Fundamental Mathematics - 347 G1
Homework 5 – Solutions

1. **5.1.** There are several ways to approach this problem, and I will show two.

   Before we count the number of ways to obtain an even number, let us first count the total number of possibilities. We are rolling n dice, each can come up one of 6 ways, and so by the Rule of Product the total number of possibilities is 6^n. We define E_n to be the number of ways to get an even sum when we roll n dice, and O_n to be the number of ways to get an odd sum when we roll n dice. We are looking for a formula for E_n, but notice from above that $E_n + O_n = 6^n$ for all n, so knowing E_n gives O_n as well. Finally, the probability we are looking for is $E_n/6^n$.

   Now to count the even possibilities:

   (a) The first method is a direct count of the number of ways to get an even number. We will prove this by induction.

   Let us first consider $n = 1$. Clearly there are 3 ways to get an even and 3 ways to get an odd, so $E_1 = O_1 = 3$.

   Let us now consider $n > 1$. If we have a list of n numbers whose sum is even, then one of two things are true: the last number is even, and the sum of the first $n - 1$ numbers is even, or the last number is odd, and the sum of the first $n - 1$ numbers is odd. Thus we have the equation

   $$E_n = 3E_{n-1} + 3O_{n-1}.$$

   Using these equations for the first few n, we obtain

   $$\begin{aligned} E_2 &= 3 * 3 + 3 * 3 = 6 * 3 = 18, \\ O_2 &= 6^2 - 18 = 18, \\ E_3 &= 3 * 18 + 3 * 18 = 6 * 18 = 108, \\ O_3 &= 6^3 - 108 = 216 - 108 = 108, \\ E_4 &= 3 * 108 + 3 * 108 = 6 * 108 = 648, \ldots \end{aligned}$$

   The pattern seems to be, so far, that

   $$E_n = O_n = 3 * 6^{n-1}.$$

   We prove this by induction. We have verified this for $n = 1$, now, if we assume that it is true for any n, then

   $$E_{n+1} = 3E_n + 3O_n = 3 * 3 * 6^{n-1} + 3 * 3 * 6^{n-1} = 6 * 3 * 6^{n-1} = 3 * 6^n.$$

   and

   $$O_{n+1} = 6^{n+1} - E_{n+1} = 6^{n+1} - 3 * 6^n = 3 * 6^n,$$

   so the formula is proved. Finally,

   $$\frac{E_n}{6^n} = \frac{3 * 6^{n-1}}{6^n} = \frac{1}{2}.$$

   (b) The second method is to show that the number of ways to obtain an odd is the same as the number of ways to obtain an even, or that $E_n = O_n$. From this it follows that $E_n/6^n = 1/2$.

   How do we show that $E_n = O_n$? One way to do this is to write down a bijection between the set of all odd sums to the set of all even sums; if we can do this, then the sets must have the same size. So, fix n and define

$$S = \{x_i \in [6], i = 1, \ldots, n : \sum_{i=1}^{n} x_i \text{ is even}\},$$
$$T = \{x_i \in [6], i = 1, \ldots, n : \sum_{i=1}^{n} x_i \text{ is odd}\}.$$

We now write down a map from S to T. If $x = (x_1, \ldots, x_n) \in S$, we define $f(x) = y$, where

$$y_1 = \begin{cases} x_1 + 1, & x_1 = 1, 2, 3, 4, 5 \\ 1, & x_1 = 6, \end{cases} \tag{1}$$

and $y_i = x_i$ for all $i \geq 2$. Now, since this map f changes only one of the numbers, and either adds one, or subtracts five from this number, then if $x \in S$ then $f(x) \in T$, since adding an odd number to an even number makes an odd number. Now, we need to show that f is a bijection.

To show it is injective, let us assume that $f(x) = f(z)$. Then clearly $x_i = z_i$ for all $i \geq 2$, and moreover notice that the function in (1) is injective, so if $f(x)_1 = f(z)_1$ then $x_1 = z_1$.

To show it is surjective, notice that the function is surjective in each component; this is clearly so for $i \geq 2$, and for $i = 1$ notice that the map in (1) is surjective as well.

So we have exhibited a bijection $f\colon S \to T$, so that the size of S is the size of T.

2. **5.2.** For each of the numbers $2, \ldots, 12$, we count the number of ways to get that sum:

Sum	2	3	4	5	6	7	8	9	10	11	12
Num. ways	1	2	3	4	5	6	5	4	3	2	1

Thus we have

$$p(2) = p(12) = \frac{1}{36},$$
$$p(3) = p(11) = \frac{2}{36} = \frac{1}{18}$$
$$p(4) = p(10) = \frac{3}{36} = \frac{1}{12}$$
$$p(5) = p(9) = \frac{4}{36} = \frac{1}{9}$$
$$p(6) = p(8) = \frac{5}{36}$$
$$p(7) = \frac{6}{36} = \frac{1}{6}.$$

3. **5.4.** For the first question, we are asking how many lists of length l can we obtain when each entry is one of m choices and we allow repeats. Using the Rule of Product, this is m^l. The second question is the same except that we do not allow repeats, and we know that this formula is $\frac{m!}{(m-l)!}$.

4. **5.6.** The number of bijections from A to B is the same as the number of permutations on $[n]$. To see this, let us list the elements of A and B as

$$A = \{a_1, a_2, \ldots, a_n\}, \quad B = \{b_1, b_2, \ldots, b_n\}.$$

Now, let us write down any bijection from A to B. We first define $f(a_1)$, and we have n choices here. We then define $f(a_2)$, and since we cannot repeat the choice from before, we have $n - 1$ choices here. Continuing this, we see that we will end up with $n(n-1)(n-2)\ldots 2 * 1 = n!$ choices.

The reason that this is the same as the number of permutations of $[n]$ can be seen by thinking of each choice of $f(a_i)$ being some b_j as a map, instead, from the indices in the A set to the indices in the B set.

5. **5.7.** This is a bit more complicated because we cannot use either of the simple rules (product or sum) directly, but must use a combination of the two. We first break the problem up into two subproblems: First we count n_1, the number of ways to choose the ace of spades, and then any card not an ace, and then we count n_2, the number of ways to choose a spade not an ace, and then any card not an ace. Then we are looking to compute $n_1 + n_2$, since any outcome which was spade then not an ace must be in exactly one of the two events described above.

   For n_1, there is only one way to choose the ace of spades, and once it has been chosen, then there are 48 cards left which are not aces, so $n_1 = 1 * 48 = 48$.

   For n_2, there are 12 ways to choose a non-ace spade, but then there are only 47 cards left which are not aces, so $n_2 = 12 * 47 = 564$.

   Thus the total number of ways to make this choice is $48 + 564 = 612$.

6. **5.10.** First we count the number of ways to get n heads if we flip $2n$ coins. Think of a list of $2n$ letters, each H or T, and ask, how many lists are there with exactly n H's? This is equivalent to asking how many subsets of $[2n]$ have exactly n elements, which we know is $\binom{2n}{n}$.

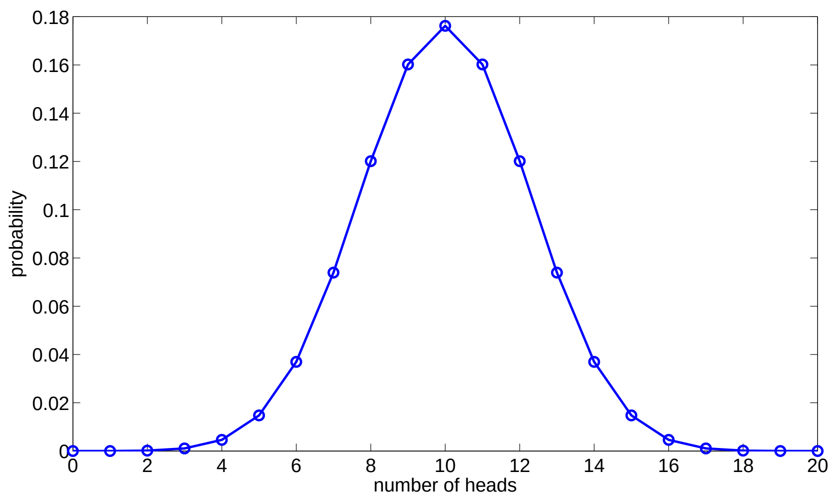

Figure 1: A plot of the probability of obtaining k heads when flipping 20 coins; this is a plot of $\binom{20}{k}/2^{20}$

   Now, the number of ways to flip $2n$ coins is, using the Rule of Product, 2^{2n}.

   Therefore our probability is

$$p = \frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{(n!2^n)^2}.$$

   When $n = 10$, we are asking for the probability of flipping 20 coins and obtaining exactly 10 heads. Plugging in $n = 10$ to the above formula gives approximately 0.1762, so there is roughly a 17.62% probability of getting exactly 10 heads.

   It should be noted that the probability of getting exactly 10 heads is not that large at all, specifically less than one in five odds. This may be counterintuitive, because we expect that half of the coins are heads. We can see from the plot in Figure 1 that there is a high probability of getting some number near 10 (e.g. the probability of getting between 8 and 12 heads is roughly 75%), but getting exactly 10 is not so large.

One could imagine a betting game where the person who knew that exactly 10 heads out of 20 coin flips had a probability of much less than 1/2 would have a large advantage over the person who didn't know that, not that I am suggesting anything.

7. **5.12.** We first count the number of ways to obtain eleven after rolling three dice. Call the three rolls n_1, n_2, n_3, and we have $n_i \in [6]$ for $i = 1, 2, 3$ and $n_1 + n_2 + n_3 = 11$.

   Now, if $n_1 = 1$, then $n_2 + n_3 = 10$. However, we computed above that the number of ways to roll 10 with 2 dice is 3. We can do this for all possible rolls, using the chart below:

n_1	1	2	3	4	5	6
$n_2 + n_3$	10	9	8	7	6	5
num. ways	3	4	5	6	5	4

   So the number of ways to get eleven is then $3 + 4 + 5 + 6 + 5 + 4 = 27$.

   The total number of ways to roll three dice is $6^3 = 216$, so the probability is then

$$p = \frac{27}{216} = \frac{1}{8} = 0.125.$$

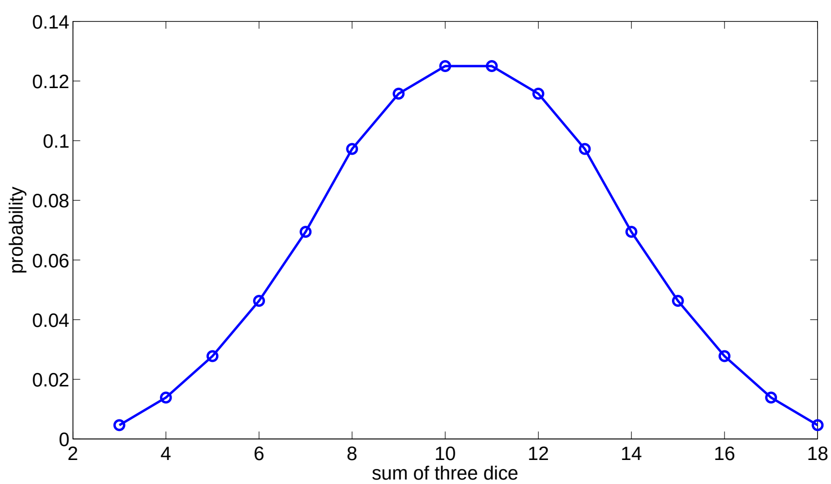

Figure 2: Probability of obtaining a given sum when rolling three dice

   In Figure 2 we plot the probability of obtaining any sum on the three dice (from 3 to 18). Notice that 10 and 11 are the most probable sums (we will talk more about this sort of thing in Chapter 9).

8. **5.13.** We first count the number of ways to do each of these, and then divide by 6^4 to compute the probability.

   $k = 0$. We are asking the number of ways to get four non-sixes. There are five non-sixes on each die, so clearly the number of ways to do this is $5^4 = 625$.

   $k = 1$. We have one six and three non-sixes. Let us first choose where the six is, and then choose the remaining numbers. Clearly, there are four places to put the six, and then once the six is placed, we need to choose three non-sixes. Thus the total number of ways to do this is $4 * 5^3 = 500$.

$k = 2$. We first choose the location of the sixes, and there are $\binom{4}{2} = 6$ ways to do this. Once the sixes have been placed, we then have to determine the two non-sixes, and there are 5^2 ways to do this. So the total number is $6 * 25 = 150$.

$k = 3$. There are $\binom{4}{3} = 4$ ways to choose where the three sixes are, and then we are left to choose one non-six, so we have $4 * 5 = 20$.

$k = 4$. There is only one way to do this.

Now, to make sure the probabilities add up to one, we need to have that

$$625 + 500 + 150 + 20 + 1 = 6^4,$$

but this is true (both sides are 1296). Another way to see why this should be true, consider the following use of the Binomial Theorem:

$$\begin{aligned} 6^4 = (5+1)^4 &= \sum_{i=0}^{4} \binom{4}{i} 5^i 1^{4-i} \\ &= 5^4 + 4 * 5^3 + 6 * 5^2 + 4 * 5^1 + 1. \end{aligned}$$

To compute these probabilities, we have

$$\begin{aligned} p(0) &= \frac{625}{1296} \approx 0.482, \\ p(1) &= \frac{500}{1296} \approx 0.386, \\ p(2) &= \frac{150}{1296} \approx 0.116, \\ p(3) &= \frac{20}{1296} \approx 0.015, \\ p(4) &= \frac{1}{1296} \approx 7.72 \times 10^{-3}. \end{aligned}$$

We plot these probabilities in Figure 3.

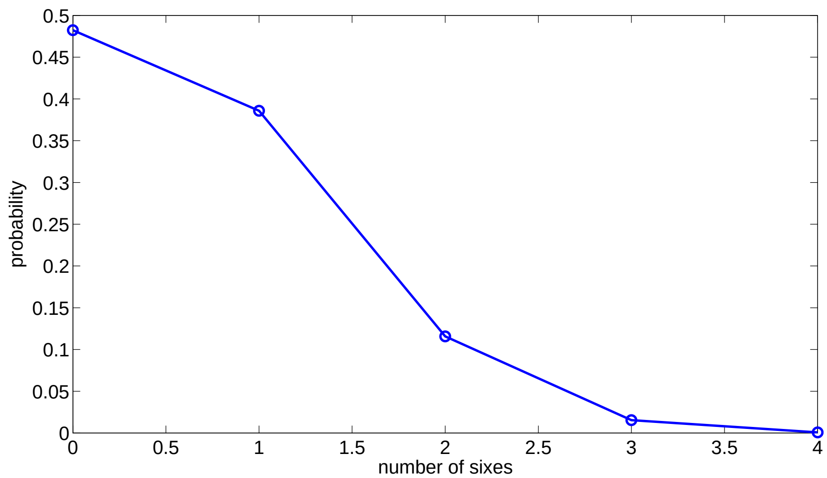

Figure 3: Probability of rolling k sixes when rolling 4 dice

9. **5.25.** We use Pascal's formula and induction. We define

$$P(n) = \text{``}\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for all } 0 \le k \le n.\text{''}$$

We first verify $P(1)$. Check that $\binom{1}{0} = \binom{1}{1} = 1$, and thus $P(1)$ is true. Now, let us assume $P(n)$. Then we need to show $P(n+1)$, or that

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n-k+1)!}, \forall 0 \le k \le n+1.$$

First assume that $1 \le k \le n$. By Pascal's Formula, we have

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \\
&= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k} + \frac{1}{n-k+1}\right) \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n-k+1+k}{k(n-k+1)} \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n+1}{k(n-k+1)} \\
&= \frac{(n+1)!}{k!(n-k+1)!}.
\end{aligned}$$

If we pick $k = 0, k = n+1$, we have $\binom{n}{k} = 1$ which also agrees with the formula, so we are done.