Fundamental Mathematics - 557 X1
Homework 2

1. Estimate, up to an error of size $O(\epsilon^2)$, the eigenvalues and eigenvectors of

   (a)
   $$\begin{pmatrix} 1 & \epsilon \\ 0 & 2 \end{pmatrix},$$

   (b)
   $$\begin{pmatrix} 1 & \epsilon & 2\epsilon \\ 0 & 2 & 0 \\ 1-\epsilon & 0 & -1 \end{pmatrix}.$$

   (c)
   $$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 2\epsilon & 0 \\ 0 & 0 & 3\epsilon \end{pmatrix}.$$

2. Find two linearly independent solutions of
   $$u'' + q(x)u = 0,$$
   where
   $$q(x) = \begin{cases} -1, & x < 0, \\ 1 & x > 0 \end{cases}$$
   for u defined on $x \in (-\infty, \infty)$. (This is problem 1 in Section 7.5 of Keener.)

3. Define the ODE by
   $$x' = 2x + \epsilon f(x, y),$$
   $$y' = -3y + \epsilon g(x, y).$$
   Describe all possible monomial choices of f and g which can be removed by a near-identity change of coordinates. Compute what the change of coordinates would be in each case. Compute the system in the new coordinates in each case.

4. Define $f(u) = u - u^3/3 + 0.1$. First show that $F(u)$ defined by $F' = -f$ is a double-well potential and determine the relative depth of each well with respect to the other. Define the minima of these wells as $u_\pm$ as in class. Consider the boundary-value problem
   $$u'' + f(u) = 0$$
   with
   $$\lim_{x\to\infty} u(x) = u_-,$$
   $$\lim_{x\to-\infty} u(x) = u_+.$$
   Describe why this problem does not have a solution using phase-plane arguments. Now consider the modified equation
   $$u'' + cu' + f(u) = 0$$
   with the same boundary conditions. Determine to four digits' accuracy the value of c which does give a solution to this boundary value problem.

   Hint. One could write original code to solve this problem, but you might find it easier to use code available on the web. Go to http://math.rice.edu/~dfield/dfpp.html and use the "pplane" software: this software allows you to type in a vector-field and plot trajectories by pointing. By playing with the value of c, you can determine which value of c gives a solution to this problem to any desired accuracy.