MA 347 Exam # 3

Name:____

- Please put all work that you want graded on these sheets.
- You may not use a calculator or any notes for this exam.
- All statements must be justified.

Number	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 pts.) Consider the function

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \\ a, & x = 0. \end{cases}$$

(a) Prove that this function continuous for $x \neq 0$.

(b) Is this function continuous at $x = 0$ for any value of a?

(c) Finally, consider $g(x) = |f(x)|$. Is there a value we can choose for a which makes $g(x)$ continuous at $x = 0$?

Solution:

(a) We will show this for $x > 0$, the proof is similar for $x < 0$. Choose and fix any $x > 0$. Notice that if we choose any z which is also positive, then $f(z) = f(x) = 1$. So, choose any $\delta < x$. Then $|x - z| < \delta$ implies that $z > 0$, and so we have, for all $\epsilon > 0$, if $|x - z| < \delta$, then $|f(x) - f(z)| = 0 < \epsilon$.

(b) There is no choice of a. It is easy to see that $\lim_{x \to 0} f(x)$ does not exist, and therefore we cannot choose a so that $f(0) = \lim_{x \to 0} f(x)$.

(c) Here we can. Notice that

$$g(x) = \begin{cases} 1, & x \neq 0, \\ |a|, & x = 0. \end{cases}$$

Clearly, choosing $|a| = 1$ makes this function constant everywhere and thus continuous (see proof of part (a)). So either $a = 1$ or $a = -1$ will work.

2. (20 pts.) Suppose that $\langle a \rangle, \langle b \rangle$ are two convergent sequences, and that

$$\lim a_n < \lim b_n.$$

Prove that there is some $N \in \mathbb{N}$ such that for all $n > N$,

$$a_n < b_n.$$

Solution: This is homework problem 13.11a, see the solutions to HW #9.

3. (20 pts.) For each case below, you are asked to give concrete examples of sequences $\langle a \rangle, \langle b \rangle$ which satisfy the condition, OR show why such a sequence cannot exist:

(a) $\lim a_n = 0$, $\lim b_n$ does not exist, $\lim(a_n b_n) = 0$.

(b) $\lim a_n = 0$, $\lim b_n$ does not exist, $\lim(a_n b_n)$ does not exist.

(c) $\lim a_n = 0$, $\lim b_n = 4$, $\lim(a_n b_n) = 2$.

(d) $\lim a_n = 0$, $\lim b_n$ does not exist, $\lim(a_n b_n) = 4$.

Solution:

(a) $a_n = 1/n^{10}, b_n = n, a_n b_n = 1/n^9$.

(b) $a_n = 1/n, b_n = n^{654}, a_n b_n = n^{653}$.

(c) Not possible, since the limits exist and are finite, we know that $\lim a_n b_n = 0 * 4 = 0$.

(d) $a_n = 1/n, b_n = 4n, a_n b_n = 4$.

4. (20 pts.) Show that $\sqrt{2}$ exists as a real number by using the Intermediate Value Theorem. (Hint: the function $f(x) = x^2$ is continuous!)

   Solution: We know that $f(1) = 1$ and $f(2) = 4$. Since $1 < 2 < 4$, by the Intermediate Value Theorem there exists at least one $x \in (1, 2)$ such that $f(x) = 2$.

   (Note that we are guaranteed a unique positive solution as well: since the function is increasing for positive x, it is injective on the positive reals.)

5. (20 pts.) In each case you are given a set S. Compute $\sup(S)$ and $\inf(S)$ or say why they do not exist.

(a) $S = \mathbb{N}$.

(b) $S = \mathbb{Z}$.

(c) $S = (0, 1)$.

(d) S is the range of $\cos(x)$.

Solution:

(a) $\inf(S) = 1$, because clearly $1 \leq n$ for any $n \in \mathbb{N}$, and moreover since $1 \in \mathbb{N}$ it must be the greatest lower bound. Since $\mathbb{N}$ is not bounded above (Archimedean Principle), it has no upper bound and thus no least upper bound, so $\sup(S)$ does not exist.

(b) no inf, no sup. It is easy to see that $\mathbb{Z}$ has neither lower bound nor upper bound.

(c) $\inf(S) = 0, \sup(S) = 1$. To see this, notice that 0 is a lower bound for S. moreover, if we choose any $x > 0$, then notice that $x/2 < x$, and $x/2 \in S$, so that x cannot be a lower bound for S. Therefore 0 is the greatest lower bound and $\inf(S) = 0$. The proof for $\sup(S) = 1$ is the same.

(d) The range of $\cos(x)$ is the set $[-1, 1]$, and it has $\inf(S) = -1$ and $\sup(S) = 1$. Again, notice that -1 is a lower bound for S, and since $-1 \in S$ it must be the greatest lower bound.