Methods of Mathematical Physics - 556 X1
Homework 3 Solutions

1. *(Problem 2.1.1 from Keener.) Verify that ℓ^2 is an inner product space. Specifically, show that if $x, y \in \ell^2$, then*

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$$

is defined and satisfies the properties of an inner product. (Here we're assuming that our sequences are real, so no need for the complex conjugate.)

Hint: *Think about how we proved Bessel's Inequality in class.*

Solution. The hardest part of this will be to show that if $x, y \in \ell^2$, then $\langle x, y\rangle$ is finite and $x + y \in \ell^2$; verifying everything else will be straightforward. We have

$$\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k y_k.$$

By the Cauchy-Schwarz inequality for the standard dot product on $\mathbb{R}^n$, we know that

$$\left(\sum_{k=1}^{n} |x_k y_k|\right)^2 \le \sum_{k=1}^{n} x_k^2 \cdot \sum_{k=1}^{n} y_k^2 \le \sum_{k=1}^{\infty} x_k^2 \cdot \sum_{k=1}^{\infty} y_k^2.$$

The right-hand side is finite, and moreover it is independent of n, and thus $\sum_{k=1}^{\infty} |x_k y_k|$ is a convergent sequence. Since we have taken absolute values, this means that the series $\sum_{k=1}^{\infty} x_k y_k$ is an absolutely convergent sequence, and thus converges as well. Moreover, note now that if $x, y \in \ell^2$, then

$$\sum_{k=1}^{\infty} (x_k + y_k)^2 = \sum_{k=1}^{\infty} x_k^2 + 2x_k y_k + y_k^2 < \infty,$$

so $x + y \in \ell^2$. Of course $\alpha x \in \ell^2$ for all $\alpha \in \mathbb{R}$ as well, and this makes ℓ^2 a vector space. Proving that the remaining axioms of an inner product are satisfied is straightforward at this point.

2. *(Problem 2.1.3 from Keener.) Show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k!}$ is a Cauchy sequence. Since the reals are complete, this means it converges. To which number does this sequence converge?*

Solution. We need to check that for any $\epsilon > 0$, there is an N such that $n, m > N$ means that

$$|x_n - x_m| < \epsilon.$$

But we have

$$|x_n - x_m| = \left|\sum_{k=m+1}^{n} \frac{1}{k!}\right| \le \left|\sum_{k=m+1}^{n} \frac{1}{k^2}\right|.$$

But we can replace a sum with an integral with only adding perhaps a constant (think of the Riemann sums, for example), and we have

$$\left|\sum_{k=m+1}^{n} \frac{1}{k^2}\right| \le \int_m^n \frac{1}{x^2}\,dx = \frac{1}{m} - \frac{1}{n} = \frac{n-m}{nm} < \frac{1}{m}.$$

So if we choose $N = 1/\epsilon$, then if $m, n > N$ then $|x_n - x_m| < \epsilon$. Therefore this is a Cauchy sequence and thus converges. (Of course, replacing the factorial with a square threw away a lot — this sum converges much faster than k^{-2}.)

Finally, we know from calculus that

$$e^x = \sum_{k=0}^{\infty} = \frac{x^k}{k!},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} - 1 = e - 1.$$

3. *Show that the sequence* $x_n = \sum_{k=1}^{n} \frac{1}{k}$ *is not a Cauchy sequence.*

   Solution. Let us try to bound $|x_n - x_m|$, where we get

   $$|x_n - x_m| = \left| \sum_{k=m+1}^{n} \frac{1}{k} \right| \geq \int_{m+1}^{n} \frac{1}{x}\, dx = \log\left(\frac{n}{m+1}\right).$$

   Choose $n = \alpha(m+1)$, then this difference is at least $\log \alpha$. This means that no matter how large m is, we can make this difference large by choose n large enough. Therefore the sequence is not Cauchy.

4. *Consider the Hilbert space* L^2. *Prove that the list of vectors*

   $$\{\cos(nx)\}_{n=0}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$$

   is an infinite orthogonal list in L^2 *with respect to the inner product*

   $$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx.$$

   Hint: *You will need some trigonometric identities to solve this problem, e.g. you will need to compute integrals like*

   $$\int_0^{2\pi} \cos(mx)\cos(nx)\, dx.$$

   Recall that we can use Euler's formula to get, for example,

   $$\begin{aligned}\cos((m+n)x) &= \mathrm{Re}(e^{i(m+n)x}) = \mathrm{Re}(e^{imx}e^{inx}) \\ &= \mathrm{Re}((\cos(mx) + i\sin(mx))(\cos(nx) + i\sin(nx)) = \cos(mx)\cos(nx) - \sin(mx)\sin(nx).\end{aligned}$$

   If you recombine these formulas in a clever way, you can do all of the integrals.

   Solution. We need to show that

   $$\begin{aligned}\int_0^{2\pi} \cos(mx)\cos(nx)\, dx &= \delta_{mn} C_m, \\ \int_0^{2\pi} \sin(mx)\sin(nx)\, dx &= \delta_{mn} S_m, \\ \int_0^{2\pi} \cos(mx)\sin(nx)\, dx &= 0,\end{aligned}$$

   where C_m and S_m are some constants, and then we are done.

   Use the formula above for $\cos((m+n)x)$, and note then that

   $$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(mx)\sin(nx)$$

   (cosine is even and sine is odd!) and then we have

   $$\cos((m+n)x) + \cos((m-n)x) = 2\cos(mx)\cos(nx).$$

Therefore

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_0^{2\pi} \cos((m+n)x) + \cos((m-n)x)\,dx \quad = \frac{1}{2}(2\pi\delta_{m+n,0} + 2\pi\delta_{m-n,0}).$$

Since $m, n \geq 0$, we can only have $m + n = 0$ if $m = n = 0$. Then we have

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 2\pi, & m = n = 0, \\ \pi, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

Similarly, we have

$$\sin(mx)\sin(nx) = \frac{1}{2}(\cos((m-n)x) - \cos((m+n)x)),$$

and thus

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_0^{2\pi} \cos((m-n)x) - \cos((m+n)x)\,dx \quad = \frac{1}{2}(2\pi\delta_{m-n,0} - 2\pi\delta_{m+n,0}).$$

Since $m, n > 0$, we cannot have $m + n = 0$. Then

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi, & m = n, \\ 0, & m \neq n. \end{cases}$$

Finally, we need the other Euler's formula, namely

$$\begin{aligned}\sin((m+n)x) &= \mathrm{Im}(e^{i(m+n)x}) = \mathrm{Im}(e^{imx}e^{inx}) \\ &= \mathrm{Im}((\cos(mx) + i\sin(mx))(\cos(nx) + i\sin(nx)) = \sin(mx)\cos(nx) + \cos(mx)\sin(nx).\end{aligned}$$

This gives

$$\sin((m+n)x) + \sin((m-n)x) = 2\sin(mx)\cos(nx),$$

so

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \sin((m+n)x) + \sin((m-n)x)\,dx = 0$$

(recall that $\sin(0x) = 0$ for all x and thus its integral is 0 as well).

5. (Problem 2.2.1 from Keener.) *Find the best quadratic polynomial fit to the function $f(x) = |x|$, where we choose as inner product*

   $$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\omega(x)\,dx,$$

   for each of the weights $\omega(x) = 1, \sqrt{1-x^2}, (1-x^2)^{-1/2}$.

   Hint: *You might find it convenient to compute some orthogonal polynomials for each weight and then compute the answer in terms of these polynomials — and we already did most of the work here on the last homework!*

   Solution. See attached Mathematica notebook/pdf.

6. (Problem 2.2.9 from Keener.) Suppose that $\{\phi_n(x)\}_{n=0}^{\infty}$ is a set of orthonormal polynomials, where we choose the inner product

   $$\langle f, g\rangle = \int_a^b f(x)g(x)\omega(x)\,dx,$$

   ($\omega(x) > 0$) and assume that $\phi_n(x)$ is a polynomial of degree n with leading coefficient k_n (specifically, we mean that

   $$\phi_n(x) = k_n x^n + (\text{terms of power } n-1 \text{ or less}).$$

   Then show:

(a) If f is a polynomial of degree less than n, then $\langle \phi_n, f \rangle = 0$.

(b) Show that every polynomial of degree n can be written in the form

$$\sum_{i=0}^{n} \alpha_i \phi_i$$

for some numbers α_i.

(c) the polynomials satisfy a recurrence relation of the form

$$\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) - C_n \phi_{n-1}(x),$$

for every n, where $A_n = k_{n+1}/k_n$. Compute B_n, C_n in terms of A_n, A_{n-1}, ϕ_n.

Hint: What do we know about $\phi_{n+1}(x) - A_n x \phi_n(x)$? Use part (b), take the inner product with ϕ_j, what do you get? Also, notice that for this inner product, $\langle xf, g \rangle = \langle f, xg \rangle$.

Solution. It's slightly more efficient to do (b) first and then (a). To prove (b), we will use induction. If $n = 0$, then $\phi_0 = k_0$, and if f is degree 0, then $f \equiv \beta$ for some $\beta \in \mathbb{R}$, so we have

$$f = \frac{\beta}{k_0}\phi_0.$$

(I'll also work out the $n = 1$ case directly to give more of the idea.)

Let $n = 1$, so we have $\phi_0 = k_0, \phi_1 = k_1 x + C$. Now, if f is degree one then

$$f(x) = \beta_1 x + \beta_0.$$

Then if we have

$$f(x) = \alpha_0\phi_0 + \alpha_1\phi_1,$$

then

$$f(x) = \alpha_0 k_0 + \alpha_1(k_1 x + C) = \alpha_1 k_1 x + (\alpha_1 C + \alpha_0 k_0),$$

so we choose α_0, α_1 to solve the two-by-two system

$$\alpha_1 k_1 = \beta_1, \quad \alpha_1 C + \alpha_0 k_0 = \beta_0.$$

Now, we use induction. Let us assume that for any polynomial f of degree n, we can write f as a linear combination of $\{\phi_0, \dots, \phi_n\}$. Now assume f is degree $n+1$, where

$$f(x) = \beta_{n+1}x^{n+1} + \dots$$

Then

$$g(x) := f(x) - \frac{\beta_{n+1}}{k_{n+1}}\phi_{n+1}$$

is a polynomial of degree n (since we picked constants to kill off the first term). By the induction hypothesis, we can then write

$$g(x) = \sum_{i=0}^{n} \alpha_i \phi_i,$$

so then

$$f(x) = \frac{\beta_{n+1}}{k_{n+1}}\phi_{n+1} + \sum_{i=0}^{n} \alpha_i \phi_i,$$

which is a linear combination of $\{\phi_0, \dots, \phi_{n+1}\}$, and we are done.

Now, to prove part (a), assume f is a polynomial of degree $k < n$. Then

$$f(x) = \sum_{i=0}^{k} \alpha_i \phi_i,$$

so

$$\langle f, \phi_n \rangle = \left\langle \sum_{i=0}^{k} \alpha_i \phi_i, \phi_n \right\rangle = \sum_{i=0}^{k} \alpha_i \langle \phi_i, \phi_n \rangle = 0.$$

Finally, we do part (c). Basically, we use the ideas above, plus a clever trick or two. First of all, we know by assumption that

$$\begin{aligned} \phi_{n+1} &= k_{n+1} x^{n+1} + O(x^n), \\ \phi_n &= k_n x^n + O(x^{n-1}). \end{aligned}$$

From this, we know

$$f(x) := \phi_{n+1} - \frac{k_{n+1}}{k_n} x \phi_n$$

is a polynomial of degree n. We then know, from part (b) above, that

$$f(x) = \sum_{i=0}^{n} \alpha_i \phi_i$$

for some constants α_i. Note further that since the ϕ_i are orthonormal, we know that

$$\alpha_j = \langle f, \phi_j \rangle = \left\langle \phi_{n+1} - \frac{k_{n+1}}{k_n} x \phi_n, \phi_j \right\rangle.$$

Since $j \leq n$, $\langle \phi_{n+1}, \phi_j \rangle = 0$, so that

$$\alpha_j = -\frac{k_{n+1}}{k_n} \langle x\phi_n, \phi_j \rangle.$$

Now, of course, we have no idea what $\langle x\phi_n, \phi_j \rangle$ is, in general. However, we have one trick up our sleeve: notice that for any functions f, g, we have

$$\langle xf, g \rangle = \langle f, xg \rangle,$$

because of the way we've defined our inner product.

NB. Of course, we cannot do this for every inner product, but this one has a special form.

So we then have

$$\langle x\phi_n, \phi_j \rangle = \langle \phi_n, x\phi_j \rangle,$$

and if $j < n - 1$, then the degree of $x\phi_j$ is less than n, and by part (a) this is zero. Therefore we know

$$f(x) = \alpha_n \phi_n + \alpha_{n-1} \phi_{n-1},$$

so define $B_n = \alpha_n, C_n = -\alpha_{n-1}$, and we have established the formula for some B_n, C_n. It remains to compute B_n, C_n. We will use the two equations

$$\langle \phi_{n+1}, \phi_n \rangle = 0, \quad \langle \phi_{n+1}, \phi_{n-1} \rangle = 0. \tag{1}$$

From the first equation in (1), we have

$$\begin{aligned} 0 = \langle \phi_{n+1}, \phi_n \rangle &= \langle A_n x\phi_n + B_n \phi_n - C_n \phi_{n-1}, \phi_n \rangle \\ &= A_n \langle x\phi_n, \phi_n \rangle + B_n \langle \phi_n, \phi_n \rangle - C_n \langle \phi_{n-1}, \phi_n \rangle. \end{aligned} \tag{2}$$

Using the fact that

$$\langle \phi_n, \phi_n \rangle = 1, \quad \langle \phi_{n-1}, \phi_n \rangle = 0,$$

equation (2) becomes

$$B_n = -A_n \langle x\phi_n, \phi_n \rangle .$$

From the second equation in (1), we have

$$\begin{aligned} 0 = \langle \phi_{n+1}, \phi_{n-1} \rangle &= \langle A_n x\phi_n + B_n\phi_n - C_n\phi_{n-1}, \phi_{n-1} \rangle \\ &= A_n \langle x\phi_n, \phi_{n-1} \rangle + B_n \langle \phi_n, \phi_{n-1} \rangle - C_n \langle \phi_{n-1}, \phi_{n-1} \rangle . \end{aligned} \tag{3}$$

Similarly, this becomes

$$C_n = A_n \langle x\phi_n, \phi_{n-1} \rangle .$$

We would now like to write this in terms of only ϕ_n. But notice that

$$\begin{aligned} \langle x\phi_n, \phi_{n-1} \rangle = \langle \phi_n, x\phi_{n-1} \rangle &= \left\langle \phi_n, x(k_{n-1}x^{n-1} + O(x^{n-2})) \right\rangle \\ &= \left\langle \phi_n, k_{n-1}x^n + O(x^{n-1}) \right\rangle = k_{n-1} \langle \phi_n, x^n \rangle . \end{aligned}$$

To compute the last term there, notice that

$$1 = \langle \phi_n, \phi_n \rangle = \left\langle \phi_n, k_n x^n + O(x^{n-1}) \right\rangle = k_n \langle \phi_n, x^n \rangle ,$$

so that

$$\langle \phi_n, x^n \rangle = \frac{1}{k_n},$$

and thus

$$\langle x\phi_n, \phi_{n-1} \rangle = \frac{k_{n-1}}{k_n} = \frac{1}{A_{n-1}}.$$

7. (Problem 2.2.10 from Keener.) **Problem fixed!** Consider the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\omega(x)\, dx$$

($\omega(x) > 0$).

Show that

$$P_n(x) := \frac{1}{\omega(x)} \frac{d^n}{dx^n} \left(\omega(x)(1-x^2)^n \right)$$

is orthogonal to every polynomial of degree less than n.

Hint: We proved this in class when $\omega \equiv 1$. Adapt that argument to this case.

Solution. Let f be a polynomial of degree $k < n$. Then we have

$$\langle f, P_n \rangle = \int_{-1}^{1} f(x) \frac{d^n}{dx^n} \left(\omega(x)(1-x^2)^n \right)\, dx.$$

Recall the two lemmas we proved in class. First of all, since $(1-x^2)$ is zero at ± 1, then if we define $g(x) = (1-x^2)^n$, g, and its first $n-1$ derivatives are as well, i.e.

$$g(\pm 1) = g'(\pm 1) = g''(\pm 1) = \cdots = g^{(n-1)}(\pm 1) = 0.$$

Now, the question is, does the function $\omega(x)g(x)$ also have the same property, i.e. are its first $n-1$ derivatives zero as well? The answer is yes: recall the product rule from calculus,

$$\frac{d^p}{dx^p}(\omega(x)g(x)) = \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) \omega^{(k)}(x) g^{(p-k)}(x),$$

and if we replace p with any integer less than n, then all the derivatives on g which appear in the sum are less than n, and these are all zero at ± 1, so therefore we know ωg and its first $n-1$ derivatives are zero at ± 1.

Now we use the other lemma we proved in class, namely that if

$$h(\pm 1) = h'(\pm 1) = \cdots = h^{(n-1)}(\pm 1) = 0,$$

then

$$\int_{-1}^{1} f(x) \frac{d^n}{dx^n} h(x)\, dx = (-1)^n \int_{-1}^{1} \frac{d^n}{dx^n} f(x) h(x)\, dx.$$

But notice that f is a polynomial of degree $k < n$, and so if we take n derivatives on f it is zero.

8. (Problem 2.2.14 from Keener.) Suppose that $f(t)$ and $g(t)$ are 2π-periodic functions with Fourier series representations

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt}, \quad g(t) = \sum_{k=-\infty}^{\infty} g_k e^{ikt}.$$

Now define

$$h(t) = \int_0^{2\pi} f(t-x) g(x)\, dx.$$

Compute the Fourier series for h.

Solution. We compute

$$\begin{aligned} h(t) &= \int_0^{2\pi} \sum_{k=-\infty}^{\infty} f_k e^{ik(t-x)} \sum_{l=-\infty}^{\infty} g_l e^{ilx}\, dx \\ &= \int_0^{2\pi} \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ik(t-x)} e^{ilx}\, dx \\ &= \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} \int_0^{2\pi} e^{i(l-k)x}\, dx. \end{aligned}$$

Now, if α is a non-zero integer, then

$$\int_0^{2\pi} e^{i\alpha x}\, dx = \left. \frac{e^{i\alpha x}}{i\alpha} \right|_{x=0}^{x=2\pi} = 0,$$

but if $\alpha = 0$ then the integral is 2π, so

$$\int_0^{2\pi} e^{i\alpha x}\, dx = 2\pi \delta_{\alpha,0}.$$

Thus we have

$$\begin{aligned} h(t) &= \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} \int_0^{2\pi} e^{i(l-k)x}\, dx \\ &= \sum_{k,l=-\infty}^{\infty} f_k g_l e^{ikt} 2\pi \delta_{l,k} \\ &= \sum_{k=-\infty}^{\infty} 2\pi f_k g_k e^{ikt}. \end{aligned}$$

So, the kth Fourier coefficient of h is $2\pi f_k g_k$, i.e. forming the convolution of f and g is equivalent (up to a constant) to multiplying their Fourier series term-by-term.