

Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 3 — due September 28

1. Here we will prove that solutions to the heat equation satisfy (some of) the invariance principles mentioned in class, or in the book in §2.4. That is, if $u(x, t)$ is a solution to $u_t = ku_{xx}$ for $x \in \mathbb{R}, t > 0$, then so are

- (a) $u(x - y, t)$ for any fixed y ,
- (b) u_x, u_t ,
- (c) $v(x, t) = \int_{-\infty}^{\infty} u(x - y, t)g(y) dy$ where g has finite support,
- (d) $v(x, t) = u(\sqrt{a}x, at)$ for any $a > 0$.

2. (**Strauss 2.4.1.**) Solve the heat equation with initial condition

$$\phi(x) = \begin{cases} 1, & |x| < L, \\ 0, & |x| \geq L. \end{cases}$$

(You can use the formula for the solution as derived in class, but there is a simpler way to build this solution using the invariance principles above.)

3. (**Strauss 2.4.8.**) Show that the tails of

$$S(x, t) = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/4kt}$$

are uniformly small for small times, i.e. that for any $\delta > 0$,

$$\lim_{t \rightarrow 0} \max_{|x| > \delta} S(x, t) = 0.$$

Interpret this in terms of speed of propagation of information for solutions of the heat equation.

4. (**Strauss 2.4.9.**) We will write down an exact solution to the heat equation

$$\begin{aligned} u_t &= ku_{xx}, \\ u(x, 0) &= x^2, \end{aligned}$$

but not using the formula derived in class. The idea is as follows.

- (a) Show that u_{xxx} solves the heat equation with initial condition zero,
 - (b) Use uniqueness to show $u_{xxx}(x, t) \equiv 0$,
 - (c) From this we can deduce that $u(x, t) = A(t)x^2 + B(t)x + C(t)$ for some functions A, B, C (Why?),
 - (d) Solve for A, B, C .
5. Generalize the previous problem to a general initial condition which is a polynomial of x . (You don't need to compute anything exactly here, just describe the algorithm which would allow you to obtain a solution.)