# Fundamental Mathematics - 556 X1
# Homework 1
# Due September 8, 2008

1. (1.1.2. from Keener) Show that in any inner product space,
$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right). \tag{1}$$
Interpret this geometrically in $\mathbb{R}^2$.

2. (Problem 1.1.3 from Keener.)
   (a) Verify that in an inner product space,
   $$\mathrm{Re}\ \langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$
   (b) Show that in any real inner product space there is at most one inner product which generates the same induced norm.
   (c) In $\mathbb{R}^n$ with $n > 1$, show that
   $$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$
   can be induced by an inner product if and only if $p = 2$. (*Hint:* Use both 1.1.2. and 1.1.3. here!)

3. (Problem 1.1.5 from Keener.) Show that
$$\langle f, g\rangle = \int_0^1 \left(f(x)\overline{g(x)} + f'(x)\overline{g'(x)}\,dx\right)$$
is an inner product on the space of all continuously differentiable functions defined on $[0,1]$, i.e. on
$$C^1([0,1]) := \{f\colon [0,1] \to \mathbb{C} : f(x), f'(x) \text{ are continuous}\}.$$

4. (Problem 1.1.8 from Keener.) Verify that the choice $\gamma = \langle x, y\rangle / \|y\|^2$ minimizes $\|x-\gamma y\|^2$. Show then that $|\langle x, y\rangle|^2 = \|x\|^2 + \|y\|^2$ if and only if $x$ and $y$ are linearly dependent.

5. (Problem 1.1.9 from Keener.) For any $w(x) > 0$, we can define the inner product
$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\,dx$$
for real continuous functions defined on $[a,b]$ and taking real values. ($w(x)$ is a "weight" function — think of it like a weighted average — we have considered the case where $w \equiv 1$ before.) Start with the basis $\{1, x, x^2, x^3, x^4\}$ for $P_4$. Use the Gram-Schmidt algorithm to generate an orthogonal set of polynomials when we choose:
   (a) $a = -1, b = 1, w(x) \equiv 1$ (Legendre polynomials),
   (b) $a = -1, b = 1, w(x) = (1-x^2)^{-1/2}$ (Chebyshev polynomials),
   (c) $a = 0, b = \infty, w(x) = e^{-x}$ (Laguerre polynomials),
   (d) $a = -\infty, b = \infty, w(x) = e^{-x^2}$ (Hermite polynomials).

   *Hint:* You might find a computer algebra system (e.g. Maple, Mathematica) useful here, but you can do it by hand with some work as well.

6. (Problem 1.2.2 from Keener.)
   (a) Prove that two symmetric matrices are equivalent if and only if they have the same eigenvalues (with the same multiplicities).
   (b) Show that if $A$ and $B$ are equivalent, then $\det A = \det B$.
   (c) Is the converse true?

7. (Problem 1.2.4 from Keener.) Show that the eigenvalues of an anti-self-adjoint matrix ($A^* = -A$) are imaginary.

8. (Problem 1.2.5 from Keener.) Find a basis for the range and null space of the following matrices:
   (a)
$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{pmatrix}$$
   (b)
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

9. (Problem 1.2.6 from Keener.) Find an invertible matrix $T$ and a diagonal matrix $\Lambda$ so that $A = T\Lambda A T^{-1}$ for each of the following matrices $A$:
   (a)
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix},$$
   (b)
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
   (c)
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix},$$
   (d)
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$
   (e)
$$A = \begin{pmatrix} 1/2 & 1/2 & \sqrt{3}/6 \\ 1/2 & 1/2 & \sqrt{3}/6 \\ \sqrt{3}/6 & \sqrt{3}/6 & 5/6 \end{pmatrix}.$$

10. (Problem 1.2.9 from Keener.) The two sets of vectors $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ in an inner product space $S$ are said to be **biorthogonal** if $\langle \phi_i, \psi_j \rangle = \delta_{ij}$. Assume that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are biorthogonal.
    (a) Show that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ each form linearly independent sets.

(b) Show that if $S$ is $n$-dimensional, then any vector $x \in S$ can be written as

$$x = \sum_{i=1}^{n} \alpha_i \phi_i \tag{2}$$

where $\alpha_i = \langle x, \psi_i \rangle$.

(c) Express (2) in matrix form, i.e. show that

$$x = \sum_{i=1}^{n} P_i x$$

where $P_i$ are projection matrices with the properties that $P_i^2 = P_i$ and $P_i P_j = 0$ whenever $i \neq j$. Express the matrix $P_i$ in terms of the vectors $\phi_i, \psi_i$.