Fundamental Mathematics - 347 G1
Homework 3 – Solutions

1. **3.1.** Example: "n has two or fewer digits in base 10."

2. **3.5.** We will prove this by induction. We define the mathematical statement

   $$P(n) = \text{"} \sum_{k=1}^{n} (2k+1) = n^2 + 2n.\text{"}$$

   We first check $P(1)$, i.e.

   $$\sum_{k=1}^{1} (2k+1) = 3,$$

   and this is true.

   Now, assume $P(k)$, namely that

   $$\sum_{i=1}^{k} (2i+1) = k^2 + 2k.$$

   We need to then show $P(k+1)$, or

   $$\sum_{i=1}^{k+1} (2i+1) = (k+1)^2 + 2(k+1).$$

   Then compute:

   $$\begin{aligned} \sum_{i=1}^{k+1} (2i+1) &= \sum_{i=1}^{k} (2i+1) + 2(k+1) + 1 \\ &= k^2 + 2k + 2k + 3 \\ &= k^2 + 2k + 1 + 2k + 2 \\ &= (k+1)^2 + 2(k+1). \end{aligned}$$

3. **3.12.** We will again use induction on n. Let

   $$P(n) = \text{"} \sum_{i=1}^{n} x = nx.\text{"}$$

   What this says in words is that if we add x to itself n times we get nx, which is not surprising.

   First check $P(1)$:

   $$\sum_{i=1}^{1} x = x.$$

   Now, assume that $P(k)$:

   $$\sum_{i=1}^{k} x = kx$$

   and check

   $$\sum_{i=1}^{k+1} x = \sum_{i=1}^{k} x + x = kx + x = (k+1)x.$$

   Thus we have shown $P(k+1)$ and we are done.

4. **3.17.** Again, induction. Define $P(n)$ in the obvious way. First prove $P(1)$:

$$\sum_{i=1}^{1} i(i+1) = 1 * 2 = 2,$$

while plugging $n = 1$ into the right-hand side gives

$$\frac{1 * 2 * 3}{3} = 2.$$

Now, assume $P(k)$ and prove $P(k+1)$. So we compute

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= (k+1)\left[\frac{k(k+2)}{3} + \frac{3(k+2)}{3}\right] \\
&= (k+1)\frac{k^2+2k+3k+6}{3} \\
&= (k+1)\frac{k^2+5k+6}{3} \\
&= (k+1)\frac{(k+2)(k+3)}{3} = \frac{(k+1)(k+2)(k+3)}{3},
\end{aligned}$$

and we have proved $P(k+1)$.

5. **3.24.** The statement is false because the conclusion, while close, is too strong. For example, pick $m = 10$. Then the statement would be "If T is a set of natural numbers such that $10 \in T$ and $n \in T \implies (n+1) \in T$, then $T = \{10, 11, 12, 13, \ldots\}$."

But this statement is obviously false. For example, the set of natural numbers have these two properties: $10 \in \mathbb{N}$, and furthermore $n \in \mathbb{N} \implies (n+1) \in \mathbb{N}$.

On the other hand, it is clear that any set T with those properties must contain all of the natural numbers larger than 10, so that we know $T \supseteq \{10, 11, 12, 13, \ldots\}$. So, if one changes the statement to "If T is a set of natural numbers such that $m \in T$ and $n \in T \implies (n+1) \in T$, then $T \supseteq \{n \in \mathbb{N} : n \geq m\}$."

3.28. We first want to find a formula for this expression, so let us work it out for a few n. We will write, for shorthand,

$$f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

We compute $f(1) = 1/2$. Then we have

$$\begin{aligned}
f(2) &= \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}. \\
f(3) &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}. \\
f(4) &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{48}{60} = \frac{4}{5}.
\end{aligned}$$

So a pattern which seems to arise is

$$f(n) = \frac{n}{n+1}.$$

Let's see if we can prove this by induction.

If $n = 1$, then this formula is true. Now, assume it is correct for k, then we compute

$$\begin{aligned} f(k+1) &= \sum_{i=1}^{k+1} \frac{1}{i(i+1)} \\ &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2}, \end{aligned}$$

and we are done.

6. **3.41.**

   (a) Choose $x = y = 0$. The formula must hold for these numbers, so we get

   $$f(0) = f(0) + f(0).$$

   Subtract $f(0)$ from both sides and we get $f(0) = 0$.

   (b) We will use induction. Now, clearly this is correct for $n = 1$. So, let us assume it is correct for k, i.e. that

   $$f(k) = kf(1).$$

   Then we have

   $$f(k+1) = f(k) + f(1) = kf(1) + f(1) = (k+1)f(1).$$

   3.47. Again, induction. First check for $n = 1$, plugging in we get

   $$5 + 5 < 5^2 = 25,$$

   which is true. Now, assume that this is true for k, namely that

   $$5^k + 5 < 5^{k+1}.$$

   Then we have

   $$5^{k+1} + 5 = 5(5^k + 1) < 5(5^k + 5) < 5 * 5^{k+1} = 5^{k+2}.$$

   3.50.

   (a) We first compute with $f(1) = 1$. It is more convenient to rewrite the formula as

   $$f(x) = f(x-y)f(y).$$

Let us try and compute $f(2)$. If we choose $x = 2, y = 1$, we get

$$f(2) = f(1)f(1) = 1.$$

Choosing $x = 3, y = 1$ gives

$$f(3) = f(2)f(1) = 1.$$

It looks like $f(n) = 1$ for any $n \in \mathbb{N}$ may be the correct hypothesis. This is of course true for $n = 1$, and notice that if we write

$$f(n) = f(n-1)f(1) = f(n-1),$$

then there is a simple induction proof: if we assume that $f(k) = 1$, then $f(k+1) = f(k) = 1$.

(b) Now assume that $f(1) = c$. Similarly, we compute

$$f(2) = f(1)f(1) = c * c = c^2.$$

We have

$$f(3) = f(2)f(1) = c^2 * c = c^3.$$

The correct conjecture seems to be $f(n) = c^n$ for all $n \in \mathbb{N}$. This is true by definition for $n = 1$, so let us assume it is true for k, i.e. that $f(k) = c^k$, and compute

$$f(k+1) = f(k)f(1) = cf(k) = c * c^k = c^{k+1}.$$