

Fundamental Mathematics - 557 X1 Homework 1

1. Define the exponential integral:

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt.$$

Follow the scheme introduced in lecture of repeated integration by parts to derive the asymptotic formula

$$E_1(x) = e^{-x} \sum_{k=0}^n \frac{(-1)^k k!}{x^{k+1}} + R_n(x),$$

$$R_n(x) = (-1)^n n! \int_x^\infty \frac{e^{-t}}{t^{n+1}} dt.$$

Show that $R_n(x) \rightarrow 0$ as $x \rightarrow \infty$ with n fixed, and $R_n(x) \rightarrow \infty$ as $n \rightarrow \infty$ with x fixed (we showed the first of these limits in class, modify that proof to show the second limit).

Hint. You might find it useful to define

$$E_k(x) = \int_x^\infty \frac{e^{-t}}{t^k} dt,$$

derive a recursive rule which looks something like

$$E_k(x) = f_k(x) + E_{k+1}(x),$$

for some function $f_k(x)$, and then apply induction.

2. (Similar to the last) Define the complementary incomplete Gamma function

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt.$$

Prove that

$$\Gamma(a, x) = \sum_{k=1}^n \frac{\Gamma(a)}{\Gamma(a-k+1)} e^{-x} x^{a-k} + \frac{\Gamma(a)}{\Gamma(a-n)} \Gamma(a-n, x),$$

where

$$\Gamma(a) = \lim_{x \rightarrow 0} \Gamma(a, x).$$

3. Consider the polynomial $ax^2 + bx + c = 0$, where $a > 0, b \neq 0, c \neq 0$. In each of three cases, place and ϵ in front of each term, and then derive an asymptotic series for the roots, namely consider the roots for the three cases

- (a) $ax^2 + bx + \epsilon c = 0$,
- (b) $ax^2 + \epsilon bx + c = 0$,
- (c) $\epsilon ax^2 + bx + c = 0$.

Show that in the first two cases one can use the sequence of gauge functions given by $\{1, \epsilon, \epsilon^2, \dots\}$ but in the third case we must take $\{\epsilon^{-1}, 1, \epsilon, \epsilon^2\}$. In each case, calculate the first three coefficients in the asymptotic series for each root. (It is ok to only consider real roots in this calculation, however, be careful to specify any necessary conditions on the coefficients a, b, c to make the roots real.)

4. We showed that the Fourier transform is an isometry of L^2 to itself, i.e. that $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ through a complicated argument involving dense subspace, etc. The analogous argument for Fourier series is a bit simpler and we will work on that here.

Let us consider $f \in L^2_{per}([0, 2\pi])$, i.e.

$$f: [0, 2\pi] \rightarrow \mathbb{C}, \quad f \in L^2, \quad f(x) \text{ is } 2\pi\text{-periodic,}$$

where we consider the usual L^2 norm

$$\|f\|_{L^2} = \int_0^{2\pi} |f(x)|^2 dx.$$

We also define the linear space ℓ^2 by all bi-infinite sequences $\{a_k\}_{k=-\infty}^{\infty}$ such that

$$\|a\|_{\ell^2} := \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty.$$

Let us write

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \tag{1}$$

where $a_k \in \mathbb{C}$. We call the a_k the Fourier coefficients of f and write

$$\hat{f}_k = a_k.$$

- (a) Show that, formally,

$$a_k = \int_0^{2\pi} f(x) e^{ikx} dx.$$

- (b) Show that f is real iff $\bar{a}_k = a_{-k}$.

- (c) Show that the map from L^2 to ℓ^2 where we send a function f to its sequence of Fourier coefficients is also an isometry, i.e.

$$\|f\|_{L^2} = \|a\|_{\ell^2},$$

where the a_k are given by (1).

- (d) Use this to prove that the Fourier representation is a bijection from L^2 to ℓ^2 and that the calculation in part (a) is not formal.

- (e) For each $f \in L^2_{per}([0, 2\pi])$, define the *upscale operator* U by

$$\widehat{Uf}_k = \hat{f}_{k+1}.$$

Prove that U is a bounded operator, i.e. that

$$\sup_{\|f\|_{L^2} \neq 0} \frac{\|Uf\|_{L^2}}{\|f\|_{L^2}} < \infty.$$

- (f) Consider the set $C^\infty_{per}([0, 2\pi])$ given by all infinitely smooth functions with period 2π . First show this set is a subset of $L^2_{per}([0, 2\pi])$. Now define the differentiation operator $D: C^\infty_{per}([0, 2\pi]) \rightarrow C^\infty_{per}([0, 2\pi])$ given by

$$Df(x) = f'(x).$$

Show D is not a bounded operator, i.e.

$$\sup_{\|f\|_{L^2} \neq 0} \frac{\|Df\|_{L^2}}{\|f\|_{L^2}} = \infty.$$

The best way to do this is to give a sequence of functions f_n such that $\|f_n\|_{L^2} = 1$ but $\|Df_n\|_{L^2} \rightarrow \infty$ as $n \rightarrow \infty$.

(g) Define the integration operator $I: C_{per}^\infty([0, 2\pi]) \rightarrow C_{per}^\infty([0, 2\pi])$ by

$$If(x) = \int_0^x f(t) dt.$$

Prove that I is a bounded operator.

5. Use the formulas from class and integration by parts to show

$$\int_{-\infty}^{\infty} x^{2k} e^{-x^2/\sigma^2} dx = \sigma^{2k+1} \Gamma(k + 1/2),$$
$$\int_{-\infty}^{\infty} x^{2k+1} e^{-x^2/\sigma^2} dx = 0,$$

where $\sigma > 0$ and k is any positive integer.