Fundamental Mathematics - 347 G1
Homework 9 – Solutions

1. **13.6.** The flaw in this reasoning is that we will never list any irrational number.

2. **13.8.** False. Let $S = \{0, 1\}$. Then $\sup(S) = 1, \inf(S) = 0$, so that $\sup(S) \in S$ and $\inf(S) \in S$, but S is not a closed interval.

3. **13.11.**

   (a) This is true. To prove this, let us say that

   $$\lim a_n = A, \quad \lim b_n = B,$$

   with $A < B$. Choose any $\epsilon < (B - A)/2$. We know from the definition of limit that there are N_1, N_2 such that for $n > N_1$, we have

   $$|a_n - A| < \epsilon,$$

   and for $n > N_2$, we have

   $$|b_n - B| < \epsilon.$$

   Choose $N = \max(N_1, N_2)$ and thus for $n > N$, we have

   $$|a_n - A| < \epsilon, \quad |b_n - B| < \epsilon.$$

   In particular, this means that $a_n < A + \epsilon$ and $b_n > B - \epsilon$. But since $A + \epsilon < B - \epsilon$, we have

   $$a_n < A + \epsilon < B - \epsilon < b_n,$$

   and we are done.

   (b) This is false. Consider

   $$a_n = 1 + \frac{1}{n}, \quad b_n = 1 - \frac{1}{n}.$$

   We have $\lim a_n = \lim b_n$ but not that $a_n \leq b_n$. The reason the argument breaks down in the first part is that if we tried to prove this like part (a) above, the above argument would have us choose $\epsilon = 0$, which is not ok.

4. **13.12.** False, use a similar example to above. Let $S = \{0, 1\}$ and let x_n be any sequence which tends to 0 and y_n any sequence which tends to 1. But $1/2 \notin S$.

5. **13.19.** Since $f(x)$ is bounded, so is $-f(x)$ (Why?) and therefore has an infimum and a supremum, as does f itself. Let us write S for the range of f and $-S$ for the range of $-f$. Thus

   $$-S = \{x \in \mathbb{R} : -x \in S\}.$$

   Let us write

   $$\alpha = \sup(-S).$$

   By Proposition 13.15, this means that α is an upper bound for $-S$, and that there is a sequence $\langle x \rangle \subseteq -S$ such that $x_n \to \alpha$. But if we then define

   $$y_n = -x_n,$$

   then clearly $y_n \to -\alpha$. Moreover, if α is an upper bound for $-S$, then $-\alpha$ is a lower bound for S. So $-\alpha$ is a lower bound for S, and there is a sequence in S which converges to it. Again by Proposition 13.15, this means that $-\alpha = \inf(S)$.

6. **13.20.**

   (a) $x_n = 1 - 1/n, \quad x_n = 1/n.$

   (b) Notice that $\inf(S) = 0$ and $\sup(S) = 3/2$. To get a sequence which converges to $\inf(S)$, just take the defining sequence itself, namely $x_n = (2 + (-1)^n)/n$. To get a sequence which converges to $\sup(S)$, take $x_n = 3/2$ for all n. (Not the most subtle sequence, but it works.)

7. **13.24.**

   (a) $f(x) = \sin(x), \quad g(x) = \sin(x) + 40.$

   (b) $f(x) = \sin(x), \quad g(x) = \sin(x) + 2.$

   (c) $f(x) = \sin(x), \quad g(x) = \sin(x) + 0.1.$

8. **13.29.** We define

$$x_n = \frac{1+n}{1+2n}.$$

We will show that $x_{n+1} < x_n$ for all n. To see this, let us write down the inquality we are trying to prove and do some algebra to see what we would need to make it true. So:

$$\begin{aligned} \frac{1+n}{1+2n} &> \frac{n+2}{2n+3}, \\ (2n+3)(n+1) &> (2n+1)(n+2), \\ 2n^2+5n+3 &> 2n^2+5n+2, \\ 3 &> 2. \end{aligned}$$

Well, clearly the last inequality is always true, thus so is the first one. By Monotone Convergence, we have a limit.

To see why we would expect the limit to be $1/2$, look at the formula and think of n as being really large. Then the $+1$ doesn't matter, so the fraction is roughly $n/2n = 1/2$. (Of course this is not yet a proof.)

So, let us try to prove this: we need to show that for every $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that $n > N$ implies

$$\left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \epsilon.$$

Let us do some algebra, this gives

$$\begin{aligned} \left| \frac{2(n+1)}{2(2n+1)} - \frac{2n+1}{2(2n+1)} \right| &< \epsilon, \\ \left| \frac{1}{2(2n+1)} \right| &< \epsilon, \\ \frac{1}{2(2n+1)} &< \epsilon, \\ 2(2n+1) &> \frac{1}{\epsilon}. \end{aligned}$$

For any positive ϵ, we can choose N so that

$$2(2N+1) > \frac{1}{\epsilon},$$

and therefore if $n > N$, we have

$$\left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \epsilon.$$