In each of the following problems, you should give the general solution of the differential equation (i.e. do steps 1 & 2 as described in class)

   (a) $y'' - 2y' + y = 0$,
   (b) $y'' - 3y' + y = 0$,
   (c) $y'' - y' + y = 0$,
   (d) $y' - 2y = 0$,
   (e) $y'' - 2y' = 0$,
   (f) $y''' - 2y'' = 0$.

   Solution: In each case, we compute the characteristic equation and then find the roots which gives us the solutions.

   (a) $r^2 - 2r + 1$ has repeated root $r = 1, 1$, so our two solutions are e^x, xe^x, and the general solution is

   $$y(x) = C_1 e^x + C_2 x e^x.$$

   (b) We get $r^2 - 3r + 1 = 0$. Using the quadratic formula gives

   $$r = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2},$$

   so our general solution is

   $$y(x) = C_1 e^{(3+\sqrt{5})x/2} + C_2 e^{(3-\sqrt{5})x/2}.$$

   (c) We get $r^2 - r + 1 = 0$, and using the quadratic formula gives

   $$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

   Thus our general solution is

   $$y(x) = C_1 e^{x/2} \cos(3x/2) + C_2 e^{x/2} \sin(3x/2).$$

(d) We get $r - 2 = 0$, which has one root, $r = 2$, so our general solution is

$$y(x) = Ce^{2x}.$$

(e) We get $r^2 - 2r = 0$, which has roots $r = 0, 2$, so we get

$$y(x) = C_1e^{0x} + C_2e^{2x} = C_1 + C_2e^{2x}.$$

(f) We get $r^3 - 2r^2 = 0$, which has roots $r = 0, 0, 2$, so our three solutions should be

$$y_1(x) = e^{2x}, \quad y_2(x) = e^{0x}, \quad y_3(x) = xe^{0x},$$

so we obtain

$$y(x) = C_1e^{2x} + C_2 + C_3x.$$