Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 3 Solutions

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1. Here we will prove that solutions to the heat equation satisfy (some of) the invariance principles mentioned in class, or in the book in §2.4. That is, if $u(x,t)$ is a solution to $u_t = ku_{xx}$ for $x \in \mathbb{R}, t > 0$, then so are

   (a) $u(x-y,t)$ for any fixed y,

   (b) u_x, u_t,

   (c) $v(x,t) = \int_{-\infty}^{\infty} u(x-y,t)g(y)\,dy$ where g has finite support,

   (d) $v(x,t) = u(\sqrt{a}x, at)$ for any $a > 0$.

   Solution:

   (a) Let $v(x,t) = u(x-y,t)$. Then

$$\begin{aligned}\frac{\partial v}{\partial t}(x,t) &= \frac{\partial u}{\partial t}(x-y,t)\cdot 1 = \frac{\partial u}{\partial t}(x-y,t),\\ \frac{\partial v}{\partial x}(x,t) &= \frac{\partial u}{\partial x}(x-y,t)\cdot 1 = \frac{\partial u}{\partial x}(x-y,t),\\ \frac{\partial^2 v}{\partial x^2}(x,t) &= \frac{\partial}{\partial x}\frac{\partial v}{\partial x}(x,t) = \frac{\partial}{\partial x}\frac{\partial u}{\partial x}(x-y,t) = \frac{\partial^2 u}{\partial x^2}(x-y,t).\end{aligned}$$

   Then

$$\frac{\partial v}{\partial t}(x,t) - k\frac{\partial^2 v}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x-y,t) - k\frac{\partial^2 u}{\partial x^2}(x-y,t) = 0,$$

   since u solves the heat equation.

   (b) We compute for u_x, the other is similar. Denoting $v = u_x$ gives

$$\begin{aligned}v_t &= (u_x)_t = u_{xt},\\ v_{xx} &= (u_x)_{xx} = u_{xxx}.\end{aligned}$$

   Then

$$v_t - kv_{xx} = u_{xt} - u_{xxx} = u_{tx} - u_{xxx} = (u_t - u_{xx})_x = 0_x = 0.$$

   (c) Since g has compact support, we can exchange derivatives and integration (see e.g. Theorem A.3.2 from Strauss), and thus we have

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x-y,t)g(y)\,dy = \int_{-\infty}^{\infty}\frac{\partial u}{\partial t}(x-y,t)g(y)\,dy$$

   and

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} u(x-y,t)g(y)\,dy = \int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}(x-y,t)g(y)\,dy.$$

   But then

$$\begin{aligned}v_t - kv_{xx} &= \int_{-\infty}^{\infty}\frac{\partial u}{\partial t}(x-y,t)g(y)\,dy - \int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}(x-y,t)g(y)\,dy\\ &= \int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial t}(x-y,t) - \frac{\partial^2 u}{\partial x^2}(x-y,t)\right)g(y)\,dy = \int_{-\infty}^{\infty} 0\,dy = 0.\end{aligned}$$

(d) We have

$$\begin{aligned}
\frac{\partial v}{\partial t}(x,t) &= a\frac{\partial u}{\partial t}(\sqrt{a}x, at),\\
\frac{\partial v}{\partial x}(x,t) &= \sqrt{a}\frac{\partial u}{\partial x}(\sqrt{a}x, at),\\
\frac{\partial^2 v}{\partial x^2}(x,t) &= a\frac{\partial^2 u}{\partial x^2}(\sqrt{a}x, at).
\end{aligned}$$

So then

$$v_t - kv_{xx} = au_t(\sqrt{a}x, at) - au_{xx}(\sqrt{a}x, at) = a(u_t(\sqrt{a}x, at) - u_{xx}(\sqrt{a}x, at)) = a \cdot 0 = 0.$$

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2. **(Strauss 2.4.1.)** Solve the heat equation with initial condition

$$\phi(x) = \begin{cases} 1, & |x| < L, \\ 0, & |x| \geq L. \end{cases}$$

(You can use the formula for the solution as derived in class, but there is a simpler way to build this solution using the invariance principles above.)

Solution: We will solve two ways, the first using the formula. We have

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy,$$

where

$$S(x,t) = \frac{1}{4\pi kt}e^{-x^2/4kt}.$$

We can then write

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{-L} S(x-y,t)\phi(y)\,dy + \int_{-L}^{L} S(x-y,t)\phi(y)\,dy + \int_{L}^{\infty} S(x-y,t)\phi(y)\,dy \\
&= \int_{-\infty}^{-L} 0\,dy + \int_{-L}^{L} S(x-y,t)\,dy + \int_{L}^{\infty} 0\,dy \\
&= \int_{-L}^{L} S(x-y,t)\,dy,
\end{aligned}$$

so we need to evaluate

$$\frac{1}{\sqrt{4\pi kt}}\int_{-L}^{L} e^{-(x-y)^2/4kt}\,dy.$$

Changing variables with $s = (x-y)/\sqrt{4kt}, ds = -1/\sqrt{4kt}\,dy$, gives

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{\frac{x+L}{\sqrt{4kt}}}^{\frac{x-L}{\sqrt{4kt}}} e^{-s^2}(-\sqrt{4kt})\,ds \\
&= \frac{1}{\sqrt{\pi}}\int_{\frac{x-L}{\sqrt{4kt}}}^{\frac{x+L}{\sqrt{4kt}}} e^{-s^2}\,ds \\
&= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x+L}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-L}{\sqrt{4kt}}\right)\right).
\end{aligned}$$

A completely different method is to use the transformations in Question #1, particularly (a). We know that if the initial condition is the Heaviside function $H(x)$, then we get the solution

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right).$$

We can see that our initial condition $\phi(x)$ can be written

$$\phi(x) = H(x+L) - H(x-L).$$

(One can either draw this, or check it algebraically: if $x > L$, then $H(x+L) = H(x-L) = 1$, if $x < -L$, then $H(x+L) = H(x-L) = 0$, and if $-L < x < L$, then $H(x-L) = 0$ but $H(x+L) = 1$.) From 1(a) above, we know that $Q(x+L,t)$ and $Q(x-L,t)$ are both solutions to the heat equation, and clearly they have initial conditions $H(x+L)$ and $H(x-L)$, respectively. By linearity, we know that

$$u(x,t) = Q(x+L,t) - Q(x-L,t) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x+L}{\sqrt{4kt}}\right) - \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{x-L}{\sqrt{4kt}}\right)$$

is also a solution, and it clearly satisfies $u(x,0) = \phi(x)$.

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3. **(Strauss 2.4.8.)** Show that the tails of

$$S(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt}$$

are uniformly small for small times, i.e. that for any $\delta > 0$,

$$\lim_{t\to 0} \max_{|x|>\delta} S(x,t) = 0.$$

Interpret this in terms of speed of propagation of information for solutions of the heat equation.

Solution: We first compute the inner term, namely

$$\max_{|x|>\delta} S(x,t).$$

First, note that for $x > 0$ and any fixed $t > 0$, $S(x,t)$ is monotone decreasing, because

$$\frac{\partial S}{\partial x}(x,t) = \frac{-2x}{\sqrt{\pi}(4kt)^{3/2}} e^{-x^2/4kt} < 0.$$

Therefore,

$$\max_{x>\delta} S(x,t) = S(\delta,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\delta^2/4kt}.$$

By evenness of S, we also have that $\max_{x<-\delta} S(x,t) = \max_{x>\delta} S(x,t)$, so

$$\max_{|x|>\delta} S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\delta^2/4kt}.$$

So it remains to compute

$$\lim_{t\to 0} \frac{e^{-\delta^2/4kt}}{\sqrt{4\pi k t}}.$$

This is an indeterminate form, since when we plug in $t = 0$ we obtain $0/0$. The first guess might be to try l'Hopital's rule, but this will actually not work out, because every time we differentiate the numerator, we will get a higher power of t in the denominator.

To simplify the expression, rewrite this limit as

$$\lim_{t\to 0} \frac{e^{-C_1/t}}{C_2\sqrt{t}}$$

and make the change of variables $s = 1/t$, which then gives

$$\lim_{s\to\infty} \frac{\sqrt{s}e^{-C_1 s}}{C_2} = \lim_{s\to\infty} \frac{\sqrt{s}}{C_2 e^{C_1 s}}.$$

This is still an indeterminate form of ∞/∞, but now using l'Hopital's Rule gives

$$\lim_{s\to\infty} \frac{s^{-1/2}/2}{C_1 C_2 e^{C_1 s}} = \frac{0}{\infty} = 0.$$

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4. **(Strauss 2.4.9.)** We will write down an exact solution to the heat equation

$$\begin{aligned} u_t &= k u_{xx}, \\ u(x,0) &= x^2, \end{aligned}$$

but not using the formula derived in class. The idea is as follows.

(a) Show that u_{xxx} solves the heat equation with initial condition zero,

(b) Use uniqueness to show $u_{xxx}(x,t) \equiv 0$,

(c) From this we can deduce that $u(x,t) = A(t)x^2 + B(t)x + C(t)$ for some functions A, B, C (Why?),

(d) Solve for A, B, C.

Solution:

(a) This part is similar to problem #1. If we write $v = u_{xxx}$, then

$$\begin{aligned} v_t &= u_{xxxt} = u_{txxx}, \\ v_{xx} &= u_{xxxxx}, \end{aligned}$$

and thus

$$v_t - k v_{xx} = u_{txxx} - k u_{xxxxx} = (u_t - k u_{xx})_{xxx} = 0_{xxx} = 0.$$

Moreover, notice that $v(x,0) = u_{xxx}(x,0) = (x_2)_{xxx} = 0$.

(b) We know solutions to the heat equation are unique. Moreover, we know that v solves the heat equation with $v(x,0) = 0$. However, it is easy to see that if we define $w(x,t) \equiv 0$ for all x, t, then w satisfies the heat equation and $w(x,0) = 0$. Therefore $v \equiv w$ and $v(x,t) \equiv 0$.

(c) We know that

$$\begin{aligned} u_{xxx}(x,t) &= 0, \\ u_{xx}(x,t) &= A(t), \\ u_x(x,t) &= A(t)x + B(t), \\ u(x,t) &= \frac{1}{2}A(t)x^2 + B(t)x + C(t), \end{aligned}$$

where $A(t), B(t), C(t)$ are arbitrary functions of t. Now redefine A to get rid of the 1/2 since it's arbitrary anyway.

(d) We know that u satisfies the heat equation, so we have

$$\begin{aligned} u_t &= A'(t)x^2 + B'(t)x + C'(t), \\ u_{xx} &= 2A(t), \end{aligned}$$

and if these are equal as functions, this gives

$$C'(t) = 2A(t), \quad B'(t) = 0, \quad A'(t) = 0.$$

Solving the last two are easy ($A(t) = A_0$,$B(t) = B_0$) and then the first becomes

$$C(t) = 2A_0 t + C_0.$$

Putting this together gives

$$u(x,t) = A_0 x^2 + B_0 x + (2A_0 t + C_0).$$

Plugging in the initial condition gives

$$u(x,0) = A_0 x^2 + B_0 x + C_0,$$

which means that $A_0 = 1, B_0 = C_0 = 0$, so the solution is

$$u(x,t) = x^2 + 2t.$$

We could, alternately, plug in the initial conditions as soon as we have the equations, namely say that

$$\begin{aligned} A'(t) &= 0, \quad A(0) = 1, \\ B'(t) &= 0, \quad B(0) = 0, \\ C'(t) &= 2A(t), \quad C(0) = 0, \end{aligned}$$

and directly solve to get $A(t) = 1$ and $C(t) = 2t$.

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5. Generalize the previous problem to a general initial condition which is a polynomial of x. (You don't need to compute anything exactly here, just describe the algorithm which would allow you to obtain a solution.)

Solution: The general idea is as follows. Let's say that $u(x,0) = p(x)$, where $p(x)$ is a polynomial of degree n:

$$p(x) = \sum_{k=0}^{n} \alpha_k x^k.$$

Then notice that if we take $n+1$ derivatives of p we get zero. Therefore, if u satisfies the heat equation with initial condition $p(x)$, then $\dfrac{\partial^{n+1} u}{\partial x^{n+1}}$ solves the heat equation with initial condition zero. Therefore we have $\dfrac{\partial^{n+1} u}{\partial x^{n+1}} \equiv 0$, and by the same argument we know

$$u(x,t) = \sum_{k=0}^{n} A_k(t) x^k.$$

Plugging this into the heat equation gives

$$\begin{aligned} u_t(x,t) &= \sum_{k=0}^{n} A_k'(t)x^k, \\ u_{xx}(x,t) &= \sum_{k=0}^{n} k(k-1)A_k(t)x^{k-2} = \sum_{k=0}^{n-2}(k+2)(k+1)A_{k+2}(t)x^k. \end{aligned}$$

Setting these equal gives

$$A_n'(t) = A_{n-1}'(t) = 0, \quad A_k'(t) = (k+2)(k+1)A_{k+2}(t),$$

and plugging in initial conditions gives $A_k(0) = \alpha_k$ for all k. The first two equations can be solved easily:

$$A_n(t) = \alpha_n, \quad A_{n-1}(t) = \alpha_{n-1},$$

and then the other equations can be solved recursively, e.g.

$$A_{n-2}'(t) = n(n-1)A_n(t) = n(n-1)\alpha_n,$$

so

$$A_{n-2}(t) = n(n-1)\alpha_n t + \alpha_{n-2},$$

etc.