Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 1 — due September 11

1. Determine which of the following operators are linear:
   (a) $Lu = u_{xx} + u_{xy}$
   (b) $Lu = uu_x$
   (c) $Lu = 4x^2u_y - 4y^2u_{yy}$

2. (Strauss, 1.1.4) Show that the difference of two solutions to $Lu = g$ is a solution to $Lu = 0$, when L is any linear operator.

3. Solve:

$$\begin{aligned} 2u_x + 3u_t &= 0, \\ u(x,0) &= x^2. \end{aligned}$$

4. Consider the heat equation with initial condition given as

$$\begin{aligned} u_t &= u_{xx}, \\ u(x,0) &= \alpha x + \beta, \end{aligned}$$

   where α, β are real numbers. Make an educated guess for the solution to this PDE and check that it is correct. Interpret this as a statement about the evolution of temperature in a 1D object.

5. Determine the type of the following equations:
   (a) $u_{xx} + u_{xy} + u_{yy} + 3u_y = 0$,
   (b) $9u_{xx} - u_y = 0$.
   (c) Now, for the equation $u_{xx} + 3u_{xy} + \alpha u_{yy} = 0$, determine which values of α make the equation elliptic.

6. We define the operator ∂_x by the equation $\partial_x u = \dfrac{\partial u}{\partial x}$, ∂_{xy} by $\partial_{xy} u = \frac{\partial^2 u}{\partial x \partial y}$, and similarly for other independent variables. Moreover, when we concatenate operators, we take it to mean composition, i.e.

$$LMu := L(M(u)).$$

   (a) Show that $\partial_x^2 = \partial_{xx}$.
   (b) Show that

$$(\alpha\partial_x + \beta\partial_y)^2 = \alpha^2\partial_{xx} + 2\alpha\beta\partial_{xy} + \beta^2\partial_{yy},$$

   i.e. that they multiply like polynomials.
   (c) From this, prove that if α, β, γ are complex numbers, then there exist λ_1, λ_2 complex such that

$$\alpha\partial_{xx} + \beta\partial_{xy} + \gamma\partial_{yy} = \alpha(\partial_x - \lambda_1\partial_y)(\partial_x - \lambda_2\partial_y).$$

   Compute λ_1, λ_2 in terms of α, β, γ.