Methods of Mathematical Physics - 556 X1
Homework 4
Due October 31, 2008

1. (Problem 3.1.1 from Keener.) Verify that the solution of

$$u''(x) = f(x), \quad u(0) = 0, \quad u(1) = 0,$$

is given by

$$u(x) = \int_0^1 k(x,y) f(y)\, dy,$$

where

$$k(x,y) = \begin{cases} y(x-1), & 0 \le y < x \le 1, \\ x(y-1), & 0 \le x < y \le 1. \end{cases}$$

2. (Problem 3.2.1 from Keener.) Consider the functional $T\colon L^2[0,1] \to \mathbb{R}$ by $T(f) = f(0)$. Prove that this is a linear functional, but show that it is unbounded on L^2 (using the standard definition of L^2 norm).

However, now consider the functional $T\colon C^0[0,1] \to \mathbb{R}$ defined in the same way, i.e. $Tf = f(0)$, and use the "uniform norm" on C^0, i.e.

$$\|f\| = \sup_{x \in [0,1]} f(x).$$

Show that T is *now* a bounded linear functional.

(However, note that the Riesz theorem doesn't do anything for us here, since C^0 is not a Hilbert space — this norm does not come from an inner product.)

3. Let A be an $n \times n$ matrix, and consider the linear transformation $A\colon \mathbb{R}^n \to \mathbb{R}^n$.

(a) First assume that A is self-adjoint. Show that we can reorder the eigenvalues of A so that

$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|,$$

and show

$$\|A\| = |\lambda_1|.$$

(b) Now let A be general. Show that A^*A is self-adjoint, and moreover that the eigenvalues of A^*A are non-negative. (Then from (a), $\|A^*A\|$ is the largest eigenvalue of A^*A.)

Hint: Let x be an eigenvector of A^*A, consider the inner product

$$\langle A^*Ax, x\rangle.$$

(c) Prove that $\|A\| = \sqrt{\|A^*A\|}$.

Hint: The best way to do this is in two steps. First show $\left\|A^2\right\| = \|A\|^2$ (think about the definition of the operator norm). Then show $\|A^*A\| = \left\|A^2\right\|$.

NB. Thus to compute $\|A\|$, we simply have to compute the square root of the largest eigenvalue of A^*A. (In fact, the eigenvalues of A^*A are typically called the *singular values* of A for this and other reasons; we will see these again.)

4. (Problem 3.4.4 from Keener.) Find the adjoint kernel for an integral operator when we define the inner product to be

$$\langle f, g \rangle = \int_a^b f(x)g(x)\omega(x)\,dx.$$

More specifically, what we mean here is that if we assume that K is the integral operator

$$(Ku)(x) = \int_a^b k(x,y)u(y)\,dy,$$

then the adjoint of K (under the inner product above!) should have the representation

$$(K^*u)(x) = \int_a^b k^*(x,y)u(y)\,dy$$

for some function $k^*(x,y)$. Determine k^*. (To make things simpler, assume everything is real-valued.) Work out explicitly what the formula for K^*K is, and compute $\|K^*K\|$.

5. Consider the differential equation

$$x' = ax, \quad x(0) = x_0.$$

We know that the solution to this differential equation is $x(t) = x_0 e^{at}$, but let us prove this using Neumann iterates. Prove that the differential equation is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t ax(s)\,ds,$$

and then solve this integral equation iteratively.

6. (Problem 3.6.6 from Keener.)

(a) Show that the solution of

$$u(x) = 1 + \int_0^x s\ln(s/x)u(s)\,ds$$

satisfies the differential equation

$$u'' + x^{-1}u' + u = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

(b) Use Neumann iterates to show that

$$u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2}.$$

(This is actually the zeroth Bessel function.)